# Cost of Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

# Cost of Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

⇒

```
(interp {with {y 2}
          {+ 100 {+ 99 {+ 98 ... {+ y 1}}}}})
```

# Cost of Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

⇒

```
(interp {with {y 2}
          {+ 100 {+ 99 {+ 98 ... {+ y 1}}}}})
```

⇒

```
(interp {+ 100 {+ 99 {+ 98 ... {+ 2 1}}}})
```

# Cost of Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

⇒

```
(interp {with {y 2}
          {+ 100 {+ 99 {+ 98 ... {+ y 1}}}}})
```

⇒

```
(interp {+ 100 {+ 99 {+ 98 ... {+ 2 1}}}})
```

With **n** variables, evaluation will take $O(\mathbf{n}^2)$ time!

# Deferring Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

# Deferring Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

⇒

x = 1

```
(interp {with {y 2}
          {+ 100 {+ 99 {+ 98 ... {+ y x}}}}})
```

# Deferring Substitution

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}})
```

⇒

```
(interp {with {y 2}
          {+ 100 {+ 99 {+ 98 ... {+ y x}}}}})
```

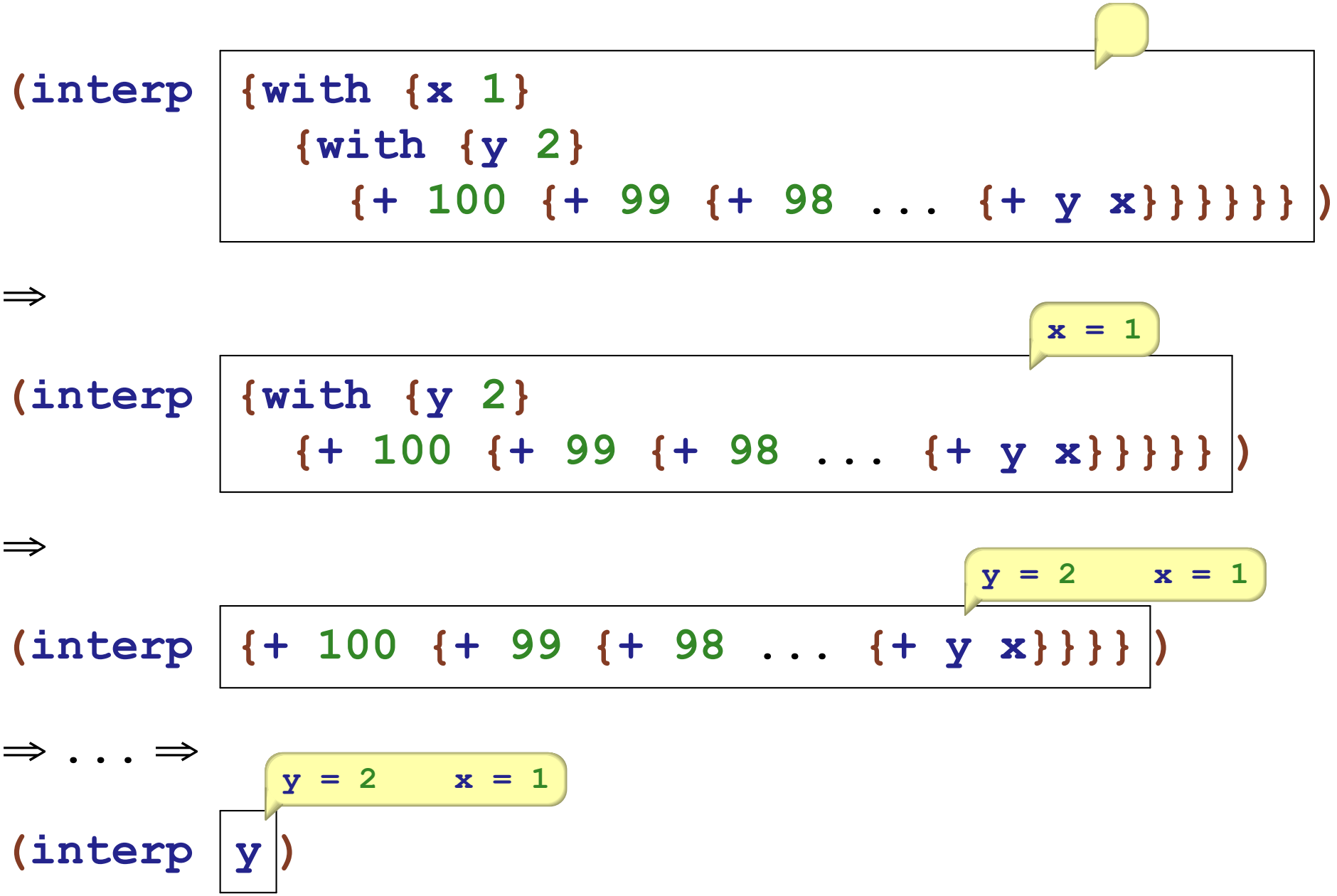
x = 1

⇒

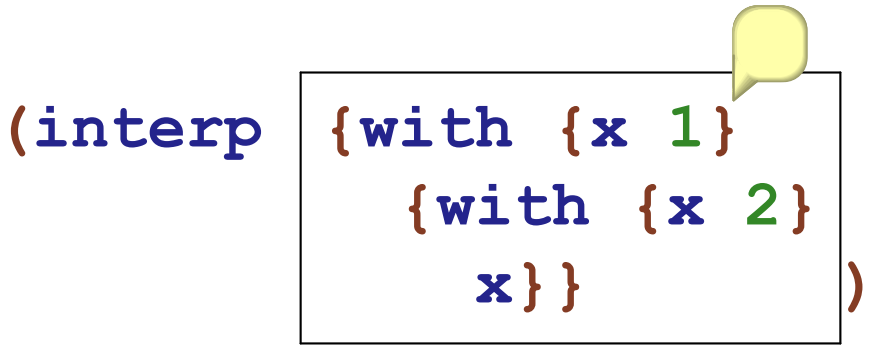
```
(interp {+ 100 {+ 99 {+ 98 ... {+ y x}}}})
```

y = 2  x = 1

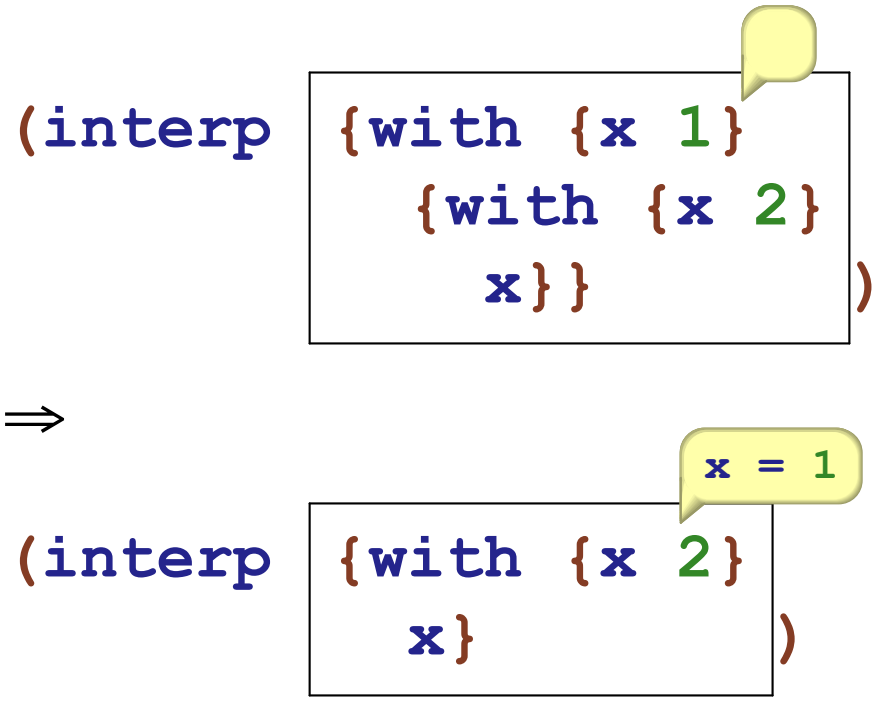
# Deferring Substitution



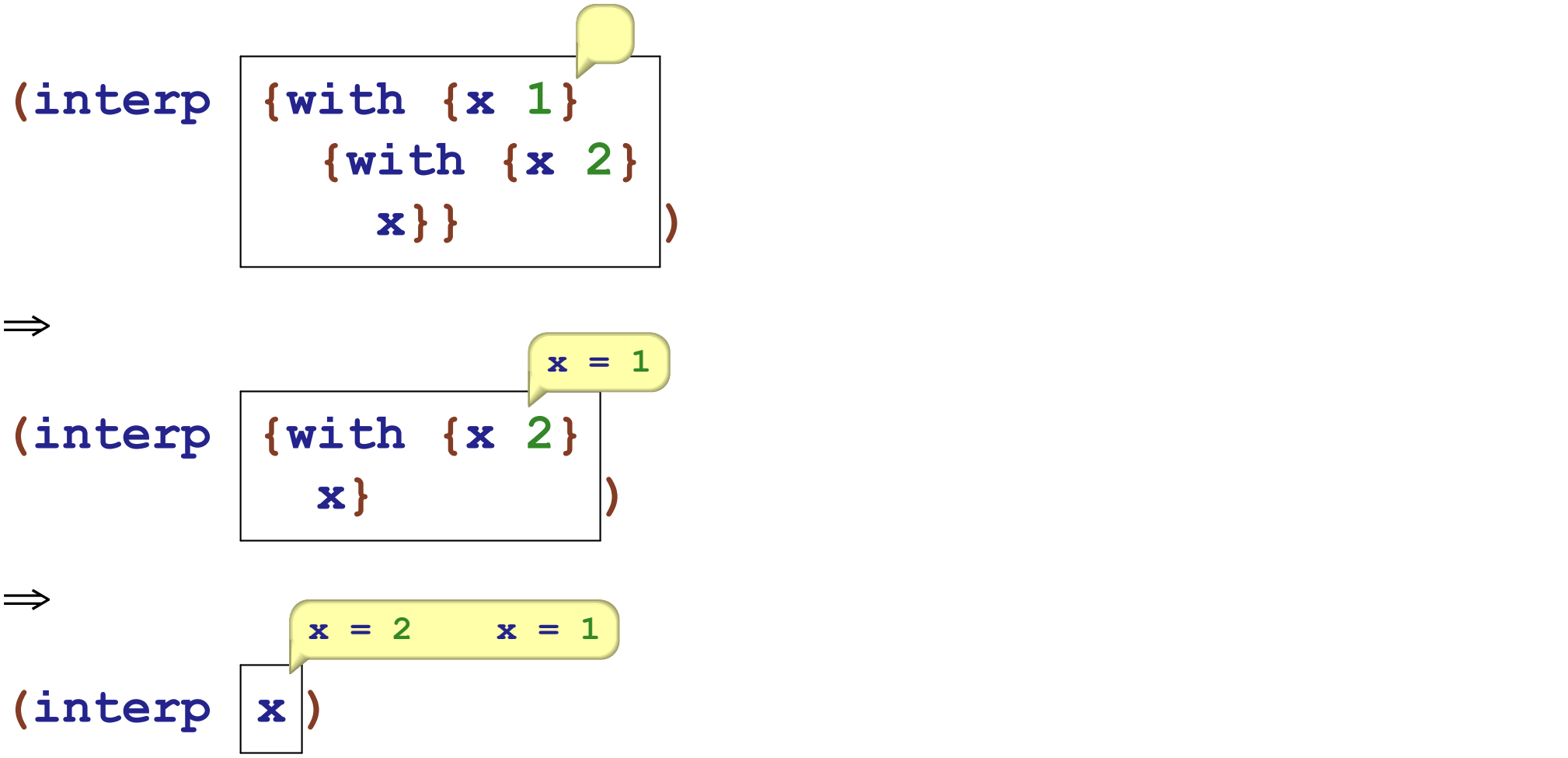
# Deferring Substitution with the Same Identifier



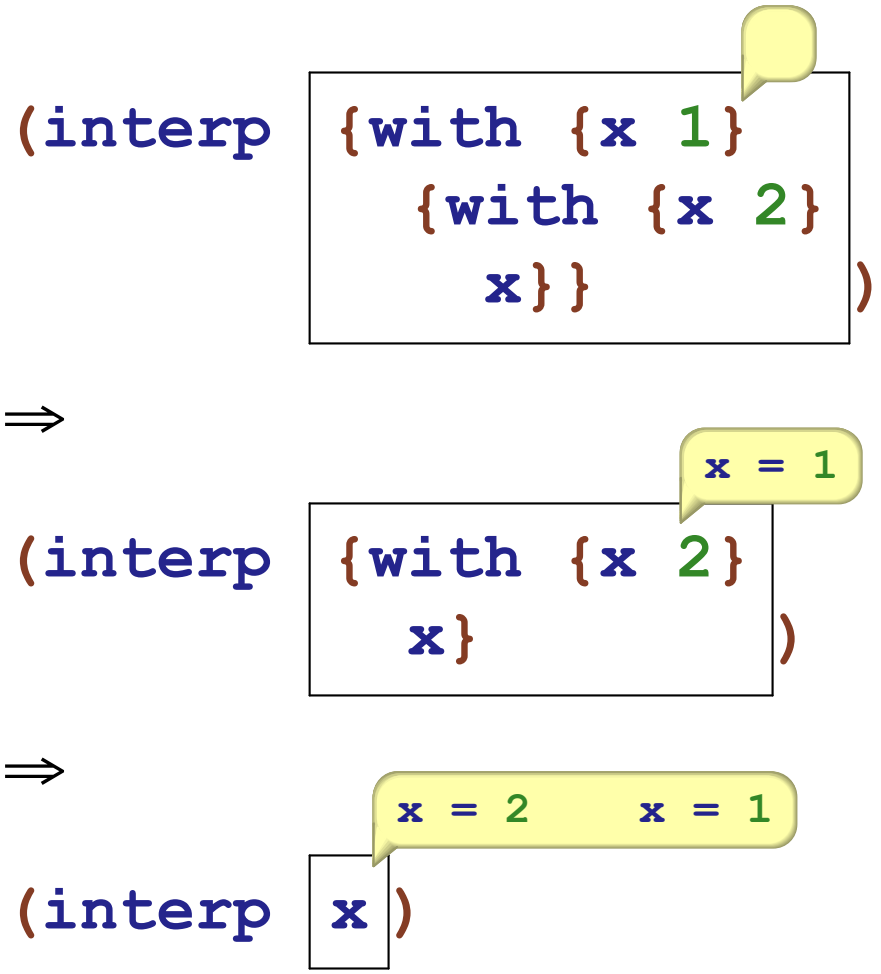
# Deferring Substitution with the Same Identifier



# Deferring Substitution with the Same Identifier

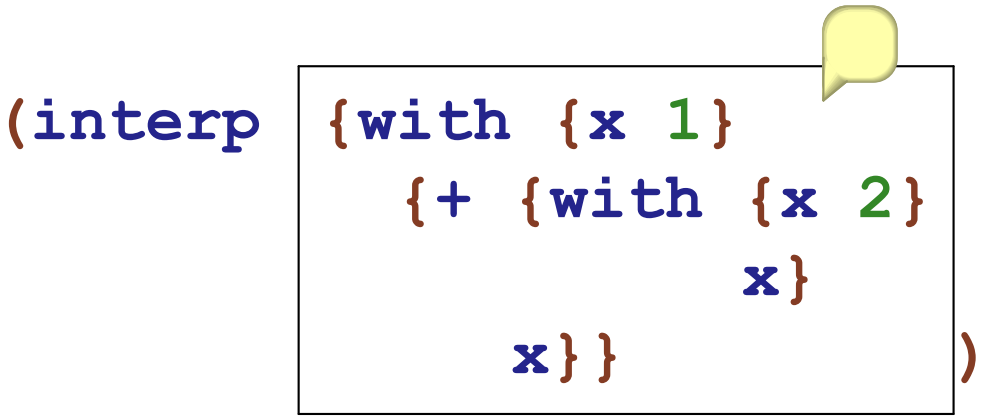


# Deferring Substitution with the Same Identifier

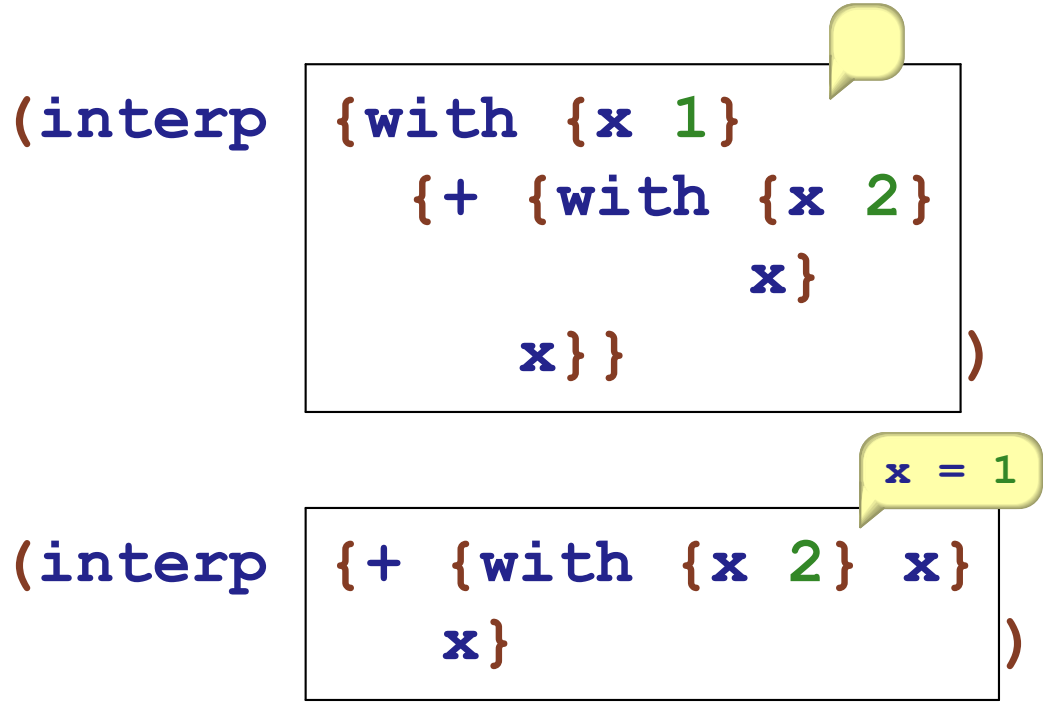



Always add to start, then always check from start

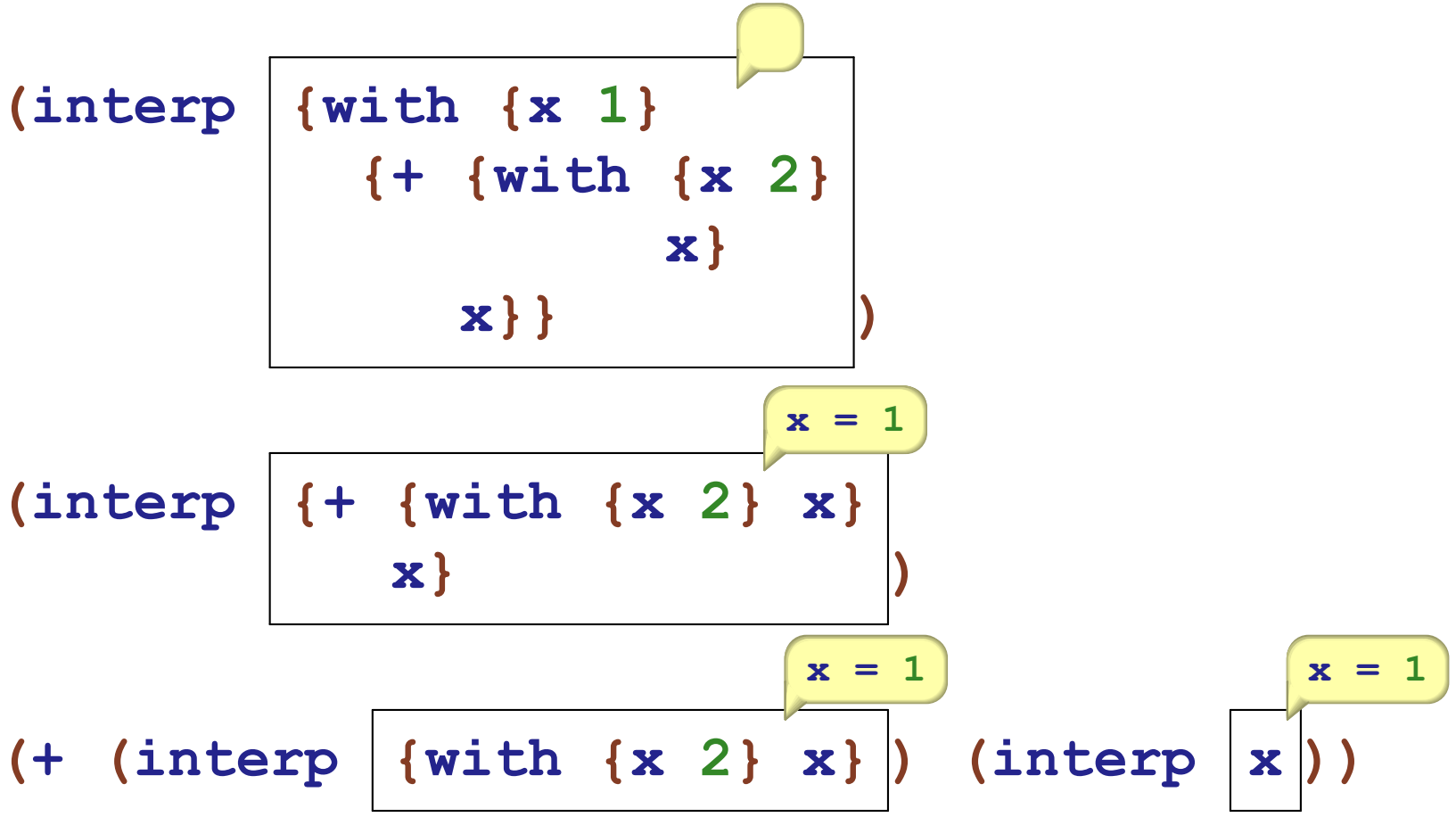
# Deferring Substitution with the Same Identifier



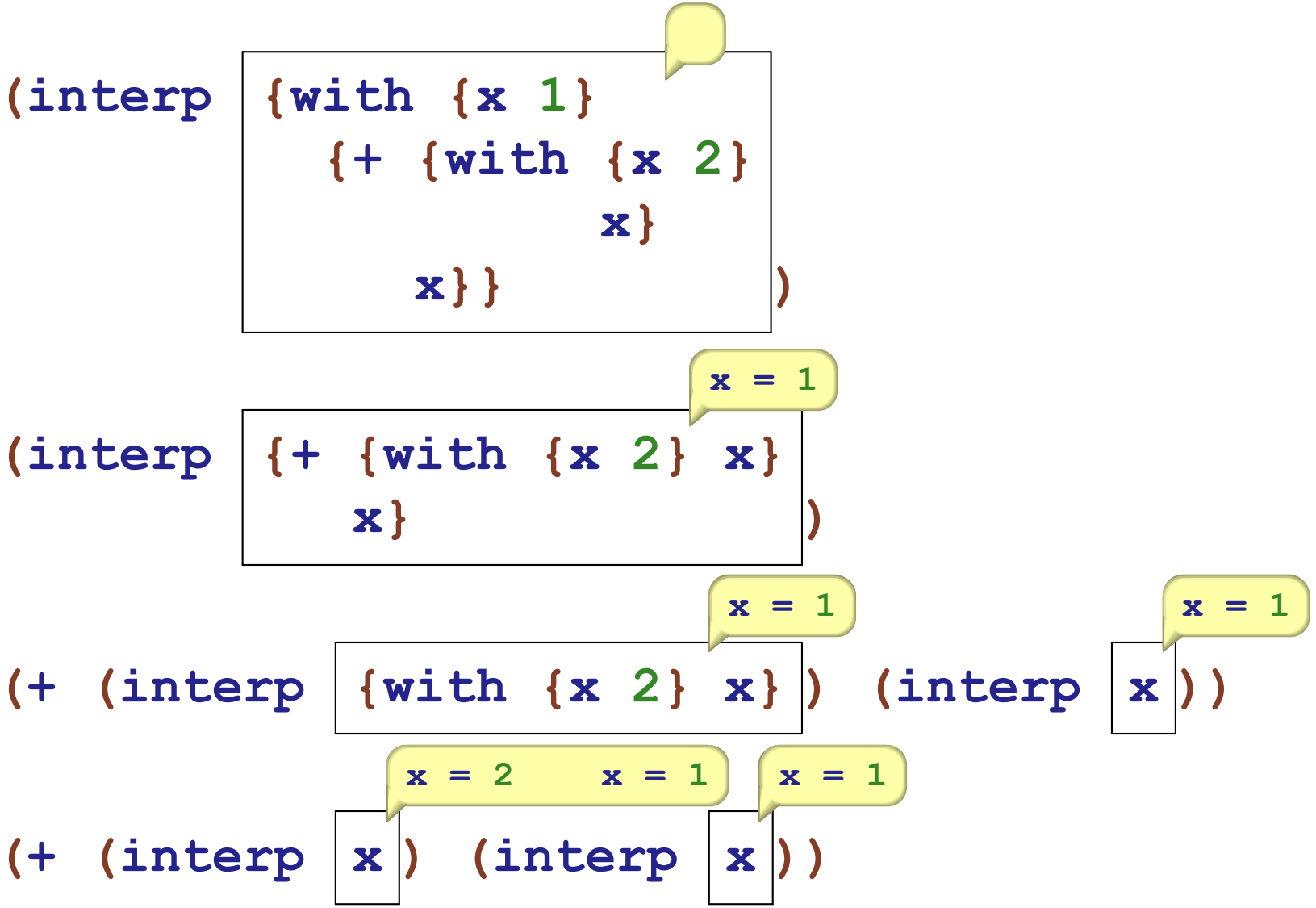
# Deferring Substitution with the Same Identifier



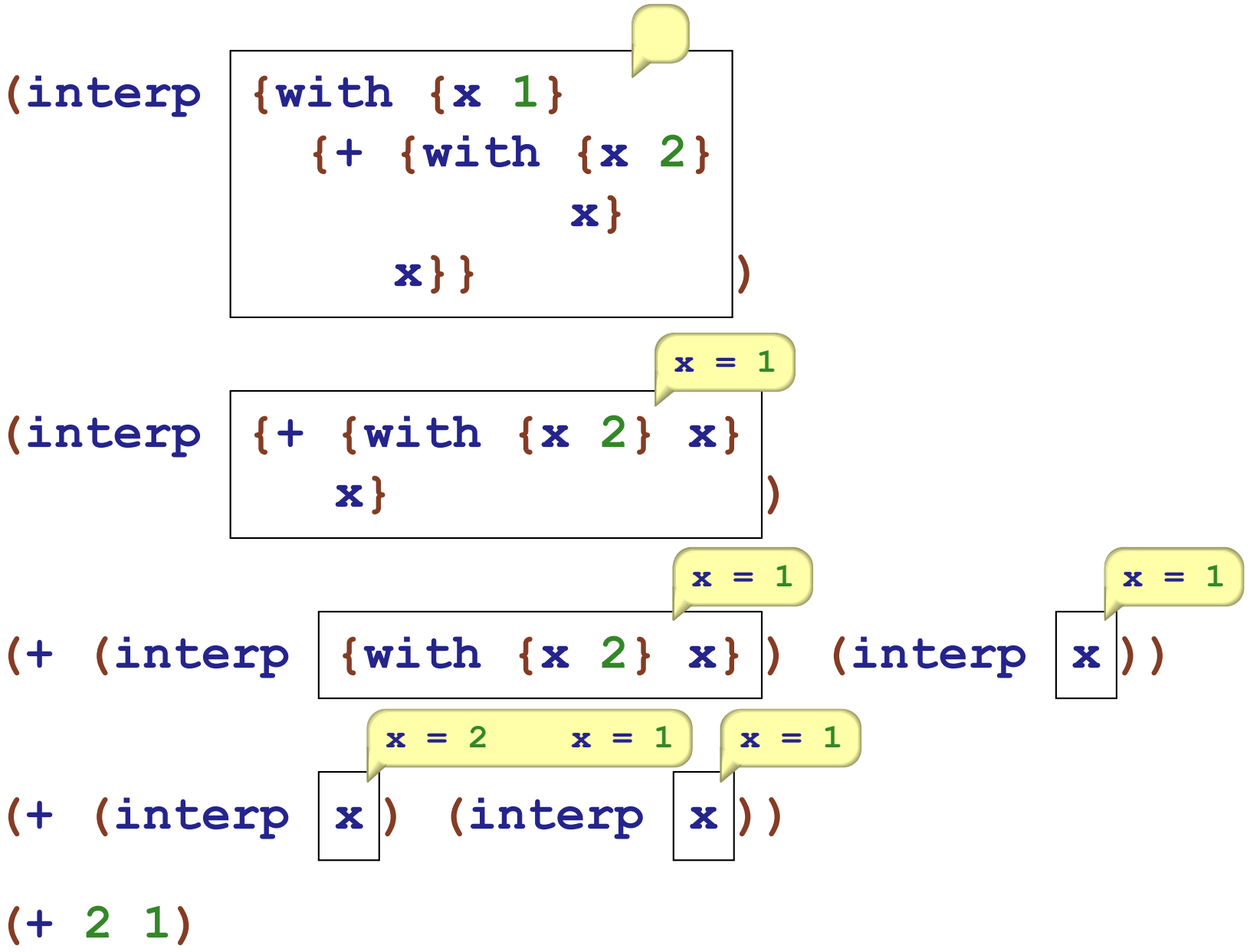
# Deferring Substitution with the Same Identifier



# Deferring Substitution with the Same Identifier



# Deferring Substitution with the Same Identifier



# Representing Deferred Substitution

Change

```
; interp : WAE -> num
```

to

```
; interp : WAE DefrdSub -> num
```

# Representing Deferred Substitution

Change

```
; interp : WAE -> num
```

to

```
; interp : WAE DefrdSub -> num
```

```
(define-type DefrdSub
  [mtSub]
  [aSub (name symbol?)
        (value number?)
        (rest DefrdSub?)])
```

# Interp with DefrdSub

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}}
        (mtSub))
```

# Interp with DefrdSub

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}}
        (mtSub))
```

```
⇒ (interp {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}
          (aSub 'x 1 (mtSub)))
```

# Interp with DefrdSub

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}}
        (mtSub))
```

```
⇒ (interp {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}
          (aSub 'x 1 (mtSub)))
```

```
⇒ (interp {+ 100 {+ 99 {+ 98 ... {+ y x}}}}
          (aSub 'y 2 (aSub 'x 1 (mtSub))))
```

# Interp with DefrdSub

```
(interp {with {x 1}
          {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}}
        (mtSub))
```

```
⇒ (interp {with {y 2}
            {+ 100 {+ 99 {+ 98 ... {+ y x}}}}}
          (aSub 'x 1 (mtSub)))
```

```
⇒ (interp {+ 100 {+ 99 {+ 98 ... {+ y x}}}}
          (aSub 'y 2 (aSub 'x 1 (mtSub))))
```

```
⇒ ...
```

```
⇒ (interp y (aSub 'y 2 (aSub 'x 1 (mtSub))))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      ...]
    [id (name) ...]))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      ...]
    [id (name) (lookup name ds)]))
```

# WAE Interpreter with Deferred Substitutions

```
; lookup : symbol DefrdSub -> num
(define (lookup name ds)
  (type-case DefrdSub ds
    [mtSub () (error 'lookup "free variable")]
    [aSub (sub-name num rest-ds)
          (if (symbol=? sub-name name)
              num
              (lookup name rest-ds))]))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      ...]
    [id (name) (lookup name ds)]))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      ... (interp named-expr ds) ...]
    [id (name) (lookup name ds)]))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      ...
      (aSub bound-id (interp named-expr ds) ds)
      ...]
    [id (name) (lookup name ds)]))
```

# WAE Interpreter with Deferred Substitutions

```
; interp : WAE DefrdSub -> num
(define (interp a-wae ds)
  (type-case WAE a-wae
    [num (n) n]
    [add (l r) (+ (interp l ds) (interp r ds))]
    [sub (l r) (- (interp l ds) (interp r ds))]
    [with (bound-id named-expr body-expr)
      (interp
       body-expr
       (aSub bound-id (interp named-expr ds) ds))]
    [id (name) (lookup name ds)]))
```

# Function Calls

```
{deffun {f x} {+ 1 x}}

(interp {with {y 2}
          {f 10}})
```

# Function Calls

```
{deffun {f x} {+ 1 x}}

(interp {with {y 2}
          {f 10}})
```

⇒

```
(interp {f 10})
```

y = 2

# Function Calls

```
{deffun {f x} {+ 1 x}}

(interp {with {y 2}
          {f 10}})
```

⇒

```
(interp {f 10})
```

y = 2

⇒

```
(interp {+ 1 x})
```

...

# Function Calls

```
{deffun {f x} {+ 1 x}}

(interp {with {y 2}
          {f 10}})
```

⇒

```
(interp {f 10})
```

y = 2

⇒

```
(interp {+ 1 x})
```

x = 10

Interpreting the function body starts with only one substitution

# Function Calls

What goes wrong if you extend the old substitution?

```
{deffun {f x} {+ y x}}

(interp {with {y 2}
          {f 10}})
```

# Function Calls

What goes wrong if you extend the old substitution?

```
{deffun {f x} {+ y x}}

(interp {with {y 2}
          {f 10}})
```

⇒

```
(interp {f 10})
```

y = 2

# Function Calls

What goes wrong if you extend the old substitution?

```
{deffun {f x} {+ y x}}

(interp {with {y 2}
          {f 10}})
```

⇒

```
(interp {f 10})
```

y = 2

⇒

```
(interp {+ y x})
```

x = 10 y = 2

⇒ 12 wrong!

# Function Calls

What goes wrong if you extend the old substitution?

```
{deffun {f x} {+ y x}}

(interp {with {y 2}
          {f 10}})

⇒

(interp {f 10})        ; y = 2

⇒

(interp {+ y x})       ; x = 10

⇒ "free var: y"
```

# F1WAE Interpreter with Deferred Substitutions

```
; interp : F1WAE list-of-FunDef DefrdSub -> num
(define (interp a-f1wae fundefs ds)
  (type-case F1WAE a-f1wae
    ...
    [app (name arg-expr)
         ...]))
```

# F1WAE Interpreter with Deferred Substitutions

```
; interp : F1WAE list-of-FunDef DefrdSub -> num
(define (interp a-f1wae fundefs ds)
  (type-case F1WAE a-f1wae
    ...
    [app (name arg-expr)
         (local [(define a-fundef
                   (lookup-fundef name fundefs))]
           (interp (fundef-body a-fundef)
                   fundefs
                   ...
                   (interp arg-expr fundefs ds)
                   ...))]))
```

# F1WAE Interpreter with Deferred Substitutions

```
; interp : F1WAE list-of-FunDef DefrdSub -> num
(define (interp a-f1wae fundefs ds)
  (type-case F1WAE a-f1wae
    ...
    [app (name arg-expr)
         (local [(define a-fundef
                   (lookup-fundef name fundefs))]
           (interp (fundef-body a-fundef)
                   fundefs
                   (aSub (fundef-arg-name a-fundef)
                         (interp arg-expr fundefs ds)
                         (mtSub))))]))
```

# Timing tests

```
(define (mk-sums n)
  (cond
    [(zero? n) 1]
    [else
     (let ([varn (string->symbol (format "x~a" n))])
       `{+ ,varn ,(mk-sums (- n 1))})]))

(define (mk-withs n body)
  (cond
    [(zero? n) body]
    [else
     (let ([varn (string->symbol (format "x~a" n))])
       `{with {,varn 1}
              ,(mk-withs (- n 1) body)})]))
```

# Timing tests, 2

```
(define (mk-exp n) (mk-withs n (mk-sums n)))

(test (mk-exp 2)
      `{with {x2 1}
             {with {x1 1}
                   {+ x2 {+ x1 1}}}})

(define (run n)
  (let ([expr (parse (mk-exp n))])
    (time (interp-expr expr '()))))
```

# Timing tests, 3

With the substitution-based interpreter, expect the difference between adjacent timings to be growing linearly. With the environment-based one, you will also see linear growth, but if you make the environment use a more efficient data structure, that'll go away

(you may need to make the numbers bigger or smaller to see what is going on here)

```
(collect-garbage) (collect-garbage)
(collect-garbage) (collect-garbage)
(run 100) (run 110) (run 120)
(run 130) (run 140) (run 160)
```

Note: always run your timing tests with **`racket`** at the commandline, not in DrRacket.

# Evenness

```
; even? : num[pos] -> num[bool]
{deffun {even? n}
  {if0 n
       0
       {if0 {- n 1}
            1
            {even? {- n 2}}}}}
; div2 : even? -> num[bool]
{deffun {div2 n}
  {if0 n
       0
       {+ 1 {div2 {- n 2}}}}}
```

# Evenness tests

```
(test (interp (parse `{even? 0})) 0)
(test (interp (parse `{even? 1})) 1)
(test (interp (parse `{even? 2})) 0)
(test (interp (parse `{div2 0})) 0)
(test (interp (parse `{div2 2})) 1)
(test (interp (parse `{div2 12})) 6)
```

# Collatz numbers

n → n/2, if n is even

n → 3n+1, if n is odd

terminate when n is 1

```
(test (interp (parse `{orbit 1})) 0)
(test (interp (parse `{orbit 2})) 1)
(test (interp (parse `{orbit 3})) 7)
(test (interp (parse `{orbit 30})) 18)
(test (interp (parse `{orbit 31})) 106)
(test (interp (parse `{orbit 32})) 5)
```

# Collatz numbers

```
; orbit : number[>0] -> number
; return the length of the collatz orbit
{deffun {orbit n}
  {if0 {- n 1}
       0
       {+ 1 {if0 {even? n}
                 {orbit {div2 n}}
                 {orbit {+ {+ n n}
                           {+ n 1}}}}}}}
```