2.1. PROBLEM SOLUTIONS

(d) The convolution gives
\[ y_4(t) = \int_{-\infty}^{t} x(\lambda) d\lambda \]

Problem 2.32

(a) Using the convolution and time delay theorems, we obtain
\[
Y_1(f) = F\left[ e^{-\alpha t} u(t) * \Pi(t - \tau) \right] \\
= F\left[ e^{-\alpha t} u(t) \right] F[\Pi(t - \tau)] \\
= \frac{1}{\alpha + j2\pi f} \text{sinc}(f) e^{-j2\pi f \tau}
\]

(b) The superposition and convolution theorems give
\[
Y_2(f) = F\left\{ [\Pi(t/2) + \Pi(t)] * \Pi(t) \right\} \\
= [2\text{sinc}(2f) + \text{sinc}(f)] \text{sinc}(f)
\]

(c) By the convolution theorem
\[
Y_3(f) = F\left[ e^{-\alpha |t|} * \Pi(t) \right] \\
= \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \text{sinc}(f)
\]

(d) By the convolution theorem (note, also, that the integration theorem can be applied directly)
\[
Y_4(f) = F[x(t) * u(t)] \\
= X(f) \left[ \frac{1}{j2\pi f} + \frac{1}{2} \delta(f) \right] \\
= \frac{X(f)}{j2\pi f} + \frac{1}{2} X(0) \delta(f)
\]

Problem 2.33

(a) The normalized inband energy is
\[
\frac{E_1(|f| \leq W)}{E_{\text{total}}} = \frac{2}{\pi} \tan^{-1}\left( \frac{2\pi W}{\alpha} \right)
\]
(b) The result is
\[ \frac{E_1(|f| \leq W)}{E_{\text{total}}} = 2 \int_0^{\pi W} \sin^2(u) \, du \]
The integration must be carried out numerically.

**Problem 2.34**

(a) By the modulation theorem
\[ X(f) = \frac{A T_0}{4} \left\{ \text{sinc} \left(\frac{f - f_0}{2} T_0\right) + \text{sinc} \left(\frac{f + f_0}{2} T_0\right) \right\} \]
\[ = \frac{A T_0}{4} \left\{ \text{sinc} \left(\frac{1}{2} \left(\frac{f}{f_0} - 1\right)\right) + \text{sinc} \left(\frac{1}{2} \left(\frac{f}{f_0} + 1\right)\right) \right\} \]

(b) Use the superposition and modulation theorems to get
\[ X(f) = \frac{A T_0}{4} \left\{ \text{sinc} \left(\frac{f}{2f_0}\right) + \frac{1}{2} \left[ \text{sinc} \frac{1}{2} \left(\frac{f}{f_0} - 2\right) + \text{sinc} \frac{1}{2} \left(\frac{f}{f_0} + 2\right) \right] \right\} \]

**Problem 2.35**

Combine the exponents of the two factors in the integrand of the Fourier transform integral, complete the square, and use the given definite integral.

**Problem 2.36**

Consider the development below:
\[ x(t) \ast x(-t) = \int_{-\infty}^{\infty} x(-\lambda) x(t - \lambda) \, d\lambda = \int_{-\infty}^{\infty} x(\beta) x(t + \beta) \, d\beta \]
where \( \beta = -\lambda \) has been substituted. Rename variables to obtain
\[ R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(\beta) x(t + \beta) \, d\beta \]

**Problem 2.37**

The result is an even triangular wave with zero average value of period \( T_0 \). It makes no difference whether the original square wave is even or odd or neither.
Problem 2.40
Use the transform pair for a sinc function to find that
\[ Y(f) = \Pi \left( \frac{f}{2B} \right) \Pi \left( \frac{f}{2W} \right) \]

(a) If \( W < B \), it follows that
\[ Y(f) = \Pi \left( \frac{f}{2W} \right) \]
because \( \Pi \left( \frac{f}{2W} \right) = 1 \) throughout the region where \( \Pi \left( \frac{f}{2W} \right) \) is nonzero.

(b) If \( W > B \), it follows that
\[ Y(f) = \Pi \left( \frac{f}{2B} \right) \]
because \( \Pi \left( \frac{f}{2W} \right) = 1 \) throughout the region where \( \Pi \left( \frac{f}{2B} \right) \) is nonzero.

Problem 2.41
(a) Replace the capacitors with \( 1/j\omega C \) which is their ac-equivalent impedance. Call the junction of the input resistor, feedback resistor, and capacitors 1. Call the junction at the positive input of the operational amplifier 2. Call the junction at the negative input of the operational amplifier 3. Write down the KCL equations at these three junctions. Use the constraint equation for the operational amplifier, which is \( V_2 = V_3 \), and the definitions for \( \omega_0, Q, \) and \( K \) to get the given transfer function.

(d) Combinations of components giving \( RC = 2.3 \times 10^{-4} \) seconds

and

\[ \frac{R_a}{R_b} = 2.5757 \]

will work.

Problem 2.42
(a) By long division
\[ H(f) = 1 - \frac{R_1/L}{R_3 + R_2 + j2\pi f} \]
Using the transforms of a delta function and a one-sided exponential, we obtain
\[ h(t) = \delta(t) - \frac{R_1}{L} \exp \left( -\frac{R_1 + R_2}{L} t \right) u(t) \]
Problem 2.45
The energy spectral density of the output is

\[ G_y(f) = |H(f)|^2 |X(f)|^2 \]

where

\[ X(f) = \frac{1}{2 + j2\pi f} \]

Hence

\[ G_y(f) = \frac{100}{9 + (2\pi f)^2} \left[ 4 + (2\pi f)^2 \right] \]

Problem 2.46
Using the Fourier coefficients of a half-rectified sine wave from Table 2.1 and noting that those of a half-rectified cosine wave are related by

\[ X_{cn} = X_{sn} e^{-j\pi/2} \]

The fundamental frequency is 10 Hz. The ideal rectangular filter passes all frequencies less than 13 Hz and rejects all frequencies greater than 13 Hz. Therefore

\[ y(t) = \frac{3A}{\pi} - \frac{3A}{2} \cos(20\pi t) \]

Problem 2.47
(a) The 90% energy containment bandwidth is given by

\[ B_{90} = \frac{\alpha}{2\pi} \tan (0.45\pi) = 1.0055\alpha \]

(b) For this case, using \( X_2(f) = \Pi(f/2W) \), we obtain

\[ B_{90} = 0.9W \]

(c) Numerical integration gives

\[ B_{90} = 0.85/\tau \]
Solution of Extra Problem

a. \[ x(t) = \sum_{n=-\infty}^{\infty} 100 \prod_{n=0}^{\infty} (4000t - 4n) = [100 \ \Pi(4000t)]^* \sum_{n=-\infty}^{\infty} \delta(t - n / 1000) \]

\[ X(f) = F\{[100 \ \Pi(4000t)]\} \ F\{ \sum_{n=-\infty}^{\infty} \delta(t - n / 1000) \} = \]

\[ = (100/4000) \ \text{Sinc}(f/4000) \ {1000 \ \sum_{n=-\infty}^{\infty} \delta(f - 1000n)} = \]

\[ = 25 \ \sum_{n=-\infty}^{\infty} \text{Sinc}(0.25n) \delta(f - n / 1000) \]

b. \[ Y(f) = H(f)X(f) – \text{it is not zero only for } |f|=2000 \text{ and } |f|=3000 \]

The phase of \( H(f) \) at those frequencies is 1.

\[ Y(f) = 25(\pi^2/2) \ {\text{Sinc}(0.5)[\delta(f-2000)+ \delta(f+2000)]+} \]

\[ +\text{Sinc}(0.75)[ \delta(f-3000)+ \delta(f+3000)]\]

\[ = 25(\pi^2/2)\{(2/\pi)[\delta(f-2000)+ \delta(f+2000)]+[4/(3\pi)](\sqrt{2}/2) \ [ \delta(f-3000)+ \delta(f+3000)]\}\]

\[ = 25\pi\{[\delta(f-2000)+ \delta(f+2000)]+ (\sqrt{2} /3)[\delta(f-3000)+ \delta(f+3000)]\}\]

\[ y(t) = 50\pi \ \{ \cos (4000\pi t) + (\sqrt{2} /3)\cos (6000\pi t) \} \]