(b) The result is
\[ \frac{E_1 (|f| \leq W)}{E_{\text{total}}} = 2 \int_0^W \sin^2 (u) \, du \]
The integration must be carried out numerically.

**Problem 2.34**
(a) By the modulation theorem
\[ X(f) = \frac{AT_0}{4} \left\{ \sin \left[ \frac{f - f_0}{2} \left( T_0 \right) \right] + \sin \left[ \frac{f + f_0}{2} \left( T_0 \right) \right] \right\} \]

(b) Use the superposition and modulation theorems to get
\[ X(f) = \frac{AT_0}{4} \left\{ \sin \left[ \frac{f}{2f_0} \right] + \frac{1}{2} \sin \left[ \frac{1}{2} \left( \frac{f}{f_0} - 1 \right) \right] + \sin \left[ \frac{1}{2} \left( \frac{f}{f_0} + 1 \right) \right] \right\} \]

**Problem 2.35**
Combine the exponents of the two factors in the integrand of the Fourier transform integral, complete the square, and use the given definite integral.

**Problem 2.36**
Consider the development below:
\[ x(t) \ast x(-t) = \int_{-\infty}^{\infty} x(-\lambda) x(t - \lambda) \, d\lambda = \int_{-\infty}^{\infty} x(\beta) x(t + \beta) \, d\beta \]
where \( \beta = -\lambda \) has been substituted. Rename variables to obtain
\[ R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(\beta) x(t + \beta) \, d\beta \]

**Problem 2.37**
The result is an even triangular wave with zero average value of period \( T_0 \). It makes no difference whether the original square wave is even or odd or neither.
Problem 2.52
In terms of the input spectrum, the output spectrum is

\[ Y(f) = X(f) + 0.2X(f) \ast X(f) \]

\[ = 4 \left[ \Pi \left( \frac{f-20}{6} \right) + \Pi \left( \frac{f+20}{6} \right) \right] \]
\[ + 3.2 \left[ \Pi \left( \frac{f-20}{6} \right) + \Pi \left( \frac{f+20}{6} \right) \right] \ast \left[ \Pi \left( \frac{f-20}{6} \right) + \Pi \left( \frac{f+20}{6} \right) \right] \]
\[ = 4 \left[ \Pi \left( \frac{f-20}{6} \right) + \Pi \left( \frac{f+20}{6} \right) \right] \]
\[ + 3.2 \left[ 6 \Lambda \left( \frac{f-40}{6} \right) + 12 \Lambda \left( \frac{f}{6} \right) + 6 \Lambda \left( \frac{f+40}{6} \right) \right] \]

where \( \Lambda(t) \) is an isosceles triangle of unit height going from -1 to 1. The student should sketch the output spectrum given the above analytical result..

Problem 2.53
(a) The output of the nonlinear device is

\[ y(t) = 1.075 \cos(2000\pi t) + 0.025 \cos(6000\pi t) \]

The steadystate filter output in response to this input is

\[ z(t) = 1.075 |H(1000)| \cos(2000\pi t) + 0.025 |H(3000)| \cos(6000\pi t) \]

so that the THD is

\[ \text{THD} = \frac{(1.075)^2 |H(1000)|^2}{(0.025)^2 |H(3000)|^2} \]
\[ = \frac{43^2}{1 + 4Q^2 (1000 - 3000)^2} \]
\[ = \frac{1849}{1 + 16 \times 10^6 Q^2} \]

where \( H(f) \) is the transfer function of the filter.

(b) For \( \text{THD} = 0.005\% = 0.00005 \), the equation for \( Q \) becomes

\[ \frac{1849}{1 + 16 \times 10^6 Q^2} = 0.00005 \]
2.1. PROBLEM SOLUTIONS

or

\[
6 \times 10^6 Q^2 = 1849 \times 2 \times 10^4 - 1
\]

\[
Q = 1.52
\]

**Problem 2.54**

Frequency components are present in the output at radian frequencies of 0, \(\omega_1, \omega_2\), \(2\omega_1, 2\omega_2, \omega_1 - \omega_2, \omega_1 + \omega_2, 3\omega_1, 3\omega_2, 2\omega_2 - \omega_1, \omega_1 + 2\omega_2, 2\omega_1 - \omega_2, 2\omega_1 + \omega_2\). To use this as a frequency multiplier, pass the components at \(2\omega_1\) or \(2\omega_2\) to use as a doubler, or at \(3\omega_1\) or \(3\omega_2\) to use as a tripler.

**Problem 2.55**

Write the transfer function as

\[
H(f) = H_0 e^{-j2\pi f t_0} - H_0 \Pi \left( \frac{f}{2B} \right) e^{-j2\pi f t_0}
\]

Use the inverse Fourier transform of a constant, the delay theorem, and the inverse Fourier transform of a rectangular pulse function to get

\[
h(t) = H_0 \delta(t - t_0) - 2BH_0 \text{sinc}[2B(t - t_0)]
\]

**Problem 2.56**

(a) The Fourier transform of this signal is

\[
X(f) = A\sqrt{2\pi b^2} \exp(-2\pi^2 f^2)
\]

By definition, using a table of integrals,

\[
T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)| \, dt = \sqrt{2\pi \tau}
\]

Similarly,

\[
W = \frac{1}{2X(0)} \int_{-\infty}^{\infty} |X(f)| \, df = \frac{1}{2\sqrt{2\pi \tau}}
\]

Therefore,

\[
2WT = \frac{2}{2\sqrt{2\pi \tau}} \sqrt{2\pi \tau} = 1
\]
where
\[ X_\delta (f) = f_s \sum_{n=-\infty}^{\infty} X (f - nf_s) \]
and
\[ H (f) = \tau \text{sinc} (f \tau) \exp (-j \pi f \tau) \]
The latter represents the frequency response of a filter whose impulse response is a square pulse of width \( \tau \) and implements flat top sampling. If \( W \) is the bandwidth of \( X (f) \), very little distortion will result if \( \tau^{-1} \gg W \).

**Problem 2.60**
(a) The sampling frequency should be large compared with the bandwidth of the signal.
(b) The output spectrum of the zero-order hold circuit is
\[
Y (f) = \text{sinc} (T_s f) \sum_{n=-\infty}^{\infty} X (f - nf_s) \exp (-j \pi f T_s)
\]
where \( f_s = T_s^{-1} \). For small distortion, we want \( T_s \ll W^{-1} \).

**Problem 2.61**
The lowpass recovery filter can cut off in the range \( 1.9^+ \) kHz to \( 2.1^- \) kHz.

**Problem 2.62**
For bandpass sampling and recovery, all but (b) and (e) will work theoretically, although an ideal filter with bandwidth exactly equal to the unsampled signal bandwidth is necessary. For lowpass sampling and recovery, only (f) will work.

**Problem 2.63**
The Fourier transform is
\[
Y (f) = \frac{1}{2} X (f - f_0) + \frac{1}{2} X (f + f_0) + [-j \text{sgn} (f) X (f)] * \left[ \frac{1}{2} \delta (f - f_0) e^{-j \pi/2} + \frac{1}{2} \delta (f + f_0) e^{j \pi/2} \right]
\]
\[
= \frac{1}{2} X (f - f_0) [1 - \text{sgn} (f - f_0)] + \frac{1}{2} X (f + f_0) [1 + \text{sgn} (f + f_0)]
\]
2.1. PROBLEM SOLUTIONS

Problem 2.64
(a) \( \hat{x}_a (t) = \cos (\omega_0 t - \pi/2) = \sin (\omega_0 t) \), so

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \hat{x}(t) \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sin (\omega_0 t) \cos (\omega_0 t) \, dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{2} \sin (2\omega_0 t) \, dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \cos (2\omega_0 t) \bigg|_{-T}^{T} = 0
\]

(b) Use trigonometric identities to express \( x(t) \) in terms of sines and cosines. Then find the Hilbert transform of \( x(t) \) by phase shifting by \( -\pi/2 \). Multiply \( x(t) \) and \( \hat{x}(t) \) together term by term, use trigonometric identities for the product of sines and cosines, then integrate. The integrand will be a sum of terms similar to that of part (a). The limit as \( T \to \infty \) will be zero term-by-term.

(c) Use the integral definition of \( \hat{x}(t) \), take the product, integrate over time to get

\[
\int_{-\infty}^{\infty} x(t) \hat{x}(t) \, dt = A^2 \int_{-\infty}^{\infty} \Pi (t/\tau) \left[ \int_{-\infty}^{\infty} \frac{\Pi (\lambda/\tau)}{\pi (t - \lambda)} \, d\lambda \right] \, dt
\]

\[
= A^2 \int_{-\tau/2}^{\tau/2} \left[ \int_{-\tau/2}^{\tau/2} \frac{1}{\pi (t - \lambda)} \, d\lambda \right] \, dt
\]

\[
= A^2 \int_{-\tau/2}^{\tau/2} \frac{1}{\pi} \ln\left| \frac{t - \tau/2}{t + \tau/2} \right| \, dt = 0
\]

where the result is zero by virtue of the integrand of the last integral being odd.

Problem 2.65
(a) Note that \( F[j\hat{x}(t)] = j[-\text{sgn}(f)] X(f) \). Hence

\[
x_1(t) = \frac{3}{4} x(t) + \frac{1}{4} j\hat{x}(t) \rightarrow X_1(f) = \frac{3}{4} X(f) + \frac{1}{4} j[-\text{sgn}(f)] X(f)
\]

\[
= \left[ \frac{3}{4} + \frac{1}{4} \text{sgn}(f) \right] X(f)
\]

\[
= \begin{cases} 
\frac{3}{4} X(f), & f < 0 \\
X(f), & f > 0 
\end{cases}
\]

A sketch is shown in Figure 2.4.
(b) It follows that
\[ x_2(t) = \left[ \frac{3}{4}x(t) + \frac{3}{4}j\hat{x}(t) \right] \exp(j2\pi f_0 t) \]
\[ \rightarrow \quad X_2(f) = \frac{3}{4}[1 + \text{sgn}(f - f_0)]X(f - f_0) \]
\[ = \begin{cases} 
\frac{3}{2}X(f - f_0), & f > f_0 \\
0, & f < f_0
\end{cases} \]
A sketch is shown in Figure 2.4.
(c) This case has the same spectrum as part (a), except that it is shifted right by \( W \) Hz. That is,
\[ x_3(t) = \left[ \frac{3}{4}x(t) + \frac{1}{4}j\hat{x}(t) \right] \exp(j2\pi W t) \]
\[ \rightarrow \quad X_3(f) = \left[ \frac{3}{4} + \frac{1}{4}\text{sgn}(f - W) \right]X(f - W) \]
A sketch is shown in Figure 2.4.
(d) For this signal
\[ x_4(t) = \left[ \frac{3}{4}x(t) - \frac{1}{4}j\hat{x}(t) \right] \exp(j\pi W t) \]
\[ \rightarrow \quad X_4(f) = \left[ \frac{3}{4} - \frac{1}{4}\text{sgn}(f - W/2) \right]X(f - W/2) \]
A sketch is shown in Figure 2.4.

**Problem 2.66**
(a) The spectrum is
\[ X_p(f) = X(f) + j[-\text{sgn}(f)]X(f) = [1 + \text{sgn}(f)]X(f) \]
The Fourier transform of \( x(t) \) is
\[ X(f) = \frac{1}{2} \Pi \left( \frac{f - f_0}{2W} \right) + \frac{1}{2} \Pi \left( \frac{f + f_0}{2W} \right) \]
Thus,
\[ X_p(f) = \Pi \left( \frac{f - f_0}{2W} \right) \text{ if } f_0 > 2W \]
(b) The complex envelope is defined by
\[ x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} \]
Figure 2.4: