

Power and Delay Trade-offs in Fading Channels

by

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Submitted to the

Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2000

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Abstract

Energy is a constrained resource in mobile wireless networks. In such networks, communication takes place over fading channels. By varying the transmission rate and power based on the current fading level, a user in a wireless network can more efficiently utilize the available energy. For a given average transmission rate, information theoretic arguments provide the optimal power allocation. However, such an approach can lead to long delays or buffer overflows. These delays can be reduced but at the expense of higher transmission power. The trade-offs between the required power and various notions of delay are analyzed in this thesis.

We consider a user communicating over a fading channel. Arriving data for this user is stored in buffer until it is transmitted. We develop several buffer control problems which fit into a common mathematical framework. In each of these problems, the goal is to both minimize the average transmission power as well as the average “buffer cost”. In two specific examples, this buffer cost corresponds to the probability of buffer overflow or the average buffer delay. These buffer control problems are analyzed using dynamic programming techniques. Several structural characteristics of optimal policies are given. The relationship of this model to the delay-limited capacity and outage capacity of fading channels is discussed. We then analyze the asymptotic performance in two cases – the probability of buffer overflow case and the average delay case. In both cases, we bound the asymptotic performance and provide simple policies which are asymptotically optimal or nearly optimal. Finally we extend this analysis to a model with multiple users communicating over a multiple-access channel to a common receiver. The single user results for the probability of buffer overflow case are generalized to this multiple user situation. Extensions to other multi-user models are also discussed.

Thesis Supervisor: Robert G. Gallager

Title: Professor of Electrical Engineering

For Mom, Dad and Ju Chien

Acknowledgments

My thesis advisor, Professor Robert Gallager, has my sincerest gratitude; both this thesis and my personal development have benefited greatly from his guidance and insight. His enthusiasm and dedication to his students are truly inspiring; I feel fortunate to have been one of them.

I want to thank my thesis committee members, Professor John Tsitsiklis, Dr. Steve Finn and Dr. James Mills. Their suggestions have helped to make this a better thesis.

The ideas behind this thesis emerged during weekly group meetings with Hisham Kassab, Prof. Gallager, Dr. Finn, and Dr. Mills. This interaction was very stimulating and enjoyable. I want to thank Hisham for his input and friendship; we made much of this journey together.

I would like to thank Professor Sanjoy Mitter, Professor Eytan Modiano, and Dr. Charles Rohrs for their interest in my work and career. I also want to thank the other faculty and staff at LIDS for creating such a wonderful environment in which to be a student.

I am very thankful for the many friends and officemates I have known while at LIDS, including Ibrahim Abou-Faycal, Sae-Young Chung, Aaron Cohen, Diane Dabby, Anand Ganti, Diane Ho, Hisham Kassab, Thierry Klein, Can Emre Koksal, Asuman Koksal, Peter Marbach, Michael Neely, Anant Sahai, Brett Schein, Sekhar Tatikonda, Ben Van Roy, Edmund Yeh, and Won Yoon (I am sure I have forgotten some). It has been a privilege to know such a great group of people.

Finally, I want to thank my parents and my wife, Ju Chien, for their love and support; I could not have accomplished any of this without them. I can not find the words to express how grateful I am that they are a part of my life.

This research was supported in part by the ATIRP Consortium sponsored by the U.S. Army Research Lab under Cooperative agreement DAAL01-96-2-0002 and in part by the ARO under grants DAAH04-94-G-0221, DAAH04-95-I-0103, and DAAG55-97-1-0305.¹

¹The views and conclusions contained in the document are those of the author and should not be interpreted as representing official policies, either expressed or implied of the Army Research Laboratory or the U.S. Government.

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Notation

We use capital letters to denote random variables and small letters to denote sample values. For a random variable, X , we denote its probability measure by π_X . Thus, for example, if X is discrete, we have $\pi_X(x) = \Pr(X = x)$. If it is clear which probability measure we are using, we will drop the subscript. The set of sample values of a random variable X will be denoted by \mathcal{X} .

The following is some notation which is often used:

β	weighting factor
$\Gamma(\theta)$	parallel channel with gains corresponding to θ
$\Delta^\mu(s)$	average drift in buffer state s using policy μ .
μ	a control policy
θ	a sequence of K channel states
A_n	number of bits that arrived between time $n - 1$ and n
\bar{A}	average arrival rate in bits/block
$b(s)$	buffer cost
\bar{b}^μ	average buffer cost using policy μ
\bar{D}^μ	average buffer delay under policy μ
G_n	channel state at time n
K	number of channel blocks per codeword
N	number of channel uses per fading block
L	buffer size
$P(g, u)$	power to transmit u bits when channel gain is g
$\mathcal{P}_a(\bar{A})$	minimum average power to transmit $\bar{A}W/N$ bps.
$P^*(B)$	optimum power/delay curve.
\bar{P}^μ	average power using policy μ

p_{of}^μ	steady state probability of buffer overflow under policy μ
$Q_{g,g'}$	channel transition probability
S_n	buffer state at time n
U_n	number of bits transmitted at time n
W	bandwidth of continuous time channel

CHAPTER 1

Introduction

In this thesis we consider several resource allocation problems which arise in mobile wireless communication networks. Examples of such networks include cellular networks, wireless LAN's, satellite networks, and packet-radio networks.¹ Mobile wireless networks are primarily deployed as access networks; that is, users communicate over the wireless network to gain access to a high speed back-bone network which is typically wired. However there are cases, such as packet-radio networks, in which an entire network may be wireless. The obvious advantage of a wireless network is the ability to provide users with “anywhere, anytime access”. Additionally, the infrastructure cost for building out a wireless network may be less than for a wired alternative, particularly when considering “last mile” technologies. In some situations, such as a military setting, there are clear reasons for not wanting to rely on a fixed communication infrastructure. Finally, improvements in the capability and size of portable communication and computation devices are helping to make wireless networks more attractive.

In any communication network, many fundamental problems involve the management and allocation of resources. In a wired network, the crucial resources are the nominal data rates² available on each link in the network. Techniques such as

¹In the literature, packet-radio networks are referred to by many different names including *ad hoc* networks, peer-to-peer networks, and all-wireless networks.

²In most of the networking literature, what we refer to as nominal data rate is called bandwidth; we reserve bandwidth for referring to the amount of frequency spectrum available for communication.

flow control, routing, and admission control are all centered around allocating these resources. In this work we focus on resource allocation problems which are unique to wireless situations. These problems are motivated by two characteristics of mobile wireless communication. The first characteristic is the wireless channel itself. The wireless channel is a time-varying multi-path channel; furthermore this channel generally must be shared among multiple users. These issues will be discussed in greater detail in the following chapters. The second characteristic is that users in a mobile wireless network typically rely on a battery with a limited amount of energy. As a consequence it can be desirable to adjust the transmission rate and/or power to conserve energy. On a wired point-to-point link, there is little reason to adjust the transmission rate and power in this way.³ Based on these characteristics, we argue that in a wireless network the critical resource is the energy or power transmitted by the users rather than the nominal data rates on each link. Due to fading and the ability to vary a user's transmission rate, the nominal data rate on a link in general will not even be well defined.

The resource allocation problems we consider concern how to efficiently allocate power in a wireless network. We approach this by considering the following question: *what is the minimum power required for the users in a wireless network to obtain acceptable service?* It is worth mentioning some of the benefits of reducing the required power. Reducing the needed power translates directly into either longer battery life or smaller batteries. Additionally, reducing the transmitted power can reduce the interference between users; this often results in being able to accommodate more users in the network. Finally, lower transmitted power can reduce the probability that the user is detected or intercepted, which is of particular importance in military networks.

In any wireless network, a variety of aspects influence the required power, including the over-all network topology, routing within the network, the multiple-access technique, the modulation and coding which are used, and the power required by the

³On a wired link there are typically power constraints such as those due to regulatory concerns. Such constraints usually restrict the average power over short time periods, for example the time to send a single codeword. In such cases there is no savings from the user's perspective for using less than the maximum average power per codeword.

electronics within the handset. We will not consider all of these issues in this work. In particular, we focus on “single hop” situations, and ignore higher layer issues such as the network topology and routing. We also ignore issues related to the power required by the electronics within the handsets. It should be emphasized that these other issues are equally if not more important in reducing the required power. We can not ignore the higher layer issues completely, for design choices at one layer influence the other layers; *i.e.*, there is no separation theorem for the network layers. The issues which we do address involve both physical layer topics as well as some network issues such as buffer control and multiple-access. Indeed the possible interaction between these layers is at the heart of this work. A theme we want to emphasize is that in wireless networks this region of intersection is larger and more interesting than in traditional wire-line networks.

By single hop situations we mean a transmitter or a group of transmitters that are communicating directly to a receiver or a group of receivers. Initially we consider only a single transmitter transmitting to a single receiver over a fading channel. This models a single “link” in a wireless network and ignores any issues involving interference from other users. In other words, such a model focuses on the time varying nature of the channel and avoids issues related to sharing the channel between multiple users. This is appropriate if we assume that the other users are scheduled at a higher layer in such a way that their interference is minimal. Even if this is not the case, we can regard this as a simplified model. This link model is studied in the first six chapters. Issues related to multiple users will be considered in Chapter 7.

As noted above, we want to minimize the average transmission power required for a user to attain acceptable service. Two key issues that need to be clarified are what is meant by “acceptable service” and whether other constraints are placed on the transmitter. Suppose acceptable service means to provide a given average rate with small probability of error and there are no other constraints on the transmitter. Then, at the link level, this problem is equivalent to finding the capacity of the fading channel. Such problems have received much attention in the information theory literature ([BPS98] is a recent comprehensive survey of this work). For most applications, the above assumptions are not true; acceptable service entails more than simply an average rate with small probability of error, and there are other constraints on the

transmitter. In particular, there are often constraints relating to the delay experienced by data, such as an average delay constraint or a maximum delay constraint. There may also be constraints that limit the number of channel uses allowed per codeword. We refer to all of these considerations as “delay constraints”; these are discussed in more detail in the next chapter.

In fading channels, such delay constraints can limit the usefulness of many of the standard capacity results. By “useful”, we mean that capacity gives a good indication of the rates that are practically achievable. For example, in an additive white Gaussian noise channel (with no fading), one can often send at rates near capacity with an acceptable error rate and a tolerable delay. To send at rates near capacity in a fading channel requires enough delay to average over the channel fading; this can be longer than the tolerable delay. In such a case, a user must either use more power than indicated by capacity arguments or accept a higher probability of error corresponding to those cases when the fading is severe. Thus for a given error probability, there is a trade-off between the average power used and the acceptable delay.

Some of these issues have been addressed in work on *outage capacity* [OSW94] and *delay-limited capacity* [HT98]. Both these works consider the situation where one desires to send every codeword at a constant rate and each codeword is limited by a strict delay constraint. These ideas are examined more closely in Chapter 3. For traffic such as data, the above assumptions may not be appropriate. For example, it may be either desirable or required to use code words that are much shorter than the delay constraint, or an average delay constraint may be more appropriate. Also, with bursty traffic it may not be desirable to transmit every codeword at a constant rate. The majority of this thesis focuses on these cases.

We illustrate the possibilities in the above situation by considering a model as shown in Fig. 1-1. Here data arrives from some higher layer application and is placed into a transmission buffer. Data is periodically removed from the buffer, encoded and transmitted over a fading channel. After sufficient information is received, the data is eventually decoded and sent to a peer application at the receiver. By using more power, data can be removed from the buffer at a faster rate, thus reducing the delay. Furthermore, suppose that the transmitter can adjust its rate based on the

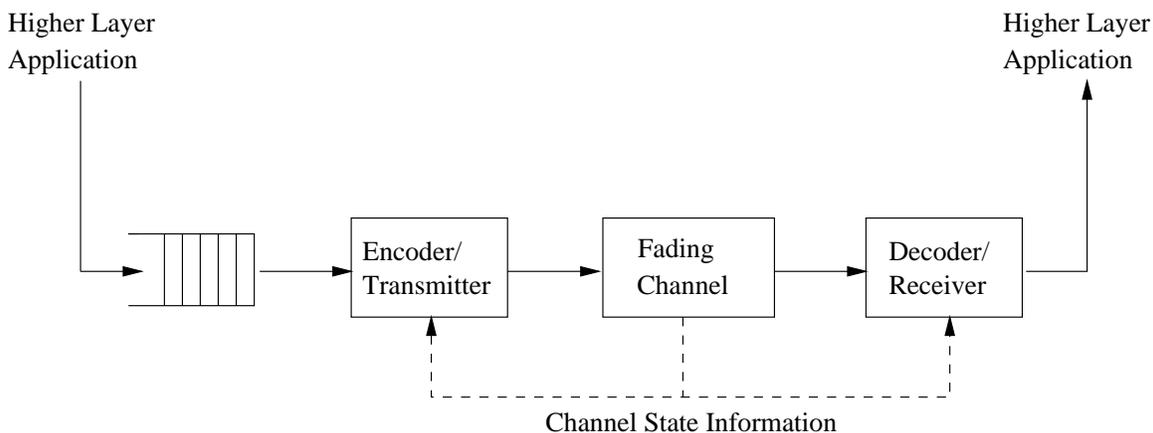


Figure 1-1: System model.

buffer occupancy. For example, as the buffer occupancy increases, the transmitter can transmit at a higher rate to ensure that the delay constraint is satisfied, but again at the cost of more power. We will see that adjusting the transmission rate in this manner is usually preferable to a fixed rate transmission. Often, the transmitter in a wireless network has some knowledge about the channel’s fading level. This knowledge can also be used in choosing the transmission rate and transmission power. Once again it is usually advantageous to do this if possible. We will consider several variations of this buffer control problem. Analysis of these models will provide us with some understanding of trade-off between the required power and delay in such channels. We will also gain insight into the optimal buffer control strategies in these situations.

1.1 Related Work

As we have already stated, the problem of communicating reliably over a fading channel has received much attention in the information theoretic literature; we will summarize some of this work in Chap. 3 for single user channels and in Chap. 7 for multi-user channels. We point out some other related work below.

In wireless networks, most of the work on resource allocation has been in the multi-user setting and centered around “power control” problems in a cellular net-

work. These problems are largely motivated by CDMA systems such as IS-95, where competing users are treated as noise. Power control is necessary in such systems to avoid the “near-far” problem in which one user’s received power is much higher than another’s [IS993]. The emphasis of this work is on providing each user with a desired Signal to Interference Ratio (SIR) which is assumed to indicate the quality of service desired by the user. Here the SIR is the ratio of the power transmitted by a given user to the interference power - which includes the transmitted power of the other users as well as the background noise. This problem has been addressed in a single cell setting [NA883], [Mit93]; in a multi-cell setting, combined power control and base station assignment has also been studied [Han95], [Yat95]. Power control has been considered for TDMA or FDMA systems to reduce co-channel interference [BE64] or inter-cell interference [Zan92]. Finally, power control has also been addressed in the context of multi-user receivers [TH99], [VAT99]. Again this work assumes that each user requires a constant transmission rate - this is reflected in the fixed SIR requirement. For the models we consider, a user’s rate and thus the required SIR is allowed to vary depending on the user’s buffer state as well as the channel conditions.

For the single user case, in [CC99] a model similar to that in Fig. 1-1 was studied with the assumption that the transmitter can vary the transmission rate and power, with the restriction that the received E_b/N_0 must be constant at all times. Here N_0 is the power spectral density of the additive noise and E_b is energy per bit, which is the received power divided by the bit rate. Thus the required transmission power is linear in the transmission rate. The assumption of constant E_b/N_0 is appropriate for systems with low spectral efficiency, *i.e.*, systems operating in the power-limited regime. The authors consider minimizing the average power for a given average delay constraint. A dynamic programming formulation of this problem similar to that in Chap. 5 is given.

Buffer control problems or, more generally, the control of queuing systems have been researched extensively in the stochastic dynamic control literature; [Ber95] and [Sen99] contain discussions of such problems and extensive bibliographies. The problems we examine can be considered a generalization of a service rate controlled queuing system (*cf.* [Ber95], Sect. 5.2). In such systems it is assumed that there is a cost $q(\mu)$ for using a given service rate μ , where q is a continuous, increasing function.

The difference between this and the problem we consider is that in our case, the cost depends on both the service rate and the channel's fading state, which has uncontrollable dynamics. The mathematical structure underlying the results in Chap. 6 is closely related to the buffer control problem addressed in [Tse94]. This work considers buffer control for variable rate lossy compression and will be discussed in more detail in Chap. 6.

1.2 Organization of Thesis

The remainder of this thesis is organized as follows. In Chapter 2 we discuss single user fading channels and develop the model that is used in Chapters 3-6. We also discuss the type of delay constraints that can arise in a wireless network. In Chapter 3 we discuss some of the information theoretic results for fading channels. In particular we discuss the work on delay-limited capacity, capacity vs. outage, and average capacity. This work is discussed in a slightly different context than elsewhere and several extensions are included. We also demonstrate the limitations of using these approaches. In particular, we argue that they do not provide an adequate answer for the type of situation depicted in Fig. 1-1. In Chapter 4, we develop several models for the buffer dynamics in Fig. 1-1. One of these models can be viewed as a generalization of the delay-limited capacity and capacity vs. outage framework discussed in Chap. 3. Another is derived from Telatar's model for multiple-access communication in [TG95]. Using these models, we formulate several buffer control problems to illustrate the trade-offs between the average power needed and some "buffer cost" which is related to the user's delay constraints. We consider two primary examples of such buffer costs; these correspond to probability of buffer overflow and average delay. In Chapter 5, we consider this problem from a dynamic programming point of view. We prove some structural characteristics of the optimal policies. We also look at this problem as a multi-objective optimization problem and show that all of the Pareto optimal policies of interest can be found by solving a dynamic programming problem. In Chapter 6 we analyze asymptotic versions of these problems as the delay constraint becomes relaxed. We do such an analysis for both the probability of overflow and average delay cases. In such cases we can find the limiting power and bound the rate

at which this limit is approached. We also show that a class of “simple policies” are asymptotically nearly optimal. Finally, in Chapter 7, we discuss extending these ideas to models with more than one user. We study a multiple-access situation in detail. For the probability of overflow case, we show that the single user results generalize directly. Extensions to other multi-user situations are also discussed.

CHAPTER 2

Fading Channels and Delay Constraints

In Chapters 3-6 we consider a single user communicating over a fading channel subject to various delay constraints. In this chapter we will give some background both on fading channels and delay constraints. First we discuss some features of fading channels, with an emphasis on the model we will use in the remainder of this thesis. We focus on single-user channels; models of multiple user channels will be discussed in Chap. 7. Since there are many treatments of fading channels available in the literature, we will not give a detailed exposition of this material. We also discuss the type of delay constraints that can arise in a wireless environment. Again our goal is to motivate the models we use in this thesis.

2.1 Fading Channels

The transmitted signal in a wireless network usually reaches the receiver via multiple paths and these paths change with time due to the mobility of the users and/or the reflectors in the environment. The changing strength of each path and the changing interference between these paths result in fading. In this section we discuss models for single user fading channels. Comprehensive references on fading channels include [Jak74], [Ste92] and [Ken69]. Treatments of fading channels similar to this section

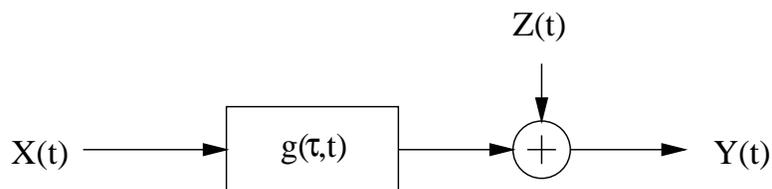


Figure 2-1: Continuous time channel model.

include [Gal99] and [Med95]. As noted above we focus on single user channels. Also, we restrict ourselves to models in which both the transmitter and receiver have a single antenna.

A fading channel is typically modeled as a randomly time-varying linear filter. Let $g(\tau, t)$ be the impulse response of a realization of this filter at baseband,¹ where this represents the response of the channel at time t to an input τ seconds earlier. Since we are looking at a baseband model, $g(\tau, t)$ will be complex valued. Let $X(t)$ be the baseband representation of the channel input. We assume that $X(t)$ is band-limited to the bandwidth $[-W/2, W/2]$. Thus this model corresponds to a passband system with a bandwidth of W Hz. For a given realization $g(\tau, t)$, the received signal $Y(t)$, also at baseband, is given by

$$Y(t) = \int X(t - \tau)g(\tau, t)d\tau + Z(t), \quad (2.1)$$

where $Z(t)$ is the baseband additive noise process. This is illustrated in Fig. 2-1. We assume that $Z(t)$ is a complex, circularly symmetric white Gaussian noise process with (double-sided) power spectral density N_0 . In other words, the real and imaginary parts of $Z(t)$ are independent and identically distributed white Gaussian processes with zero mean and power spectral density $N_0/2$.

Since $g(\tau, t)$ is time varying with t , the bandwidth of the output will be larger than the bandwidth of the input. Specifically the bandwidth of the output (at baseband)

¹Of course, the actual channel will be a passband channel around some carrier frequency f_c , but for our purposes we will only deal with the baseband equivalent model. For a discussion of the relationship between the baseband and passband models we refer to [Gal94b] and [Med95].

will be equal to $W/2 + B_D/2$, where $B_D/2$ is equal to the bandwidth of the Fourier transform of $g(\tau, t)$ with respect to t . In a wireless channel B_D is referred to as the Doppler spread and is related to the difference between the maximum and minimum Doppler shift on the received paths. The Doppler shift for a specific path is given by $\frac{f_c v}{c}$ where f_c is the carrier frequency, v is the speed at which the path length is changing and c is the speed of light. For current commercial wireless systems the maximum Doppler spread is on the order of 100 Hz, which is much less than the bandwidth of these systems. We assume that W is chosen large enough to account for any Doppler spreading. Using the sampling theorem both $X(t)$ and $Y(t)$ can be expressed in terms of the orthonormal basis $\{\phi_n(t)|n \in \mathbb{Z}\}$, where

$$\phi_n(t) = \sqrt{W} \frac{\sin(\pi(Wt - n))}{\pi(Wt - n)}.$$

Thus we can represent $X(t)$ and $Y(t)$ by the complex random sequences $\{X_n\}$ and $\{Y_n\}$ where, for example $X_n = \langle X(t), \phi_n(t) \rangle$. These can be interpreted as complex sampled time process with sample rate W .

For a given realization of the channel, $\{X_n\}$ and $\{Y_n\}$ are related by

$$Y_n = \sum_m X_{n-m} g_{m,n} + Z_n \quad (2.2)$$

where

$$g_{m,n} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_m(t - \tau) g(\tau, t) \phi_n(t) dt d\tau.$$

and $Z_n = \langle Z(t), \phi_n(t) \rangle$ is a complex circularly symmetric Gaussian random variable with zero mean and $\mathbb{E}Z_n Z_n^* = N_0$. Equation (2.2) can be interpreted as a discrete time tapped delay line model of the channel. Note $\{Z_n\}$ is a sequence of i.i.d. random variables.

The multi-path delay spread of a fading channel is the difference in propagation delay between the shortest and longest path to the receiver. This depends on a number of factors including the surrounding environment and the antenna height and location. In typical cellular systems the multi-path delay spread can range from for a few μsec to over 20 μsec . In a packet radio network the delay spreads may be much

smaller. Let L denote the multi-path delay spread of the channel and assume that the time reference is set at the receiver as the time at which the shortest delayed path is received.² Then we can assume that $g_{m,n}$ is approximately 0 for all $m < 0$ and all $m > LW$.³

Let $\mathbf{g}_n = [g_{0,n}, g_{1,n}, \dots, g_{[LW],n}]^T$ be the sequence of channel taps at time n , and let $\mathbf{X}_n = [X_n, \dots, X_{n-[LW]}]^T$. The channel model in (2.2) can then be rewritten as

$$Y_n = \mathbf{g}_n^T \mathbf{X}_n + Z_n. \quad (2.3)$$

Over intervals of interest, the vector sequence $\{\mathbf{g}_n\}$ is often modeled as a realization of a stationary random process, \mathbf{G}_n . There are many models for the statistics of both this process and the corresponding continuous time process in the literature (*e.g.* see [Jak74] or [Ste92]). The appropriateness of these models depends upon factors such as the environment and the motion of the users. A particular broad class of models we will consider are referred to as Markov or state-space models. In this case, we assume that $\{\mathbf{G}_n\}$ is a Markov chain⁴ with state space $\mathcal{G} \subset \mathbb{C}^{[LW]+1}$. We assume that this Markov chain is stationary and ergodic with steady-state distribution $\pi_{\mathcal{G}}$. The process $\{\mathbf{G}_n\}$ is assumed to be independent of the additive noise process $\{Z_n\}$. Also, we assume that conditioned on the previous state, the current state \mathbf{G}_n is independent of the channel inputs through time m . Thus for all $m \geq 1$,

$$\begin{aligned} \Pr(\mathbf{G}_m \in B | \mathbf{G}_{m-1} = \mathbf{g}_{m-1}, \dots, \mathbf{G}_1 = \mathbf{g}_1, X_m = x_1, \dots, X_1 = x_1) \\ = \Pr(\mathbf{G}_m \in B | \mathbf{G}_{m-1} = \mathbf{g}_{m-1}) \end{aligned}$$

for any measurable set $B \subset \mathcal{G}$. We have mentioned two general characteristics which are used to classify fading channels – the multi-path delay spread and the Doppler spread. There are two other common characteristics which are the time/frequency duals of these quantities – the coherence bandwidth and the coherence time. The

²This delay is changing with, but typically slow enough that it can be tracked and ignored.

³Actually slightly more than LW taps may be needed due to our choice of sampling functions, [Med95] contains a more detailed discussion of this.

⁴More generally we could assume that $\{\mathbf{G}_n\}$ is determined by an underlying Markov chain A_n with state space \mathcal{A} . For example, this generalization would let us use an n th order Markov model for $\{G_n\}$. We focus on the above simpler case mainly to simplify notation.

coherence bandwidth is defined to be the reciprocal of the delay spread. This gives a measure of the frequency band over which the fading will be highly correlated. When $W \ll 1/L$, the fading can be considered flat, *i.e.* the magnitude of the Fourier transform of $g(\tau, t)$ with respect to τ is approximately flat over the frequencies $[-W/2, W/2]$. Such a channel is referred to as a flat or narrow band fading channel; when $W > 1/L$, the channel is referred to as a wide band or frequency selective fading channel. Notice that in the narrow band case, at each time n , there is only one significant channel tap, *i.e.* $\mathbf{g}_n = g_{0,n}$. We will primarily consider a flat fading model, but we will at times also look at frequency selective fading channels. Our motive for this is that the flat fading model is analytically easier to deal with, and yet it captures some of the basic trade-offs that are present in the wide-band case. In the narrow band case we denote the channel gain at time n by G_n . This is simply a time-varying gain and phase shift. The channel model thus becomes

$$Y_n = G_n X_n + Z_n. \quad (2.4)$$

The coherence time is defined to be the reciprocal of the Doppler spread; it is a measure of how long the channel's frequency response remains relatively constant. In a 900 MHz cellular system, the coherence time can range from approximately 2 msec for a user driving at 60mph to an arbitrarily large value for a user who is stationary. At higher frequencies, systems will have larger Doppler spreads and smaller coherence times. The coherence time is incorporated into the Markov channel model through the transition probabilities of the Markov chain; the longer the coherence time, the longer the memory in the Markov chain. As defined above, the coherence time is a measure of the time-scale of "fast fading"⁵— this is fading which results from the time varying interference between the signals received over different paths. In a wireless channel there are also fading effects that occur on a slower time scale, for example due to "shadowing" that occurs when a user enters a tunnel or moves behind an obstacle. Such effects can also be incorporated into a Markov channel model by grouping states

⁵We want to emphasize that the name fast fading refers to the speed of the fading relative to the slower fading effects. The fast fading experienced by a user walking may be slower than the slow fading seen by a user driving.

that correspond to a particular slow fading level; the transitions between these groups will occur on a slower time-scale than transitions within the group.

Another way of modeling the coherence time is through the use of a *block fading* channel model. In such a model, over each block of N consecutive channel uses the channel gain is assumed to stay fixed; N is thus proportional to the channel's coherence time. For the narrow band block fading case, let G_m denote the channel gain during the m th block of N channel uses. Let $\mathbf{X}_m = [X_{(m-1)N+1}, \dots, X_{mN}]^T$, $\mathbf{Y}_m = [Y_{(m-1)N+1}, \dots, Y_{mN}]^T$, and $\mathbf{Z}_m = [Z_{(m-1)N+1}, \dots, Z_{mN}]^T$. These are related by:

$$\mathbf{Y}_m = G_m \mathbf{X}_m + \mathbf{Z}_m \quad (2.5)$$

We are still assuming that $\{G_m\}$ is a Markov chain. This type of model is frequently used to model a slowly fading channel (see for example [OSW94] or [CTB98]) and is a generalization of the block interference channel introduced by McEliece and Stark [MS84]. Such a model may be appropriate when the underlying system has a TDMA or frequency hopping structure. This can also be used to model communication in the presence of a pulsed jammer.⁶ Otherwise this model may simply be considered an approximation of the Markov model in 2.4. In this case it appears to be somewhat arbitrary to decide what portion of the coherence time is modeled by the block length and what portion is modeled with the transition probabilities. The main reason for making the block assumption is mathematical convenience. In general we will consider the block fading model in (2.5). At times we will point out that we could assume that $N = 1$ and account for all the memory with the Markov chain. At other times having $N \gg 1$ is more appropriate. Also, using a block fading model will facilitate making connections with some previous work.

The block fading model in (2.5) can be considered a discrete time vector input/vector output channel where the time samples occur at rate W/N . To avoid confusion denote the N components of \mathbf{X}_m by $X_{m,1}, \dots, X_{m,N}$; these corresponds to inputs for a discrete time scalar channel with sample rate W . The components of \mathbf{Y}_m and \mathbf{Z}_m will be denoted in a similar manner.

⁶Note the model in (2.5) is equivalent to a model where $\mathbf{Y}_m = \mathbf{X}_m + \mathbf{W}_m$, and \mathbf{W}_m is a Markov modulated additive noise processes. By equivalent we mean that the capacity of the two channels are the same.

Next we want to comment on the state space \mathcal{G} and the steady-state distribution $\pi_{\mathcal{G}}$ for the fading process. In the narrow band case \mathcal{G} will be a subset of \mathbb{C} . For most physically motivated channel models the cardinality of \mathcal{G} will be infinite. For example, with Rayleigh or Ricean fading $\mathcal{G} = \mathbb{C}$. We will not assume a particular fading distribution but we do assume some additional structure on \mathcal{G} and $\pi_{\mathcal{G}}$; in particular we assume that either \mathcal{G} is infinite and $\pi_{\mathcal{G}}$ has a continuous density with respect to Lebesgue measure on \mathbb{C} or that \mathcal{G} is finite. Such assumptions are mathematically convenient and indeed most fading models in the literature satisfy such conditions. While most physically motivated fading models have an infinite state space, at times it is expedient to assume a finite state space model. Such a model can be used to approximate a more general fading model, where the state space represents a quantization of the true fading levels. The transition probabilities for this Markov chain can then be chosen based on the particular fading environment. For example, in [WM95] a finite state Markov model is used to accurately model a Rayleigh fading channel. For calculation or simulation it is generally necessary to use a finite state model. When considering the infinite state space model, we will at times make the additional assumption that \mathcal{G} is a compact subset of \mathbb{C} . Again, this is mathematically convenient and can be viewed as an approximation of a more general model.

In addition to the statistical description of the fading process, another important characteristic of the channel model is the amount of channel state information (CSI) that is assumed to be available to the transmitter and receiver. In commercial systems the transmitter and receiver often have some such state information. This can be gained from pilot signals, training sequences, a signal received on a reverse link, and/or some form of explicit feedback. Obtaining this information requires estimating the state of the channel from noisy observations. Clearly this estimate will be better as the coherence time gets longer. Also when the the CSI is obtained from a signal transmitted on a reverse link, this signal must be transmitted within the same coherence time and coherence bandwidth.

We have already assumed that the receiver can track the delay of the various received paths, this is a form of CSI. In general we will assume that the receiver has perfect CSI, that is at each time n the receiver has perfect knowledge of the current realization of G_n , both the phase and the gain. This is clearly an idealized

assumption. In particular, in real systems it is often difficult to track the phase, which changes at a much faster rate than the gain. Many wireless systems avoid tracking the phase by using non-coherent modulation. On a typical fading channel with high signal-to-noise ratio, non-coherent modulation will perform only slightly worse than coherent modulation and does not require phase tracking [Gal99].

We must also specify the state information available at the transmitter. We will focus on the two limiting cases, which are the case where the transmitter has perfect state information and the case where there is no state information available. When available, the transmitter can use this information to adjust the transmission power and/or rate according to the fading conditions. The transmission rate can be adjusted in a variety of ways including changing the constellation size, changing the coding rate and changing the spreading factor in a spread-spectrum system. Many third generation cellular standards have provisions for such rate adjustments. Though the case of perfect transmitter state information is clearly not satisfied in a practical system, it provides an upper bound on what improvement is possible when using any state information. It also is an analytically simpler problem than dealing with imperfect state information.

In summary the usual model we consider is a block fading channel model as in (2.5) with flat fading. The fading sequence $\{G_m\}$ is assumed to be a stationary ergodic Markov chain. We assume that the receiver has perfect state information. Both the case of perfect transmitter state information and no transmitter state information will be considered. We will also comment at several places on the wide-band fading case.

2.2 Delay Constraints:

In this section we take a closer look at the type of delay constraints that can arise in a wireless network. We consider constraints which limit the overall delay experienced by data in Fig. 1-1, including both the buffer delay and the delay from when data leaves the buffer until it is decoded. We also consider other constraints on the transmitter which can arise due to architectural considerations. Specifically we consider constraints that may limit the number of channel uses over which one can send a codeword or,

in the case of convolutional codes, a constraint length. As noted above we refer to each of these as delay constraints. Delay constraints arise for a variety of reasons; we classify these into two general categories: *quality of service* constraints and *system* constraints. Each of these categories is discussed next.

Quality of service constraints: Quality of service (Q.O.S.) constraints are constraints that must be satisfied for the user to receive acceptable service.⁷ Such constraints could be an inherent characteristic of an application, such as real time voice, or simply related to the price a user pays for service. We consider both maximum delay constraints and average delay constraints. With a maximum delay constraint, we assume that to get an acceptable quality of service, data must reach the higher layer protocol at the receiver within D seconds from the time it arrives at the transmission buffer. With an average delay constraint, we assume that the average total delay⁸ from when data arrives at the transmission buffer until it reaches the higher layer application at the receiver is less than D . Of course one way to insure that the average delay is less than D seconds is to keep the maximum delay less than D seconds, but this is typically an overly conservative approach.

There are several components to the overall delay including the time spent in the buffer, the encoding time, the propagation time, the decoding time and any additional processing time. The contribution of each of these components depends on the overall architecture. We illustrate some of the possibilities with the following example. These issues will be developed further in Chap. 4.

Example: Consider the situation in Fig. 1-1 with the following assumptions. Data arrives at a constant rate of \bar{R} bits per second and has a maximum delay constraint of D seconds. This data is encoded into a block code. Before this block code can be decoded, the entire codeword must be received at the receiver. Recall we are assuming that there are W/N uses of the block fading channel per second, which corresponds to W complex valued channel uses per second. To satisfy the delay constraint each

⁷We are only considering Q.O.S. constraints which are related to the delay experienced by data. There may be other constraints such as delay jitter, but we ignore such issues.

⁸Of course to precisely define this quantity we need to specify a probabilistic model for the system; this will be done in the latter chapters.

codeword must be sent in WD/N or fewer channel blocks which corresponds to WD complex valued channel uses. For a given modulation scheme, this results in a limit on the block length of the code. Not all decoding techniques require one to receive the entire code word before decoding. For example if one uses a convolutional code with a Viterbi decoder, a bit can typically be decoded after 4-5 constraint lengths have been received [Vit79]. In this case these constraint lengths must be sent in WD or fewer scalar channel uses. These constraints can be tightened further by taking into account the buffering delay, the encoding time, the propagation delay and the processing time of the decoder. For example suppose that data arrives at a constant rate of \bar{R} bits/sec and is to be encoded into one of M codewords. Thus $\log M$ bits are required to choose a codeword; assume that all of these bits must be accumulated before a codeword is selected, this will take $\log M/\bar{R}$ seconds. Additionally let D_P denote the propagation delay and processing time per codeword. In this case each codeword must be sent in fewer than $W(D - \log M/\bar{R} - D_p)$ channel uses to satisfy the delay constraint.⁹ Again, not all encoding techniques require all of the data bits to begin encoding; in these cases the above arguments would need to be adjusted appropriately.

Networks usually carry a variety of different traffic types with a variety of different service definitions. Conceivably one could use a different coding and modulation strategy for each different service type, but such an approach would be overly complex. If the same modulation and coding are required for each service type, then the number of channel uses per codeword for all traffic would be constrained by the traffic with the strictest delay requirements.

System constraints: Besides quality of service, there are other factors that may constrain the number of channel uses one can use. These are related to the network architecture and other design parameters. We provide several examples:

1. *Interaction with higher level protocols:* Even if an application does not require a delay constraint for quality of service, higher layer protocols may require such a constraint to function properly. For example, if a higher layer protocol, such

⁹To avoid buffer overflow, the time to send a codeword must also be less than or equal to $\log M/\bar{R}$.

as TCP, is doing flow control, then excessive delays can be detrimental. For another example, consider an ad-hoc packet radio network where connectivity is frequently changing. In such a network, the higher layer protocols may require that the codeword length be short enough to be sent before the topology changes.

2. *Decoder complexity and memory:* Decoder complexity and memory requirements usually increase with the block length for block codes or the constraint length for convolutional codes. For example Viterbi decoding of a rate b/n , constraint length K convolutional code requires $2^{b(K-1)}$ comparisons per bit [Vit79]. Iterative decoding of graph based codes such as low-density parity check codes can achieve a much slower $O(\log n)$ growth in complexity per bit with block length n . However, such codes require long block lengths, and the entire codeword must be stored before decoding, thus requiring more memory, and, of course, more delay.
3. *Trade-offs between FEC and ARQ:* For applications which require very low error rates, such as data traffic, both forward error correction (FEC) and some form of ARQ are usually used in a wireless channel. The reason for this is that for a given complexity, ARQ strategies can provide a higher reliability than using only forward error control [LDC83]. This can be implemented in several ways. In the simplest setting the FEC is thought of as being at a lower layer than the ARQ; thus this can be thought of as a special case of the first point above. We describe a simple implementation. At the transmitter a CRC is added to a given packet, then the packet plus CRC are encoded into a codeword. At the receiver, the received codeword is decoded, then the CRC is checked. If this check fails, then the packet is thrown out and the transmitter is requested to repeat it. If longer codewords are used, the probability a packet is thrown out decreases, but, when an error does occur, more bits will have to be repeated. A simplified analysis of these trade-offs in [ZHG97] suggest that there is a limit on the maximum codeword length one should use. This analysis does not consider more elaborate hybrid ARQ strategies such as code-combining [Cha85] or incremental redundancy [LY83] which do not disregard a codeword which is

in error. We will give a simple model for such strategies in Sect. 4.3.

4. *Adapting to channel fades:* As in the above example, suppose that data is arriving at a constant rate of \bar{R} and that fixed length block codes are used. Additionally assume that the application has a maximum delay constraint of D . Following the above example this can be translated into a constraint on the number of channel uses per codeword. If the maximum number of channel uses per codeword are used, then each codeword must have rate \bar{R} . By using shorter codewords, then some codewords may have rate less than \bar{R} . This can be useful during deep fades.
5. *Adapting to bursty data:* Again consider the above example, but assume that data arrives in bursts. Depending on the encoding technique, shorter codewords may be more flexible in adjusting to the arriving data. For example, suppose that $\log M$ bits are required to choose a codeword. Then if fewer than $\log M$ bits are in the buffer, either the transmitter must pad these bits with zeros or wait for more bits to arrive. Of course, for a given code rate, using shorter codewords reduces the number of bits per codeword.

Many of the constraints discussed above can be modeled by limiting the number of channel uses over which a codeword can be sent. With such a constraint, standard capacity results may not be meaningful. This has led to the introduction of ideas such as capacity vs. outage [OSW94] for such situations. In the next chapter these ideas are discussed in more detail.

CHAPTER 3

Capacity and Capacity vs Outage

In this chapter we discuss various definitions of capacity for the single user block fading channel from Sect. 2.1. Specifically we consider the usual Shannon capacity as well as *outage capacity*, *delay-limited capacity* and *average capacity*. These other notions of capacity have been proposed for situations in which there is a delay constraint as in Sect. 2.2; in such cases a compound channel model is often used. Initially we consider the case where there is no CSI at the transmitter. Next the case of perfect CSI is discussed. Finally extensions to frequency selective fading are also discussed. One goal of this chapter is to summarize the previous work on these topics in a unified framework. We also discuss some natural extensions of these ideas that have not appeared in the literature. The model we study in the remainder of the thesis can be viewed as a generalization of these concepts.

3.1 No Transmitter State Information

The narrow band block fading channel model from (2.5) is repeated below:

$$\mathbf{Y}_m = G_m \mathbf{X}_m + \mathbf{Z}_m \quad (3.1)$$

where Z_m is a circularly symmetric Gaussian random vector with zero mean and covariance $N_0 \mathbf{I}_N$. Here \mathbf{I}_N denotes the $N \times N$ identity matrix. The n -dimensional proper complex Gaussian distribution with mean $m \in \mathbb{C}^n$, and covariance matrix Q

is denoted by $\mathcal{CN}(m, Q)$. Thus $Z_m \sim \mathcal{CN}(0, N_0 |N)$. Recall we are assuming that each use of this vector channel corresponds to a block of N uses of an underlying complex-valued discrete time channel, which in turn represents complex samples of a continuous time baseband channel at rate $1/W$. Assume that the input to this channel must satisfy the long term average energy constraint,

$$\lim_{M \rightarrow \infty} \frac{1}{NM} \sum_{m=1}^M \mathbb{E}(\mathbf{X}_m^\dagger \mathbf{X}_m) \leq P/W,$$

or equivalently

$$\lim_{M \rightarrow \infty} \frac{1}{MN} \sum_{m=1}^M \text{tr}(\mathbb{E}(\mathbf{X}_m \mathbf{X}_m^\dagger)) \leq P/W.$$

Thus P represents average energy per second or average power used in the continuous time channel. In this section we consider the case where there is no CSI at the transmitter and perfect CSI at the receiver. Assuming perfect CSI at the receiver is equivalent to modeling the channel output at time m as the pair (G_m, \mathbf{Y}_m) . The capacity of this channel is the maximum expected mutual information rate between \mathbf{X}_m and (G_m, \mathbf{Y}_m) ; this is given by [OSW94]:

$$C_{NT} = \mathbb{E}_G \left(W \log \left(1 + \frac{|G|^2 P}{N_0 W} \right) \right) \quad \text{bits/sec.} \quad (3.2)$$

where G is a random variable with distribution π_G , the steady-state distribution of $\{G_n\}$. Capacity is achieved by choosing \mathbf{X}_m to be i.i.d. with distribution $\mathcal{CN}(0, \frac{P}{W} |N)$. For shorthand we use the notation

$$C(x) = W \log \left(1 + \frac{x}{N_0 W} \right). \quad (3.3)$$

Thus $C_{NT} = \mathbb{E}_G(C(|G|^2 P))$. At various places in the literature this quantity is sometimes referred to as the *throughput capacity*, the *ergodic capacity* or the *Shannon capacity* of the channel, mainly to differentiate it from other notions of capacity which we discuss next. We want to emphasize that C_{NT} has the usual operational significance of providing the maximum rate for which there exists codes whose rates

approach C_{NT} with arbitrarily small probability of error. This is to be contrasted with the other notions of capacity to be defined next.

As we stated above, when there is a delay constraint, this capacity expression may not be useful. By “useful” we mean that the capacity gives a good indication of the rates that are achievable with small probability of error. For example in a voice grade telephone line, the channel capacity gives a good indication of the rate that can be achieved by modern modems. The reason that capacity may not be useful in the fading case is that there are two sources of randomness in such a channel – both the randomness due to the noise and the randomness due to the fading. To approach capacity on such a channel usually requires one to use codewords whose lengths are long enough to average over both types of randomness. With respect to the fading, this means that codewords must be long enough so that the time average mutual information rate per codeword is, with high probability, near the ergodic average in (3.2). The length of codeword needed to achieve this must span many coherence times of the channel.¹ Due to the delay constraint such code lengths may not be feasible.

In situations such as these, other notions of capacity have been proposed including *capacity vs outage*, *delay-limited capacity*, and *average capacity*. Each of these quantities is intended to provide a more meaningful performance measure than (3.2) when delay constraints are present. These concepts can be defined by modeling the channel over which a codeword is sent as a compound channel; we discuss such a model next. We then look at each of these concepts and discuss when it is useful and when it is not. In the following chapters we will look at a model which in one sense is a generalization of these concepts.

We are still considering the narrow band block fading model from (3.1). Assume that there is a delay constraint as in Sect. 2.2 which requires that a codeword be sent over K channel blocks. This corresponds to KN scalar channel uses or KN/W seconds. Let G_1, \dots, G_K be the sequence of channel gains over such a set of K channel blocks. Corresponding to such a sequence, define \mathbf{G} to be a $K \times K$ diagonal matrix whose i th diagonal term is G_i . The relationship between the input and output over

¹Or if interleaving is used, the interleaving depth must span many coherence times.

such a set of K channel blocks can be written in matrix form as

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \quad (3.4)$$

where

$$\mathbf{X} = [\mathbf{X}_1 \cdots \mathbf{X}_K]^T \quad \mathbf{Y} = [\mathbf{Y}_1 \cdots \mathbf{Y}_K]^T \quad \mathbf{Z} = [\mathbf{Z}_1 \cdots \mathbf{Z}_K]^T.$$

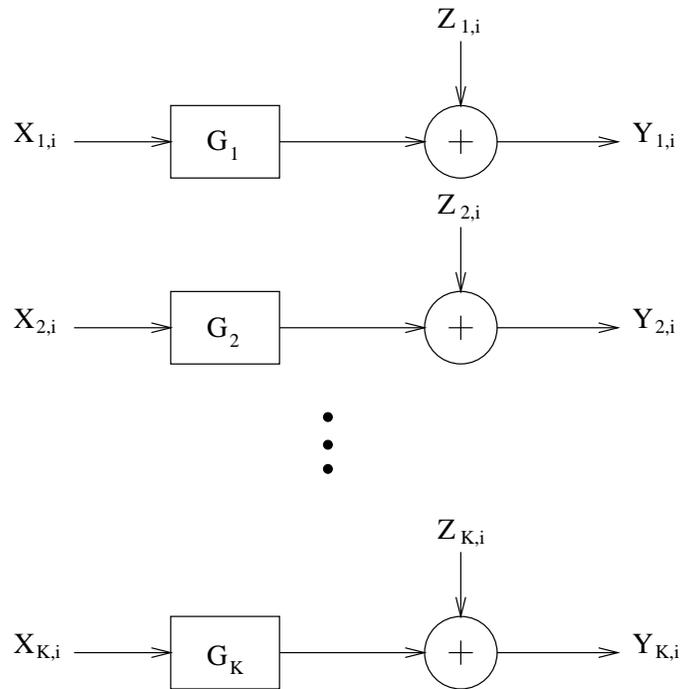
Here, for example, the j th row of \mathbf{X} corresponds to \mathbf{X}_j . For $i \in \{1, \dots, N\}$ let $\hat{X}_i = [X_{1,i} \cdots X_{K,i}]^T$ denote the i th column of \mathbf{X} , likewise define \hat{Y}_i and \hat{Z}_i to correspond to the i th column of \mathbf{Y} and \mathbf{Z} respectively. These are then related by

$$\hat{Y}_i = \mathbf{G} \hat{X}_i + \hat{Z}_i. \quad (3.5)$$

We can view this as another vector input/vector output channel. Note $\{\hat{Z}_i\}$ will be a sequence of i.i.d. random vectors with distribution $\mathcal{CN}(0, N_0 \mathbf{I}_K)$. Since \mathbf{G} is diagonal, (3.5) can in turn be viewed as a collection of K independent parallel complex Gaussian channels; one channel corresponding to each block. This is shown in Fig. 3-1. Equation (3.4) can be thought of as N uses of such a parallel channel. An advantage of viewing (3.4) in this way is that the channel in (3.5) is memoryless.

The parallel channel over which a particular codeword is sent will depend on the channel states G_1, \dots, G_K , which are random variables. This can be modeled by viewing the channel as a *compound channel*. A compound channel is a family of channels $\{\Gamma(\theta) : \theta \in \Theta\}$ indexed by some set Θ . A given codeword will then be sent over one channel from this set, the channel staying constant for the entire codeword.² For a given block fading channel and a given delay constraint K we define the compound channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$ as follows. Let Θ_K be the set of all length K sequences of channel states, $\{g_1, \dots, g_K\}$ which occur with positive steady-state

²Compound channels have long been studied in the literature, see for example [Wol78].

Figure 3-1: K parallel channel model.

probability in the original channel. Thus Θ_K can be considered a subset of \mathbb{C}^K . For each $\theta = \{g_1, \dots, g_K\} \in \Theta_K$ we associate a channel $\Gamma(\theta)$, where $\Gamma(\theta)$ corresponds to a parallel Gaussian channel as in Fig. 3-1 with gains $\{g_1, \dots, g_K\}$. Put another way, each $\theta \in \Theta_K$ can be identified with a realization of \mathbf{G} ; the channel $\Gamma(\theta)$ is then the discrete time memoryless vector channel in (3.5) corresponding to this particular realization of \mathbf{G} . Also, for each $\theta \in \Theta_K$ associate an *a priori* probability π_θ corresponding to the steady state probability the sequence $\{g_1, \dots, g_K\}$ occurs in the original channel.³

The compound channel described above corresponds to the channel over which a single codeword is sent. Because of the delay constraint, the codeword is sent over K uses of the block fading channel in (3.1); this corresponds to N uses of the compound channel. We want to emphasize that for more than N channel uses, the correspondence between these two channels no longer holds. Several notions of capacity have been defined for a compound channel with an *a priori* probability distribution. We will discuss several of these next. First these quantities will be defined as a maximum mutual information “rate”, where “rate” will be interpreted differently in each case. To give these quantities operational significance a coding theorem and a converse is required. To prove a coding theorem, *i.e.*, achievability with arbitrarily small probability of error, requires one to be able to send arbitrarily long codewords. However, for the compound channel to correspond to the original block fading channel, codewords can be no longer than N channel uses. With this restriction a coding theorem can not be proven. On the other hand if we allow arbitrarily long codewords, then coding theorems can be proven for the compound channel, but this no longer corresponds to the original channel. One approach to this, as in [CTB99] is to consider a sequence of block fading channels indexed by block length $N = 1, 2, \dots$. As N increases, for a fixed bandwidth, the coherence time of the channel must also increase. Letting $N \rightarrow \infty$, a coding theorem can be proven while keeping the correspondence between the two channel models. Note in this way one does not prove a coding theorem for the original channel but for a limiting channel with arbitrarily large coherence time. Of course, in the actual channel the coherence

³Often when compound channels are studied in the literature, there is no *a priori* probability distribution assumed on Θ . Some authors refer to a compound channel with an *a priori* probability distribution as a *Composite* channel [EG98].

time is fixed; thus this limiting operation is not physically realizable. The point we are trying to make is that such a coding theorem is a much weaker statement than the “usual” cases, such as a Gaussian channel without fading.

From a practical point of view, these capacities can be useful if N is large enough relative to the block length required for reliable communication, but N is still small relative to the coherence time of the channel. Again by useful we mean that these quantities give a good indication of the rates that are achievable with acceptably small probability of error. How large N must be for this to hold depends on the error exponents for the compound channel; we will take a more precise look at these ideas below.

3.1.1 Delay-limited Capacity

Consider an arbitrary compound channel, $\{\tilde{\Gamma}(\theta) : \theta \in \Theta\}$ where each channel $\tilde{\Gamma}(\theta)$ is a memoryless channel with input X and output Y . Assume that the receiver knows the state θ but the transmitter does not, and additionally assume that the input probability distribution must be chosen from a set Π . In this case, the capacity of the compound channel is given by [Wol78]:

$$C = \sup_{\pi_X \in \Pi} \inf_{\theta \in \Theta} I(X; Y | \theta) \quad (3.6)$$

where for each input probability distribution π_X , $I(X, Y | \theta)$ is the mutual information between X and Y over the channel $\tilde{\Gamma}(\theta)$. Thus the capacity of the compound channel is the maximum mutual information between X and Y regardless of which channel $\tilde{\Gamma}(\theta)$ is chosen. By allowing arbitrarily long codewords, one can prove a coding theorem and a converse which shows that arbitrarily small probability of error (for any channel $\tilde{\Gamma}(\theta)$) can be attained if and only if one transmits at rates less than C over the compound channel [Wol78].

For the compound channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$ defined above, let Π be the set of probability distributions for $\hat{X} \in \mathbb{C}^K$ satisfying

$$\text{tr}(\mathbb{E}(\hat{X}\hat{X}^\dagger)) \leq KP/W, \quad (3.7)$$

i.e., Π is the set of input distributions which satisfy the average power constraint. With this constraint set, consider the maximization in (3.6). Recall for any $\theta \in \Theta_K$, the channel $\Gamma(\theta)$ is a memoryless Gaussian channel in (3.5) corresponding to a particular realization of \mathbf{G} . Let \mathbf{G}_θ be the realization of \mathbf{G} corresponding to θ . Assume that we fix the input distribution to have covariance $\mathbb{E}(\hat{X}\hat{X}^\dagger) = \mathbf{Q}$ which satisfies the average power constraint. Then it is straightforward to show that using the input distribution $\mathcal{CN}(0, \mathbf{Q})$ will maximize the mutual information between \hat{X} and \hat{Y} for any $\theta \in \Theta_K$. In other words the capacity achieving distribution must have the form $\mathcal{CN}(0, \mathbf{Q})$. Using such an input distribution yields

$$I(\hat{X}; \hat{Y} | \Theta = \theta) = \log \left(\det \left(\mathbf{I}_K + \frac{1}{N_o} \mathbf{G}_\theta \mathbf{Q} \mathbf{G}_\theta^\dagger \right) \right) \quad (3.8)$$

for any $\theta \in \Theta$. By Hadamard's inequality,

$$\det \left(\mathbf{I}_K + \frac{1}{N_0} \mathbf{G}_\theta \mathbf{Q} \mathbf{G}_\theta^\dagger \right) \leq \prod_{i=1}^K (1 + \mathbf{Q}_{ii} |g_i|^2) \quad (3.9)$$

where \mathbf{Q}_{ii} and g_i are respectively the i th diagonal terms of \mathbf{Q} and \mathbf{G}_θ . Equality is attained in (3.9) when \mathbf{Q} is diagonal. Thus the capacity achieving distribution \mathbf{Q} will be diagonal. Finally, assume that the channels in Θ have the following symmetry property: for any \mathbf{G}_θ and any permutation matrix \mathbf{U} , then there exists a $\theta' \in \Theta$ such that $\mathbf{G}_{\theta'} = \mathbf{U} \mathbf{G}_\theta \mathbf{U}^\dagger$. With this assumption it can be shown that the capacity achieving distribution is $\mathcal{CN}(0, \frac{P}{W} \mathbf{I}_k)$ which results in

$$\begin{aligned} I(\hat{X}; \hat{Y} | \theta) &= \log \left(\det \left(\mathbf{I}_K + \frac{P}{N_0 W} \mathbf{G}_\theta \mathbf{G}_\theta^\dagger \right) \right) \\ &= \log \left(\prod_{i=1}^K \left(1 + \frac{|g_i|^2 P}{N_0 W} \right) \right) \\ &= \sum_{i=1}^K C(|g_i|^2 P) \end{aligned}$$

where the second equality follows since \mathbf{G}_θ is diagonal. Let $C_\theta = \frac{W}{K} I(\hat{X}; \hat{Y} | \theta)$; this is the average mutual information per second for the channel $\Gamma(\theta)$. Using this, we can

write the capacity of the compound channel, C_{DL} , as

$$C_{DL} = \inf\{C_\theta : \theta \in \Theta\} \quad (3.10)$$

In the context of block fading channels, C_{DL} is referred to as the delay-limited capacity⁴

There are several things we wish to note about this quantity. First, note that C_{DL} is simply the capacity of the channel corresponding to the worst sequence of K fades in the block fading channel. Any realistic channel model will have $Q_{g,g} > 0$ for every state $g \in \mathcal{G}$. In this case, $C_{DL} = \inf\{C(|g|^2 P) : g \in \mathcal{G}\}$. For most common continuous state space fading models, such as Rayleigh or Ricean fading, $0 \in \mathcal{G}$; in these cases $C_{DL} = 0$. If we assume that the block fading model came from quantizing a more general fading model, as discussed in the previous chapter, then the value of C_{DL} will depend on how we do this quantization. In the following we simply assume that we are given a specific block fading model, but wish to emphasize that one must always be careful if this model is intended to approximate a more general channel.

We again want to emphasize that there is no coding theorem with regard to C_{DL} for the original block fading channel, only for the compound channel when $N \rightarrow \infty$. Let $E_r(R, \theta)$ be the random coding exponent for the channel $\Gamma(\theta)$ when the input distribution is $\mathcal{CN}(0, \frac{P}{W} \mathbf{1}_K)$ [Gal68]. Thus when a codeword with rate R bits per second is sent over N channel uses of $\Gamma(\theta)$, the average probability of error per codeword, $p_{err}(\theta)$, is bounded by

$$p_{err}(\theta) < \begin{cases} e^{-NE_r(R, \theta)} & \text{if } R < C_\theta \\ 1 & \text{if } R \geq C_\theta. \end{cases} \quad (3.11)$$

Suppose we desire an average probability of codeword error less than or equal to some $p > 0$ for every channel $\Gamma(\theta)$. Then if $R < C_{DL}$ from (3.11) it follows that for N large enough there exists such a code. Likewise the strong converse to the coding theorem states that for $R > C_\theta$, $p_{err}(\theta)$ increases with N to 1. Thus for N large enough there

⁴The term delay-limited capacity was first used in [HT98] for the case where the transmitter has perfect side-information. The delay limited capacity as we have defined it would coincide with the one in [HT98] for the case of $K = 1$.

exists such a code if and only if $R < C_{DL}$. Assuming N is this large, then we can make the same statement about the block fading channel with this value of N . Thus we see, as hypothesized above, that C_{DL} is meaningful for the block fading channel provided that N is large enough.

3.1.2 Capacity vs. Outage

Up to this point, we have been assuming that the error criterion is the probability of error for any channel $\Gamma(\theta)$ and have not used the *a priori* probabilities, π_θ . A more reasonable error criterion would be the probability of error averaged over the possible realizations of Θ ; denote this quantity by $\overline{p_{err}}$. Thus

$$\overline{p_{err}} = \int_{\Theta} p_{err}(\theta) d\pi_{\Theta}(\theta). \quad (3.12)$$

As above assume that the input is chosen from the input distribution $\mathcal{CN}(0, \frac{P}{W} \mathbf{I}_k)$. Using (3.11) for a transmission rate of R bits per second, we have

$$\overline{p_{err}} \leq \int_{\{\theta: R < C_\theta\}} e^{-NE_r(R, \theta)} d\pi_{\Theta}(\theta) + \int_{\{\theta: R \geq C_\theta\}} 1 d\pi_{\Theta}(\theta) \quad (3.13)$$

The event $\{R \geq C_\theta\}$ is referred to as an *outage*. Thus the second term above gives the probability of an outage occurring. For a given rate R , if we let $N \rightarrow \infty$ then the first term above will go to zero and the probability of an outage is approximately an upper bound on the probability of error. Also, for any channel $\Gamma(\theta)$ such that $R > C_\theta$, the probability of block error, $P_{err}(\theta) \rightarrow 1$ as N grows. Thus for large N the average probability of error is approximately the probability of outage. For a given probability of outage, q , the *capacity vs. outage probability* q , C_q is defined as

$$C_q = \sup \{R : \Pr(C_\theta \leq R) \leq q\} \quad (3.14)$$

This is the maximum mutual information rate that be sent over any channel $\Gamma(\theta)$ except a subset with probability less than q . In other words, C_q is the capacity of the compound channel $\{\Gamma(\theta) : \theta \in \Theta_q\}$, where $\Theta_q = \{\theta : C_\theta > R\}$ and $\pi_{\Theta}(\Theta_q) > 1 - q$. Let $\pi_{min} = \min\{\pi_\theta : \theta \in \Theta_K\}$. Note if $q \leq \pi_{min}$, then $C_q = C_{DL}$. Thus we have

$$C_{DL} = \lim_{q \rightarrow 0} C_q.$$

The capacity vs. outage probability is related to the notion of ϵ -capacity of a channel. The ϵ -capacity of a channel is defined to be the supremum of all rates R such that there exists a sequence of block codes with rate R and block length n , where for n large enough, the average probability of error is less than or equal to ϵ . For a discrete memoryless channel the ϵ -capacity is equal to the usual capacity for all $\epsilon \in (0, 1)$. For the composite channel $\{\Gamma(\theta) : \theta \in \Theta\}$, the ϵ -capacity is identical to the capacity vs. outage probability ϵ as we have defined it above. This is shown in [CTB98], using techniques from [VH94].

As with delay limited capacity, capacity per outage probability is only meaningful for the original block fading channel if N is large enough. Specifically N must be large enough to disregard the first term of (3.13). How “large” N must be for this to hold depends on the values of $E_r(R, \theta)$. As $R \rightarrow C_\theta$, $E_r(R, \theta) \rightarrow 0$. Thus channels with C_θ near R will tend to have a higher probability of error. Note what happens as we let K grow. As $K \rightarrow \infty$, we have $C_\theta \rightarrow C_{NT}$ almost surely, by the strong law of large numbers. Thus, as shown in Fig. 3-2, as K gets large, for an outage probability ϵ , C_ϵ will approach C_{NT} . In this case, with high probability a channel will have C_θ near R , and the first term in (3.13) will no longer be negligible (for a given value of N). The above argument is somewhat complicated by the fact that as $K \rightarrow \infty$, the $E_r(R, \theta)$ will tend to increase.

If the channel has a short coherence time, then N may not be large enough for capacity vs. outage or delay limited capacity to be meaningful. In this case, the best we can do is bound the maximum achievable rate for a given probability of error, η . For example, one could attempt to optimize the right hand side of (3.13) to lower bound this rate. This appears to be a difficult problem. Note the right hand side of (3.13) is increasing in R , and thus the solution of the above optimization will be less than C_η .

Similarly, an upper bound on the achievable rate for a given N and a given allowable probability of (block) error can be attained from any lower bound on the probability of error. For example, suppose we have a lower bound on the probability of error, $p_{err}(\theta)$, given that we send at a rate of R over channel $\Gamma(\theta)$, *i.e.*, a function

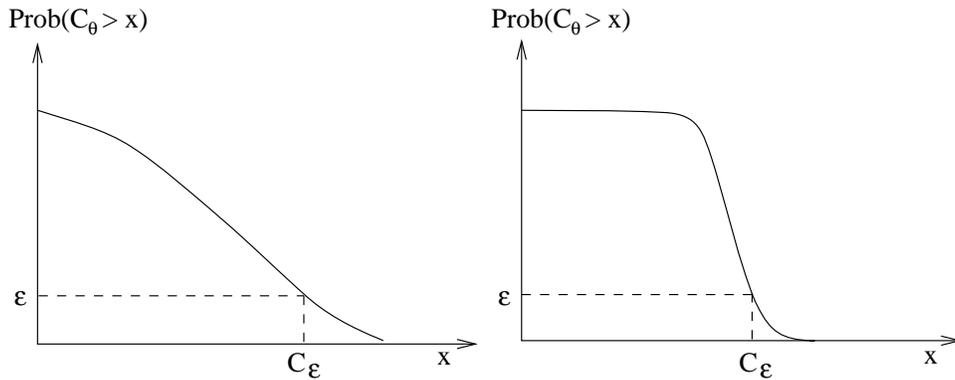


Figure 3-2: Illustration of the effect of K . The figure on the left represents the complementary distribution function of C_θ when K is small. The figure on the right represents the case where K is large.

$\ell(\theta, N, R)$ such that

$$p_{err}(\theta) > \ell(\theta, N, R) \quad (3.15)$$

Any reasonable bound, $\ell(\theta, N, R)$ will be increasing with R . In this case, the solution of the optimization problem:

$$\begin{aligned} & \text{maximize } R \\ & \text{subject to: } \sum_{\theta \in \Theta} \pi_\theta \ell(\theta, N, R) \leq \eta \end{aligned} \quad (3.16)$$

gives an upper bound on the rate that can be achieved with probability of error less than η . An example of such a lower bound on the probability of error is the sphere packing bound [Gal68].

3.1.3 Average capacity

With delay limited capacity, the assumption is that it is required to send each codeword at a given rate R regardless of the channel realization. With capacity vs. outage probability q , the assumption is that it is required to send each codeword at rate R over every channel realization except a subset with probability less than q . Over this

subset no information is received. In many situations, it may not be necessary to send a constant rate per codeword. For example, through the use of unequal error protection, instead of an outage when the channel is bad one could assume fewer bits get received reliably. Thus depending on the channel realization a variable number of bits are successfully received per codeword. The third notion of capacity we wish to consider for the compound channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$ will apply to such situations. We refer to this as the *average capacity* of the channel; in [EG98] a similar quantity is referred to as the *expected capacity*. This quantity is intended to indicate the maximum expected rate averaged over all channel realizations. Somewhat more precisely, let $R(\theta)$ be the rate that is reliably received given that a codeword is sent over the channel $\Gamma(\theta)$. The average capacity indicates the maximum of $\int_{\Theta} R(\theta) d\pi_{\Theta}(\theta)$, over all such rates $R(\theta)$. For the case of no transmitter CSI, the average capacity arises from taking a “broadcast approach” [Cov72] to communicating over the compound channel. We illustrate this with a simple example.

Example: Assume that $\mathcal{G} = \{g, \tilde{g}\}$ with $|g| > |\tilde{g}|$ and that $K = 1$ so that $\Theta = \mathcal{G}$. Also assume that the receiver uses a different decoder depending on the value of G , where G is a random variable with distribution π_G . This channel can be thought of as a degraded Gaussian broadcast channel with one “user” corresponding to each decoder. Let R_1 indicate the rate at which data is reliably received when $G = g$ and let R_2 be the rate at which data is reliably received when $G = \tilde{g}$. From [Cov72] it follows that the capacity region of such a channel is the set of rates (R_1, R_2) such that for some $\alpha \in [0, 1]$,

$$\begin{aligned} R_1 &< C(\alpha|g|^2P) \\ R_2 &< C(\alpha|g|^2P) + C\left(\frac{(1-\alpha)|\tilde{g}|^2P(N_0W)}{\alpha|\tilde{g}|^2P + N_0W}\right). \end{aligned} \tag{3.17}$$

The average capacity of this channel would be the maximum of $\pi_G(g)R_1 + \pi_G(\tilde{g})R_2$ over all rates (R_1, R_2) which lie in this capacity region.

The above example can be directly extended to larger channel state spaces. Extending this to cases where $K > 1$ requires more work. In this case the channel is

no longer a degraded broadcast channel, but a parallel Gaussian broadcast channel. Such channels have been looked at in [Tse98] and [WY00]. These results could be helpful in finding the average capacity of the compound channel $\{\Gamma(\theta) : \theta \in \Theta\}$ in such cases. We will not consider this here. Once again this quantity will only be useful for the block fading channel if N is large.

We have considered three notions of capacity for the block fading channel with a delay constraint of K channel blocks. Which of these notions is appropriate depends on the nature of the delay-constraint. For example if the delay-constraint arises because an application requires a fixed rate of R bits per K channel blocks, then delay-limited capacity or capacity vs. outage is more appropriate. If the delay constraint arises due to complexity or other system requirements, then average capacity may be more appropriate.

3.2 Perfect Transmitter State Information

Next we consider the case where both the transmitter and the receiver have perfect CSI. In this case the transmitter can adjust the transmission rate and/or energy according to the CSI. More precisely, the encoding and decoding for such a channel can depend not only on the given message sequence but on the channel state information. First we consider the capacity for the block fading channel in (3.1) with no delay constraint. We still assume that the input to the channel must satisfy the average energy constraint

$$\lim_{M \rightarrow \infty} \frac{1}{MN} \sum_{m=1}^M \text{tr}(\mathbb{E}(\mathbf{X}_m \mathbf{X}_m^\dagger)) \leq P/W$$

where N is the number of scalar channel uses in each block. In [Gol94] it is shown that the capacity of this channel is the solution of the following optimization problem:

$$\begin{aligned} & \underset{P: \mathcal{G} \rightarrow \mathbb{R}^+}{\text{maximize}} \mathbb{E}_G C(|G|^2 P(G)) \\ & \text{subject to: } \mathbb{E}_G P(G) \leq \bar{P} \end{aligned} \tag{3.18}$$

where $P(g)/W$ is the average energy per second or power used when $G_n = g$ and G is a random variable with distribution π_G . We denote the solution to this optimization

problem by C_T . The optimal power allocation for a user to achieve capacity has the form [Gol94]:

$$P(g)/W = \left[\frac{1}{\lambda} - \frac{N_0}{|g|^2} \right]^+ \quad (3.19)$$

where λ is chosen so that $\mathbb{E}_G P(G) = \bar{P}$. This allocation has the structure of the classic “water-filling” power allocation for communication over a colored Gaussian noise channel [Gal68]; here, however, power is allocated over the channel state space. With such an allocation a user transmits at a higher power when the channel is good and a lower power when the channel is bad. We note that this allocation depends only on the steady-state distribution of $\{G_n\}$. It has been shown that C_T can be achieved by using either a “single-codebook, variable power scheme” [CS98] or a “multiplexed multi-rate, variable power” transmission scheme [Gol94]. In either case approaching capacity again requires the use of codewords long enough to take advantage of the ergodic properties of the fading process $\{G_n\}$. Thus, with a delay constraint, such an approach may not be feasible. We next discuss the delay constrained case when the transmitter has perfect state information.

Assume that we are constrained to send a codeword over K blocks of the block fading channel. Once again this can be modeled as if each codeword must be sent over N uses of the composite channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$ where Θ_K denotes the set of all sequences of channel states, $\{g_1, \dots, g_K\}$ that can occur over K blocks. Recall, each channel $\Gamma(\theta)$ is a discrete memoryless channel defined by (*cf.* (3.5))

$$\begin{aligned} Y_n &= \mathbf{G}_\theta \hat{X}_n \\ &+ \hat{N}_n \end{aligned}$$

(3.20) where \mathbf{G}_θ is a $K \times K$ diagonal matrix with diagonal entries corresponding to $\theta = \{g_1, \dots, g_K\}$. We still assume that there is an *a priori* probability distribution $\pi_\Theta(\theta)$ defined on Θ_K . The difference now is that the transmitter can adjust the transmission rate and power based on the sequence of channel gains $\{g_1, \dots, g_K\}$. We will discuss precisely how this is done in the following. In this context we will again consider the notions of delay-limited capacity, capacity vs. outage and average

capacity. As in the previous section these quantities will be defined in terms of mutual information rates for the compound channel. Again proving a coding theorem requires allowing arbitrarily long codewords, in which case the compound channel no longer corresponds to the original block fading channel. For large enough N these quantities will give a useful indication of the rates that are achievable for the original block fading channel. For large N , the assumption of perfect CSI at the transmitter is also more reasonable.

3.2.1 Delay-limited Capacity and Capacity vs. Outage

As before, both delay-limited capacity and capacity vs. outage are based on the assumption that the user requires a fixed rate of R bits/sec in each group of K channel blocks for which an outage does not occur. For a block code this means that KNR/W bits are encoded into one of $2^{KNR/W}$ codewords, which is transmitted over the K channel blocks or equivalently over N uses of the compound channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$. Thus in this case the transmitter can vary the transmission rate and or power depending on the CSI but must satisfy this short term rate constraint.

Let $\mathbf{P} : \Theta_K \mapsto \mathbb{R}^K$, denote a *power allocation*, where $\mathbf{P}(\theta) = (P_1(\theta), \dots, P_K(\theta))$ and $P_i(\theta)$ represents the energy used by the i -th component of $\hat{\mathbf{X}}_n$

for the channel $\Gamma(\theta)$. In other words $P_i(\theta)$ is the i th diagonal term of $\mathbb{E}(\hat{\mathbf{X}}_n \hat{\mathbf{X}}_n^\dagger)$.

Consider a given $\theta = \{g_1, \dots, g_K\}$ and a given \mathbf{P} . Let $C_\theta(\mathbf{P})$ be the maximum mutual information rate over the channel $\Gamma(\theta)$, such that the average power of each component of $\hat{\mathbf{X}}_n$

is given by $\mathbf{P}(\theta)$. Then

$$C_\theta(\mathbf{P}) = \frac{1}{K} \sum_{i=1}^K C(P_i(\theta)|g_i|^2)$$

where we have divided by K so that this corresponds to mutual information per second in the original block fading channel. This is attained with the input distribution $\mathcal{CN}(0, Q)$ where Q is a diagonal matrix whose i th entry is $P_i(\theta)$.

Assume that the transmitter can choose any power allocation from some set Π . The delay-limited capacity of the compound channel is defined to be

$$\sup_{\mathbf{P} \in \Pi} \inf_{\theta \in \Theta} C_{\theta}(\mathbf{P}). \quad (3.21)$$

Once again, this is the maximum rate that can be sent reliably over the compound channel regardless of which channel is chosen.

For a given power allocation \mathbf{P} , let

$$C_q(\mathbf{P}) = \sup\{R : \Pr(C_{\theta}(\mathbf{P}) \leq R) \leq q\}. \quad (3.22)$$

where the probability measure is π_{Θ} . This is the capacity vs. outage probability q for the compound channel with a fixed power allocation \mathbf{P} . The capacity vs. outage probability q for the compound channel, when any power allocation from the set Π can be chosen is defined as

$$C_q(\Pi) = \sup_{\mathbf{P} \in \Pi} C_q(\mathbf{P}). \quad (3.23)$$

As before this corresponds to the ϵ -capacity of the compound channel. Also the delay limited capacity is still equal to the capacity vs. outage probability 0. These quantities will be meaningful for the block fading channel provided that N is large enough. How large N must be again requires considering the error exponents of the channels $\Gamma(\theta)$.

The above optimizations are over all power allocations in the set Π . We want to make a few comments about which power allocation should reasonably belong to Π . First every $\mathbf{P} \in \Pi$ needs to satisfy the average power constraint. By this we mean that

$$\mathbb{E}_{\theta} \left(\frac{1}{K} \sum_{i=1}^K P_i(\theta) \right) \leq \bar{P}. \quad (3.24)$$

Recall that if $\theta = \{g_1, \dots, g_K\}$, then in the original channel model the g_i 's occur sequentially in time. Thus if we allowed any power allocation which satisfied the average power constraint, this would require the transmitter to have non-causal knowledge of the channel states. Such a model has been looked at in [CTB98] but does not seem to be practical in most situations.⁵ A more reasonable set would be the set of all causal

⁵A situation where this model is applicable is if the blocks represent different frequency bands, for example in a multi-carrier system.

power allocations, *i.e.*, the power allocations, $\mathbf{P} = (P_1, \dots, P_K)$ such that

$$P_i(\theta) = P_i(g_1, \dots, g_i) \quad \forall i = 1, \dots, K \quad (3.25)$$

However, this restriction makes the above optimizations more difficult. We will look at an indirect solution to this problem for the case of delay-limited capacity next. Specifically, we assume we have a given rate, R and are using block codes. We then find the minimum average power constraint \bar{P} , such that the delay limited capacity of the channel with average power constraint \bar{P} is R . This problem will be formulated as a finite horizon dynamic programming problem.

3.2.2 Optimal causal power allocation.

Recall, we are assuming that a block code is used which must be transmitted in K blocks. Any power allocation, $\mathbf{P} = (P_1, \dots, P_k)$ which results in a delay-limited capacity of R , must satisfy

$$\frac{1}{K} \sum_{i=1}^K C(|g_i|^2 P_i(\theta)) \geq R \quad \forall \theta \in \Theta. \quad (3.26)$$

Thus the minimum average power needed for a delay limited capacity of R is the solution of:

$$\begin{aligned} & \text{minimize} \quad \frac{1}{K} \int_{\Theta} \left(\sum_{i=1}^K P_i(\theta) \right) d\pi_{\theta}(\theta) \\ & \text{subject to} \quad \sum_{i=1}^K C(|g_i|^2 P_i(\theta)) \geq KR \quad \forall \theta \in \Theta \\ & \quad \quad \quad P_i(\theta) \geq 0 \quad \forall \theta \in \Theta, i = 1, \dots, K \\ & \quad \quad \quad \mathbf{P} \text{ causal.} \end{aligned} \quad (3.27)$$

Let $u_i(\theta) = C(|g_i|^2 P_i(\theta))$ so that $u_i(\theta)$ denotes the mutual information rate sent over the i th block of the channel sequence θ . We want to write the above

optimization problem in terms of these variables⁶. Let $\mathbf{u} = (u_1, \dots, u_k)$ and let $P(g, u) = \frac{1}{|g|^2} C^{-1}(u)$, where

$$C^{-1}(u) = N_0 W (2^{u/N} - 1).$$

This is the inverse of $C(\cdot)$ given in (3.3). With these definitions we have $P_i(\theta) = P(g_i, u_i(\theta))$ and (3.27) can be rewritten as:

$$\begin{aligned} & \text{minimize } \frac{1}{K} \int_{\Theta} \left(\sum_{i=1}^K P(g_i, u_i(\theta)) \right) d\pi_{\Theta}(\theta) \\ & \text{subject to } \sum_{i=1}^K u_i(\theta) \geq KR \quad \forall \theta \in \Theta \\ & \quad \quad \quad u_i(\theta) \geq 0 \quad \forall \theta \in \Theta \\ & \quad \quad \quad \mathbf{u} \text{ causal} \end{aligned} \tag{3.28}$$

By \mathbf{u} being causal we mean that $u_i(\theta)$ only depends on g_1, \dots, g_i .

This optimization problem is equivalent to a discrete time, finite horizon, dynamic programming problem which we describe next. This dynamic programming problem has a finite horizon of K steps, each time step corresponding to a channel block. The state space of this problem is $[0, KR] \times \mathcal{G}$. The first coordinate of the state denotes how much mutual information remains to be transmitted and the second coordinate denotes the channel state. Initially, the first coordinate of the state will be KR . The control choice, u_n at each time $n < K$, is the amount of mutual information to be transmitted over that block. For a given control choice, the cost incurred is proportional to the power needed to transmit that amount of mutual information. Specifically, if the state at time $n < K$ is (x, g) and control u_n is used then the state transitions to $(x - u_n, g')$ with probability $Q_{g, g'}$ and the cost $\frac{1}{K} P(g, u_n)$ is incurred. Additionally, at time K if the state is (x, g) , a terminal cost of $\frac{1}{K} P(g, x)$ is incurred if $x > 0$ and no cost otherwise. This insures that no outage occurs. The goal is to choose controls to minimize the aggregate expected cost from any given starting

⁶The reason for this reformulation is to put this problem into a similar form to the problems that will be considered in later chapters.

state. This problem can be solved via dynamic programming. Let $J_1(KR, g)$ be the optimal cost when starting in state (KR, g) . Then $\mathbb{E}_G(J_1(KR, G))$ is the solution to (3.28), *i.e.*, the minimum average power needed for a delay limited capacity of R . The optimal controls used at each stage correspond to the optimal power allocation. We illustrate this approach with the following example.

Example: As a simple example of the above problem consider the case where $K = 2$ and $|\mathcal{G}| < \infty$. Let $J_2(x, g)$ be the terminal cost incurred if the final state is (x, g) . Then we have:

$$J_2(u, g) = \frac{N_0 W}{|g|^2} \left[\exp\left(\frac{u}{W}\right) - 1 \right]. \quad (3.29)$$

Let $J_1(RK, g')$ denote the optimal cost at time $n = 1$ given that the state is (RK, g') . From the dynamic programming algorithm, $J_1(RK, g')$ is given by the solution of the following non-linear program:

$$J_1(RK, g') = \inf_{0 \leq u \leq RK} \frac{N_0 W}{|g'|^2} \left[\exp\left(\frac{u}{W}\right) - 1 \right] + \sum_{g \in \mathcal{G}} Q_{g',g} \frac{N_0 W}{|g|^2} \left[\exp\left(\frac{RK - u}{W}\right) - 1 \right] \quad (3.30)$$

This is a minimization of a strictly convex function over the convex set $[0, RK]$, and thus it has a unique optimum. The value of u achieving this optimum is given by:

$$u^* = \begin{cases} 0 & \text{if } \sum_g Q_{g',g} \left(\frac{|g'|^2}{|g|^2}\right) \exp\left(\frac{RK}{W}\right) < 1 \\ RK & \text{if } \sum_g Q_{g',g} \left(\frac{|g'|^2}{|g|^2}\right) \exp\left(\frac{-RK}{W}\right) > 1 \\ \frac{RK}{2} + \frac{W}{2} \ln\left(\sum_g Q_{g',g} \frac{|g'|^2}{|g|^2}\right) & \text{otherwise.} \end{cases} \quad (3.31)$$

The corresponding optimal power allocation at time $n = 1$ is then given by:

$$\frac{N_0 W}{|g'|^2} \left[\exp\left(\frac{u^*}{W}\right) - 1 \right]. \quad (3.32)$$

This solution can be extended for other values of K , but as K gets large the optimization problems become more difficult and one would most likely turn to an approximate solution. We note that even for larger values of K the optimization at each step will be a convex problem. We can also modify this problem so that the

solution gives an average power and an outage probability, η . We do this by allowing an additional control choice in the final stage. Namely, we allow the option of choosing $P = 0$, and incurring a cost of β if $x_K > 0$. The optimal cost will correspond to the minimum of the average power plus β times the probability of outage. Assume that for a given choice of β , the solution to the dynamic programming problem results in an average power of \bar{P} and an outage probability of η . It follows that the capacity per outage probability η for this channel with average power constraint \bar{P} is R . By changing β we can get different outage probabilities.

This dynamic programming formulation only applies to block codes. Suppose we use convolutional codes where a constraint length must be sent in K blocks. Then it is more appropriate to require that the average rate over every window of K blocks is equal to R . In this case a different dynamic programming formulation is required. Additionally if one assumes that the overall code length is much larger than the constraint length, then an infinite horizon, average cost problem would be more appropriate. For example, again suppose that $K = 2$, then one can again formulate the problem on the state space $[0, RK] \times \mathcal{G}$. The control in each state will still represent the mutual information rate transmitted in that state. If the current state is (x, g) then, for the zero outage case, the control choice must satisfy $u > x$. If control u is chosen we transition to state $(2R - u, g')$ with probability $Q_{g,g'}$ and incur cost $P(g, u)$. We can then consider the infinite horizon, average cost problem with these dynamics. For larger values of K , we would have to increase the dimension of the state space to K where the components of the state represented the mutual information transmitted in the previous $K - 1$ channel blocks, along with the channel state. The resulting problem will quickly become intractable.

Note that we could also further restrict the set of allowable power allocations to be only those allocations which depend only on the current channel state. Such allocations are a subset of the causal allocations, and restricting to such a subset analytically simplifies the optimization and also leads to a practically simpler scheme. Also recall that the optimal power allocation without a delay constraint is in this set. This suggests that for large K such a restriction may be reasonable. Such an allocation would also be appropriate for convolutional codes. On the other hand suppose that K is not large and $Q_{g,g} > 0$ for every state $g \in \mathcal{G}$. In this case, for zero outage

probability, the optimal power allocation which depends only on the current channel state corresponds to

$$P(g) = \frac{1}{|g|^2} C^{-1}(R) \quad (3.33)$$

where R is still the short term rate constraint. This power allocation is referred to as “inverting the channel” and may result in a much higher average power than the optimal allocation without a delay constraint.

3.2.3 Average Capacity.

Now we consider average capacity for the case where the transmitter has perfect CSI. This applies when the transmitter has a constraint on the number of channel uses per codeword, but does not have a short term rate constraint. Thus the transmitter can vary the code rate depending on the CSI. The average capacity is intended to indicate the maximum average rate that can be transmitted. With no transmitter CSI, the average capacity was defined by using a broadcast model. With perfect CSI at both the transmitter and receiver, the situation is much simpler; here the transmitter can simply choose a code rate depending on the channel realization.

Let $\bar{R}(\theta)$ be the average transmission rate for the channel $\Gamma(\theta)$. Thus $KN\bar{R}(\theta)/W$ bits are encoded and transmitted during the K channel blocks corresponding to θ . Suppose we fix a power allocation \mathbf{P} as above, then $C_\theta(\mathbf{P})$ is the maximum rate at which one can reliably communicate over channel $\Gamma(\theta)$. In other words we must have $R(\theta) < C_\theta(\mathbf{P})$ for all $\theta \in \Theta_K$.

Motivated by this we define the average capacity of compound channel $\{\Gamma(\theta) : \theta \in \Theta\}$ to be

$$\sup_{\mathbf{P} \in \mathbf{\Pi}} \mathbb{E}_\theta C_\theta(\mathbf{P}). \quad (3.34)$$

This is the maximum expected mutual information rate that can be attained using any power allocation in $\mathbf{\Pi}$. As above $\mathbf{\Pi}$ is the set of allowable power allocations. Once again $\mathbf{\Pi}$ could be all power allocations which satisfy the average power constraint and are causal, or it could be defined to be some other set. Suppose $\mathbf{\Pi}$ contains only power allocations \mathbf{P} such that $P_i(\theta)$ is a function of only the current channel state, *i.e.*, for $\theta = \{g_1, \dots, g_K\}$, $P_i(\theta) = P_i(g_i)$. Additionally assume that all $\mathbf{P} \in \mathbf{\Pi}$ must satisfy

an average power constraint. In this case (3.34) is equivalent to the maximization in (3.18). The power allocation \mathbf{P} which achieves this maximum will have the form

$$P_i(\theta) = P^*(g_i) \text{ for all } i, \quad (3.35)$$

where P^* is the optimal water-filling power allocation as in (3.19). Even if $\mathbf{\Pi}$ contains any causal \mathbf{P} which satisfies the average power constraint, then (3.35) is still the optimal power allocation. This means that with perfect transmitter CSI, the average capacity is equal to the usual Shannon capacity (3.18). However, these two quantities have a different operational significance. Average capacity indicates the maximum expected rate, where codewords can have variable rates. On the other hand, for the Shannon capacity, codewords have a fixed rate. Approaching the average capacity requires N to be large, while approaching the Shannon capacity requires K to be large.

We have discussed causality issues with regard to the power allocation. For average capacity when $K > 1$, there may also be a causality issue with regard to $\bar{R}(\theta)$; this depends on some architectural details, such as how data is encoded and decoded. To achieve the average capacity, we must have $\bar{R}(\theta) = C_\theta(\mathbf{P})$ for all $\theta \in \Theta_K$. Assume that data is to be encoded into a block code with rate $\bar{R}(\theta)$ by choosing one of $KN\bar{R}(\theta)/W$ codewords at the start of the K channel blocks. In this case, the rate of the code must be known at the start of the first block, *i.e.*, based only on the first channel state. If we restrict ourselves to such a model, then we can not use the above rate allocation unless $K = 1$. Instead we need to specify a rate $\bar{R}(\theta)$ which is only a function of g_1 . Once we specify $\bar{R}(\theta)$ then we can consider the channel as having a short term rate constraint and optimize the power allocation as in the previous section. One could modify the dynamic program from Sect. 3.2.2 for this case. Specifically assume that an initial stage 0 is added in which one starts in state $(0, g)$, where g is the initial channel state. In this stage, the allowable control actions correspond to choosing a rate $\bar{R} > 0$; for a given control choice, the state transitions to $(K\bar{R}, g)$ with probability one and a cost of $-\lambda\bar{R}$ is incurred. From here on the problem proceeds as in Sect. 3.2.2. Let $J_0(0, g)$ be the minimum aggregate cost starting in state $(0, g)$. Then $\mathbb{E}J_0(0, g)$ will be the average power used minus λ times the average transmission rate.

This average transmission rate is the optimal rate for this average power constraint. We also note that if the transmission rate is required to be chosen in this manner, it is again meaningful to allow outages.

Still assume that a block code is used, but instead of selecting the codeword at the start of the given block, suppose that codewords are formed sequentially. By this we mean that the code rate $\bar{R}(\theta)$ is determined by a function $\mathbf{R}(\theta) : \Theta_K \mapsto \mathbb{R}^K$, where $\mathbf{R}(\theta) = (R_1(\theta), \dots, R_K(\theta))$ and $R_i(\theta) = R_i(g_1, \dots, g_i)$. The portion of the codeword sent over the i -th block of channel sequence θ only depends on $\sum_{j=1}^i R_j(\theta)N/W$ of the encoded bits. Thus $\bar{R}(\theta) = \frac{1}{K} \sum_{i=1}^K R_i(\theta)$. Letting $N \rightarrow \infty$ and using such a scheme one could again approach the average capacity by using the optimal power allocation as in (3.35) and letting $R_i(\theta) = C(|g_i|^2 P^*(g_i))$. However, for finite N there is a difficulty with such a scheme. For codewords formed in this fashion, bits transmitted in earlier channel blocks will have more redundancy than bits transmitted in later blocks.⁷ For example the bits transmitted over the last block are essentially only encoded into a codeword of length N . If N is not large enough to average over the additive noise then these bits may not be received reliably. This suggests that the average rate should be larger than $C(|g_i|^2 P^*(g_i))$ for small i and become less than $C(|g_i|^2 P^*(g_i))$ as the block number increases. One way to get around this problem is to use a convolutional code instead of a block code.

3.3 Frequency selective fading

In this section we comment on when and how the results of the previous sections can be extended to channels with frequency selective fading. From Sect. 2.1, recall that for a frequency selective channel the delay spread L is larger than $1/W$ or equivalently the channels coherence bandwidth is less than W . In this case the sampled channel model is (*cf.* (2.2)):

$$Y_n = \sum_{m=0}^{\lfloor LW \rfloor} X_{n-m} G_{m,n} + Z_n$$

⁷In the limiting case as $N \rightarrow \infty$ this is not an issue since one can essentially send a separate codeword over each channel block.

where the fading process $\mathbf{G}_n = [G_{0,n}, G_{1,n}, \dots, G_{\lfloor LW \rfloor, n}]^T$ is still modeled as a Markov chain. The results of the previous two sections can easily be extended to such a channel if one assumes that the channel has block fading and is also *block memoryless*. In this case block fading means that \mathbf{G}_n stays fixed for blocks of N channel uses then changes to a new value. By block memoryless we mean that the received signal in each block depends only on the transmitted signal in that block. This assumption is not generally satisfied for a frequency-selective fading channel. If the channel has a delay spread of L , then after the channel changes state, the signals transmitted in the previous block will arrive for L more seconds and thus effect the received signal in the next block. If the coherence time is much larger than the delay spread this effect may be negligible and the block memoryless assumption can be considered a reasonable approximation. Also if the system has a TDMA structure, and the block time equals the dwell time, then this assumption is again reasonable. We will only look at the block memoryless case in the following. Without this assumption, it is more difficult to generalize the above results. Some work in this direction can be found in [Med95] and [GM99].

Each channel realization $\mathbf{g} = [g_0, \dots, g_{\lfloor LW \rfloor}]$ can be thought of as a sequence of time samples at rate W . This sequence has a discrete time Fourier transform:

$$(\mathcal{F}\mathbf{g})(f) = \sum_{n=0}^{\lfloor LW \rfloor} g_n e^{j2\pi n f \frac{1}{W}}$$

for $|f| \leq W/2$. Let \mathbf{G} be a random vector with the steady-state distribution of $\{\mathbf{G}_n\}$. Then for each $f \in [-W/2, W/2]$, $|\mathcal{F}\mathbf{G}(f)|^2$ is a random variable, depending on the realization of \mathbf{G} . Consider the case where there is no transmitter CSI but perfect CSI at the receiver. Assume that for any two distinct frequencies, f_1 and f_2 in $[-W/2, W/2]$, that $|\mathcal{F}\mathbf{G}(f_1)|^2$ and $|\mathcal{F}\mathbf{G}(f_2)|^2$ are identically distributed. In other words, the fading in any narrow frequency band is identically distributed. With this assumption, the capacity of the block fading channel is given by [Gal94a]:

$$C_{NT} = W \mathbb{E}_{\mathbf{G}} \int_{-W/2}^{W/2} \log \left(1 + \frac{P|\mathcal{F}\mathbf{G}(f)|^2}{N_0 W} \right) df \quad \text{bits/sec.} \quad (3.36)$$

This is attained using i.i.d. Gaussian inputs with distribution $\mathcal{CN}(0, \frac{P}{W})$. In a similar way, one can generalize the other results in Sect. 3.1 to the case of frequency selective fading.

Next we consider the case of perfect CSI at both the transmitter and receiver. In this case the transmitter can allocate power over the frequency band $[-W/2, W/2]$ according to the channel's frequency response, $(\mathcal{F}\mathbf{g})$. Let $P(f, \mathbf{g}, \bar{P}(\mathbf{g}))$ denote a power allocation when the channel state is \mathbf{g} , which uses average power $\bar{P}(\mathbf{g})$, *i.e.*

$$\bar{P}(\mathbf{g}) = \int_{-W/2}^{W/2} P(f, \mathbf{g}, \bar{P}(\mathbf{g})) df.$$

Let $P^*(f, \mathbf{g}, \bar{P}(\mathbf{g}))$ denote the power allocation which maximizes the mutual information rate sent in this channel state with average power $\bar{P}(\mathbf{g})/W$. As in (3.19), P^* is given by a “water filling” allocation:

$$P^*(f, \mathbf{g}, \bar{P}) = \left[\frac{1}{\lambda} - \frac{N_0 W}{|(\mathcal{F}\mathbf{g})(f)|^2} \right]^+$$

where λ is chosen so that the average power is $\bar{P}(\mathbf{g})$. Let

$$C_{WB}(\mathbf{g}, \bar{P}(\mathbf{g})) = W \int_{-W/2}^{W/2} \log \left(1 + \frac{P^*(f, \mathbf{g}, \bar{P}(\mathbf{g})) |(\mathcal{F}\mathbf{g})(f)|^2}{N_0 W} \right) df. \quad (3.37)$$

This is the capacity of a time-invariant channel with impulse response \mathbf{g} and average power constraint $\bar{P}(\mathbf{g})$. The capacity of the block fading channel with perfect transmitter side-information is the solution of the optimization problem [Gol94], [TH98]:

$$\begin{aligned} & \underset{P: \mathcal{G} \rightarrow \mathbb{R}^+}{\text{maximize}} \mathbb{E}_{\mathbf{G}} C_{WB}(\mathbf{G}, \bar{P}(\mathbf{G})) \\ & \text{subject to: } \mathbb{E}_{\mathbf{G}} \bar{P}(\mathbf{G}) \leq \bar{P} \end{aligned} \quad (3.38)$$

Note the similarity of this to the optimization in (3.18). In a similar manner, the other results in Sect. 3.2 can be generalized to frequency selective fading channels.

3.4 Summary

In summary we have looked at block fading channels where the receiver has perfect state information. We looked at both the case where the transmitter has perfect CSI and where it has no CSI. The capacity of channels where the transmitter has only partial state information or delayed state information have been looked at in the literature, for example see [CS98] or [Vis98]. The above ideas also apply to these situations in an analogous manner.

Our main emphasis has been on cases where there is a delay constraint which limits the number of channel blocks over which a codeword can be sent. We formulated a compound channel model for these situations and discussed various notions of capacity for this model. We emphasized that these “capacities” constitute much weaker statements about the block fading channel than typical capacity statements. In particular they require the coherence time of the fading channel to be large relative to the error exponents of the compound channels but small relative to the delay constraint. If this is not the case, then we can still attempt to bound the average rate that is achievable for a given probability of error and a given power constraint. Equivalently we can lower bound the required average power, for a given rate and probability of error. In the following sections, it will be more convenient to think of these results in this way.

Each notion of capacity we considered is intended to signify the maximum “rate” for which there exist codes of that rate whose error probabilities are arbitrarily small. But in each case “rate” is interpreted differently. For example with delay-limited capacity, one is interested in the maximum constant rate per codeword; with average capacity one is interested in the maximum expected rate per codeword. Which notion of capacity is appropriate depends on the Q.O.S. required by the higher layer application. Each of these definitions only considered the “rate” of a single codeword. In the next chapter we will consider a model in which entire sequences of codewords are considered. This allows us to consider a much larger class of Q.O.S. requirements.

CHAPTER 4

Buffer Models

Recall the situation in Fig. 1-1; this is repeated in Fig. 4-1 below. For this situation we are interested in minimizing the power required to provide a user with acceptable service while satisfying some delay constraints as in Sect. 2.2. The work in the previous chapter provides an answer to this question if (1) any delay constraints can be mapped into a limit on the number of channel blocks that can be used per codeword, (2) the number of channel uses per block is large, and (3) the acceptable service required by a user can be stated in terms of either a constant rate or average rate per codeword. In this chapter we shall consider several models which relax each of these assumptions to various degrees. These models involve a buffer as in Fig. 4-1 and allow us to consider a larger class of service requirements. We illustrate this with an example:

Example: Consider the situation where an application generates data at a constant rate of R bits per second with the following two delay constraints – due to a system constraint a codeword must be sent in only K blocks, and, to provide acceptable quality of service, data must be received within a maximum delay of D seconds where $D \gg KN/W$. For example a user could be transmitting a video sequence which is generated at a constant rate R and transmitted in codewords which are sent over K blocks. At the receiver this sequence is to be stored in a playback buffer and then shown after a delay of D seconds. Our goal is then to find the minimum amount of power needed to transmit this sequence within these constraints. When the transmit-

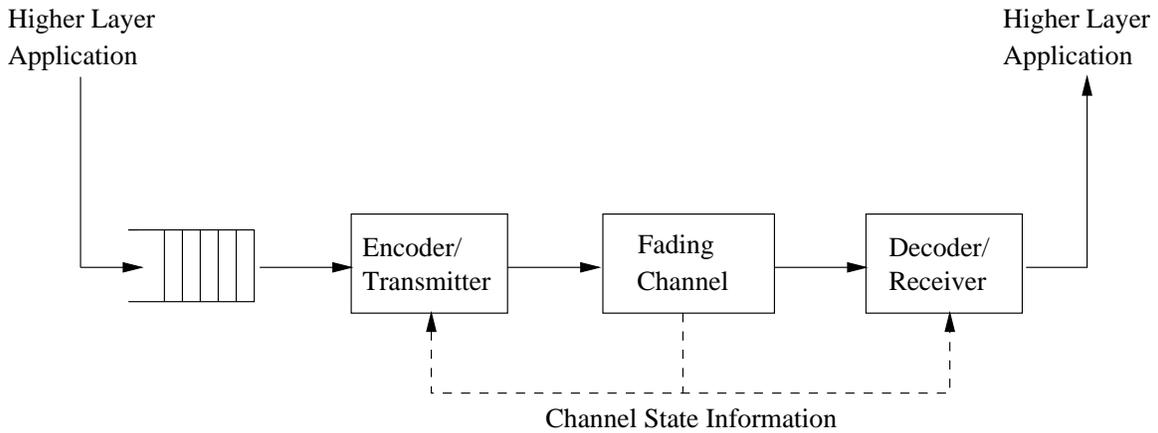


Figure 4-1: System model.

ter has perfect state information, the above constraints cannot be adequately modeled in any of the frameworks from the previous chapter. For example if we require a constant rate R per codeword, then the Q.O.S. constraint will be met, but this is more conservative than necessary. On the other hand, if we require only an average rate per codeword, then the higher layer Q.O.S. constraint may be violated. The buffer models which we consider below allow us to model such a situation as well as several others.

The remainder of the chapter will be spent describing several such models for the situation in Fig. 4-1. The difference between these models is in the types of constraints considered as well as the specific model of the transmitter/encoder and the buffer dynamics. In each case we will assume that the fading channel is a block fading channel as defined in Sect. 2.1. Furthermore we will primarily focus on the case where the transmitter has perfect CSI. Also, in each case we use a discrete time model of the buffer where the time between two adjacent samples corresponds to one block of N channel uses of the block fading channel. In other words, time n corresponds to the start of the n th block.

4.1 Mutual Information Model

The model we consider in this section can be thought of as a generalization of the capacity vs. outage framework discussed in the previous chapter. As noted above, we will consider a discrete time model for the system in Fig. 4-1. Between time $n-1$ and n assume that A_n bits arrive from the higher layer application and are placed into the buffer. At each time n the transmitter removes U_n bits from this buffer; encodes them into a codeword and transmits this codeword over the next channel block. Thus the transmission rate during this block is $U_n W/N$ bits per second. At the end of the block, the codeword is decoded and the bits are sent to the higher layer application at the receiver. Note we are assuming that each codeword is sent in N channel uses and that the entire codeword is received before it is decoded. In the notation of the previous chapter this corresponds to $K = 1$. In the following sections, we will consider relaxing some of these assumptions.

Assume that the arrival process $\{A_n\}$ is a stationary ergodic Markov chain with steady-state distribution π_A . Conditioned on A_n , we assume that A_{n+1} is independent of Z_m , G_m , and U_m , for all $m \leq n$. Denote the steady-state expected arrival rate by \bar{A} . Let S_n denote the number of bits in the buffer at time n^- , *i.e.* just before the start of the n th block. The dynamics of the buffer are then given by:¹

$$S_{n+1} = \min\{\max\{S_n + A_{n+1} - U_n, A_{n+1}\}, L\} \quad (4.1)$$

where L is the buffer size. This is illustrated in Fig. 4-2. We assume that the transmitter can choose U_n based on the buffer state S_n and the channel state G_n and the arrival state A_n .²

We also assume that the transmitter can adjust the transmission power during the block. If u bits are to be transmitted during a channel block when the fading state is g , the transmission power used is the minimum power required so that the

¹Since U_n is chosen based on S_n , it will always satisfy $U_n \leq S_n$. Thus the max is not really necessary in (4.1). We include this for clarification.

²More generally, U_n could be chosen based on the sequence of buffer, channel and arrival states up to time n . However, for the Markov decision problem considered below there is no benefit in allowing this generality.

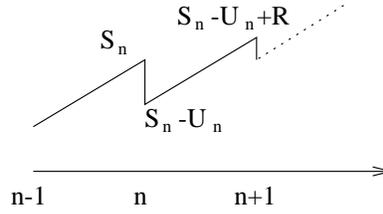


Figure 4-2: Buffer dynamics.

mutual information rate during the channel block is equal to uW/N bits/sec. This quantity is denoted by $P(g, u)$. Thus

$$C(|g|^2 P(g, u)) = uW/N \quad (4.2)$$

and therefore

$$\begin{aligned} P(g, u) &= \frac{1}{|g|^2} C^{-1}(uW/N) \\ &= \frac{N_0 W}{|g|^2} (2^{u/N} - 1). \end{aligned} \quad (4.3)$$

As in the previous chapter, this can be considered a reasonable approximation on the amount of power required provided that N is large enough to transmit near capacity with acceptably small probability of error.

For the receiver to reliably receive the transmitted message, it is beneficial for it to know the transmission rate and power used by the transmitter. At time n this depends on G_n , S_n and A_n . Thus, if the receiver knows G_n , S_n , A_n , it can calculate the transmission rate and power that the transmitter is using. By assumption, the receiver knows G_n . If the arrival rate A_n is a constant then the receiver can simply calculate the buffer state at each time n and thus will have knowledge of S_n and A_n . With random arrivals, some additional overhead may be required to convey this information. We assume that the receiver has this knowledge available.

Next we examine the following two situations. In the first case, we consider minimizing the long term average power while keeping the probability of buffer overflow

small. In the second case, we consider minimizing the long term average power while keeping the average buffer delay small. As in Sect. 3.2.2, each of these situations is formulated in a Markov decision framework, except here this will be an infinite horizon, average cost setting. In the next chapter, we will consider the solution to such problems; here we simply give a formulation. Denote the buffer state space by \mathcal{S} . At various places we will either assume that $\mathcal{S} = \{0, 1, \dots, L\}$ or $\mathcal{S} = [0, L]$, depending on which is mathematically more convenient. We consider a Markov decision problem with state space $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$, or if the arrival process is memoryless we can simply consider $\mathcal{S} \times \mathcal{G}$ to be the state space. At each time n , the possible control action corresponds to the number of bits transmitted, U_n . A given control action u in state (s, g, a) will incur a cost $P(g, u)$. A control policy is a sequence of functions $\{\mu_1, \mu_2, \dots\}$, where $\mu_n : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \mapsto \mathbb{R}^+$ specifies U_n as a function of S_n, G_n and A_n . The expected long term average power under such a policy is

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \mathbb{E} \left(\sum_{n=1}^M P(G_n, \mu_n(S_n, G_n, A_n)) \right). \quad (4.4)$$

As stated above, we are interested in minimizing this quantity, while either avoiding buffer overflows or keeping average delay small. Each of these cases is examined next.

4.1.1 Probability of overflow

In this case we want to minimize (4.4) while avoiding buffer overflows. For a constant arrival rate overflows can be completely avoided by constraining the allowable policies to those such that $\mu(s, g, a) > s + \bar{A} - L$ for all states (s, g, a) . With variable arrivals, overflow can not be avoided in general. Instead of constraining the controls so that the buffer will not overflow, we consider allowing the buffer to overflow while incurring an additional cost when this occurs – we refer to this as a “buffer cost”. By incurring a larger buffer cost, less average power is required. In the following chapters the trade-off between this buffer cost and the average power needed is considered. Note that allowing a buffer overflow is similar to allowing an outage to take place. This

additional cost is defined as:

$$b(s, a, u) = \begin{cases} 1 & \text{if } s + a - u > L, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

The time average cost corresponding to this term is

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \mathbb{E} \left(\sum_{n=1}^M b(S_n, A_{n+1}, U_n) \right).$$

Under suitable assumptions on the policy³, this is equal to the steady-state probability the buffer overflows.

When the buffer overflows at time n , anywhere between 1 and A_n bits will be lost. The above cost term assigns the same cost to any overflow regardless of the number of bits which are lost. For some applications it may make more sense to have a cost that is proportional to the number of bits lost, *i.e.*,

$$b(s, a, u) = (s + a - u - L)^+. \quad (4.6)$$

In this case the time average cost will correspond to the time average number of bits lost due to overflow. Which of these costs is more appropriate depends on the nature of the higher layer application.

Instead of (4.5) or (4.6), at times we will consider an additional cost term defined by:

$$b(s) = \mathbf{1}_L(s) = \begin{cases} 1 & \text{if } s = L, \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

the time average cost corresponding to this term corresponds to the steady-state probability that the buffer is in state L . This is an upper bound on the probability the buffer overflows and will be approximately equal for large buffers. The cost given by (4.7) depends only on the buffer state, while (4.5) and (4.6) depend on the buffer state, the control action, and the arrival process. This makes (4.7) easier to work

³We will be more precise about this in the following chapter.

with in some cases. Also note that for a constant arrival rate, if the cost in (4.7) is multiplied by \bar{A} , this will be an upper bound for (4.6).

A special case of the above situation is when the arrival rate is constant, *i.e.*, $A_n = \bar{A}$ for all n . In this case the buffer size, L , can be chosen so that a buffer overflow will correspond to a maximum delay requirement being exceeded. This can be used to model the situation in the example at the start of this chapter. The total delay experienced by a bit in Fig. 4-1 is the sum of the delay until a given bit leaves the buffer plus the delay until it is decoded once it leaves the buffer. Assume that the delay after leaving the buffer is given by $N/W + D_p$, where N/W is the time required to transmit the entire codeword and D_p accounts for the propagation delay and processing time. Recall, we are assuming that the entire codeword must be received before any bits are decoded. If this assumption is not true, then the above arguments will need to be modified; this will be discussed more in the next section.

With the above assumptions, by keeping the delay in the buffer less than $D - (N/W + D_p)$, it can be guaranteed that the total delay will be less than D . If the buffer size is chosen so that

$$L = \bar{A} \left(\frac{D - N/W - D_p}{N/W} \right), \quad (4.8)$$

the event of the buffer overflowing at time n will correspond exactly to the delay of the bit at the head of the buffer being larger than $D - N/W - D_p$. Thus

$$\Pr(\text{buffer overflows}) = \Pr(\text{Delay in Buffer} > D - N/W - D_p).$$

Note that we are considering delay in terms of a continuous time model as in Fig. 4-2 with a constant arrival rate. When the arrival rate is variable, the event of a buffer overflow no longer corresponds to a maximum delay constraint being violated. In this case we simply assume that the system has a limited buffer, and the user wishes to avoid losing data due to buffer overflows.

4.1.2 Average Delay

In this case we want to minimize (4.4) while also keeping the average delay small. For a given control policy, let \bar{D}_B indicate the time average delay of a bit in the buffer. The overall average delay experienced by a bit in Fig. 4-1, is then $\bar{D}_B + N/W + D_P$ where N/W is still the length of time it takes to transmit each codeword, and D_P is the processing and propagation delay.⁴ We assume that these last two terms are fixed and thus focus on the average delay in the buffer. Here we assume that the buffer size $L \rightarrow \infty$ so that no bits are lost due to overflow. In this case the buffer dynamics are given by⁵:

$$S_{n+1} = \max\{S_n + A_{n+1} - U_n, A_{n+1}\}. \quad (4.9)$$

As in the previous section, a buffer cost $b(s)$ is defined to account for the average delay. In this case assume that

$$b(s) = s/\bar{A}. \quad (4.10)$$

The time average cost due to this term is

$$\frac{1}{\bar{A}} \left(\limsup_{M \rightarrow \infty} \frac{1}{M} \mathbb{E} \left(\sum_{n=1}^M S_n \right) \right). \quad (4.11)$$

By Little's law this is equal to the time average delay in the buffer. The trade-off between this quantity and the average power needed will be considered.

We conclude this section with a couple of comments about this buffer cost. Let \bar{S} denote \bar{A} times the quantity in (4.11). Thus \bar{S} is the time average buffer occupancy in the discrete time system. If we assume that arrivals occur as in Fig. 4-2, then \bar{S} will be slightly larger than the time average number in the actual continuous time system. This is shown in Fig. 4-3. This figure shows a sample path of the arrivals in the continuous time system, $A_C(t)$, assuming a constant arrival rate. The equivalent arrivals in the discrete time system, $A_D(t)$ and a sample path of the departures $D(t)$ are also shown. From Little's law it follows that the average delay in the continuous

⁴Again we are assuming a block code is used and that the entire codeword must be received before it is decoded.

⁵Again, $U_n \leq S_n$ so the max is not really necessary

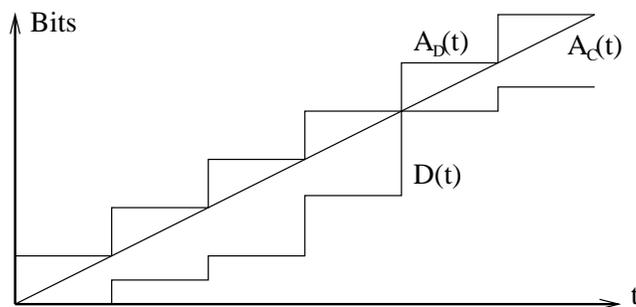


Figure 4-3: Difference between average buffer size in continuous time and discrete time model.

time system will be slightly less than \bar{S}/\bar{A} . If we assume that the actual arrival rate is a constant rate of \bar{A} per second then, from Fig. 4-3 we see that the average number in the continuous time system will be $\bar{S} - \bar{A}/2$, and thus the average delay in the continuous time system will be $\bar{S}/\bar{A} - 1/2$.

Finally, note that for \bar{S}/\bar{A} to be equal to the average buffer delay, we are assuming that no bits are lost due to overflows. When $L \rightarrow \infty$ this is true. With a finite L , if we constrain the control policies to avoid overflows, then \bar{S}/\bar{A} is still proportional to the average buffer delay. If we do allow overflows, then \bar{S}/\bar{A} will only give a lower bound on the average delay.

4.2 Model Variations

In the previous section, we assumed that each codeword is sent over one block of a block fading channel as in (2.5). Furthermore, we assumed that the power needed to transmit u bits reliably during a block with the fading state g is given by $P(g, u)$ as defined in (4.3). This is the minimum power such that the mutual information rate during that block is uW/N bits per second. In this section, we take a closer look at these assumptions and discuss modifications of the model which allow these assumptions to be relaxed.

4.2.1 Other Power Functions

First we look at the assumption that the required power is given by (4.3). For this to give a good indication of the required power, the number of scalar channel uses, N , per block needs to be large relative to the error exponent of the channel. The analysis in the following chapters depends only on a few characteristics of the cost function $P(g, u)$. Any cost function which has these characteristics is defined to be *good*. Specifically,

Definition: A function $P : \mathcal{G} \times \mathbb{R}^+ \mapsto \mathbb{R}^+$, such that $P(g, u)$ gives the power needed to transmit u bits during a block when the channel state is g , is defined to be a *good power function* if it can be written as:

$$P(g, u) = \frac{\tilde{P}(u)}{|g|^2}$$

where $\tilde{P}(u)$ depends only on u and is a monotonically increasing, strictly convex function of u .

The function $P(g, u)$ defined in (4.3) is clearly a good power function. Consider a model as in the previous section where $P(g, u)$ is not given by (4.3) but is any other good power function. The analysis in the following chapters will apply to such a model as well. We give some examples next:

Bound on required power: Still assume that each codeword has to be transmitted over a single block of N channel uses during which the fading state g remains constant. We will define $P(g, u)$ to be an upper bound on the power required to send u bits with average probability of error less than η . To do this, we use a random coding bound [Gal68]. Specifically consider a code with average power $P(g, u)/W$ and 2^u codewords. Each codeword is a sequence of N complex symbols. From the random coding bound, there exists such a code whose average probability of error⁶ P_e , after

⁶Here P_e is the probability of error averaged over the possible messages. By multiplying this bound by 4, one can get a bound which applies to the probability any codeword is in error (*cf.* [Gal68]).

transmission over a complex Gaussian channel with gain g and noise variance N_0 is bounded by

$$P_e \leq \exp(\rho u - N E_0(\rho, |g|^2 P(g, u))) \quad (4.12)$$

for any $\rho \in (0, 1]$. Here

$$E_0(\rho, |g|^2 P(g, u)) = \rho \ln \left(1 + \frac{|g|^2 P(g, u)}{N_0 W(1 + \rho)} \right). \quad (4.13)$$

From this it follows that if

$$P(g, u) = \frac{N_0 W(1 + \rho)}{|g|^2} (e^{(u/N - \ln \eta/\rho)} - 1) \quad (4.14)$$

then, for any $\rho \in (0, 1]$, this gives an upper bound on the power required as desired. Also it is clearly a good power function. Of course optimizing over ρ would give a tighter bound, *i.e.*

$$P(g, u) = \inf_{\rho \in (0, 1]} \frac{N_0 W(1 + \rho)}{|g|^2} (e^{(u/N - \ln \eta/\rho)} - 1) \quad (4.15)$$

but this bound will not necessarily be convex in u and thus not a good power function.

In a similar manner, other power functions could be derived via other bounds on the probability of error. For example, a lower bound on the probability of error can be used to find a lower bound on the required power. Any such bound which results in a good power function could be used in the following.

Specific Modulation/Coding Scheme: Suppose the transmitter is using a specific modulation/coding scheme which allows us to vary both the transmission rate and power. Let $P(g, u)$ be the power required for this transmission scheme to transmit acceptably at rate u when the channel gain is g . If P is a good power function, the following results will also apply to this modulation/coding scheme. In this case, these results provide the minimum average power needed for this particular modulation/coding scheme. A specific example of such a modulation/coding scheme is the variable rate trellis coded M-QAM proposed in [GV97]. In this case, an approximation

for the amount of transmission power required is given by

$$P(g, u) = \frac{N_0 W}{|g|^2} \left(2^{\frac{u+2r}{N}} \right) K_c, \quad (4.16)$$

where r is related to the rate of the convolutional code used and K_c is a constant that depends on the coding gain and the required bit error rate. Once again this is clearly a good power function.

Next we argue that in a certain sense any reasonable scheme which allows one to vary the transmission power and rate will “almost” have a good power function. Suppose we can choose between M different modulation/coding schemes for transmitting each codeword, and each codeword is sent over a single channel block. The i th modulation and coding scheme allows us to send one of 2^{u^i} codewords over a block. Assume that $u^1 < u^2 < \dots < u^M$. To get an adequate probability of error with the i th coding scheme, the SNR⁷ at the receiver is required to be σ^i . Thus when the i th coding scheme is used and the channel state is g , the transmission power must be $P^i = \frac{\sigma^i}{|g|^2} N_0 W$. Let $\hat{P}(g, \bar{u})$ be the minimum average power required to transmit at \bar{u} bits on average when the channel state is g . We argue that $\hat{P}(g, \bar{u})$ is convex and increasing. For any reasonable modulation and coding scheme, the σ^i 's will be increasing in i . Thus $\hat{P}(g, \bar{u})$ is increasing in \bar{u} . Assume \bar{u} is in $[0, u^M]$. To show that \hat{P} is convex in \bar{u} assume that to transmit at average rate \bar{u} in a given channel state g , the transmitter is allowed to randomly use any of the M coding schemes. In other words, we use the i th scheme with probability p_i , so that $\sum p_i u^i = \bar{u}$. Let $\hat{P}(g, \bar{u})$ be the minimum average power to transmit \bar{u} bits in this way. It can then be seen that $\hat{P}(g, \bar{u})$ will be convex. Thus \hat{P} has all of the characteristics of a good power function, except, it is convex but not strictly convex. Also in this case \bar{u} is the expected transmission rate, instead of the actual number of bits transmitted. Despite these differences, the arguments in Chap. 6 can be modified to apply in this case.

⁷signal power divided by noise power

4.2.2 Wide-band Fading

Now we look at relaxing the assumption that the fading is flat. Assume that this is not the case and that each codeword is sent over a frequency selective fading channel. As in Sect. 3.3, we will assume that this is a block memoryless fading channel. Also we still assume that each codeword is sent over one block. Recall, in this case the channel state is a vector $\mathbf{g} \in \mathbb{C}^{\lfloor LW \rfloor}$. As in Sect. 4.1, we define $P(\mathbf{g}, u)$ to be the minimum average transmission power required so that the mutual information rate during the channel block is equal to uW/N bit/sec; only now the channel is frequency selective. Thus,

$$C_{WB}(\mathbf{g}, P(\mathbf{g}, u)) = uW/N$$

where $C_{WB}(\cdot, \cdot)$ is given in (3.37). For any $\mathbf{g} \neq \mathbf{0}$, $C_{WB}(\mathbf{g}, P)$ is an increasing and strictly concave function of P . Thus the inverse function, $C_{WB}^{-1}(\mathbf{g}, x)$ is well defined, where this is the inverse with respect to the P variable. Therefore setting

$$P(\mathbf{g}, u) = \begin{cases} C_{WB}^{-1}(\mathbf{g}, uW/N) & \text{if } \mathbf{g} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{g} = \mathbf{0} \text{ and } u = 0 \\ \infty & \text{otherwise} \end{cases} \quad (4.17)$$

yields the desired function. Like a good power function, this function is strictly convex and monotonically increasing in u for all $\mathbf{g} \neq \mathbf{0}$. Most of the results in the following only rely on this structure and thus apply for the wide-band case as well. The exception is the results in Ch. 5 that describe the structure of the optimal policy with respect to the channel gain. Such results only apply for the narrow band case.

4.2.3 More Than One Channel Block per Codeword:

In this section we look at the assumption that each codeword is transmitted over only one block of a block fading channel. For this to be true each codeword must be sent in N scalar channel uses, and N/W must be less than the channel's coherence time. The constraint on the number of channel uses per codeword could be due to a system constraint as in Sect. 2.2. If there is such a system constraint and it is less than the channel's coherence time, then we can assume that the block length in the

channel model is determined by this constraint. On the other hand, if any constraint on the number of channel uses per codeword is larger than the channel's coherence time, then this model is not adequate. For example, if the system's bandwidth is 30kHz, and the coherence time is 2 msec, then we must have $N < 60$ for the above model to hold. If a larger block size is desired then the fading will change during the transmission of the codeword. One way to model this is to assume that each codeword is sent over $K > 1$ channel blocks. We look at two ways to model such an assumption.

1. The first possibility is to assume that at the start of each group of K channel blocks u bits are removed from the buffer and encoded into a codeword. This codeword is then transmitted over the group of K channel blocks. Let G_1, G_2, \dots, G_K denote the sequence of channel states during the K channel blocks. Due to causality considerations as in Sect. 3.2, the code rate must be chosen based only on the realization of the first channel state, G_1 . During the next $K - 1$ channel blocks, the code rate is fixed and the transmitter can only adjust the transmission power depending on the channel states G_2, \dots, G_K . Consider a discrete time buffer model as in Sect. 4.1, but now let the time samples correspond to groups of K channel blocks. For a given control choice of u bits, the power required to transmit these bits reliably is a random variable that depends on the realization of G_2, \dots, G_K . Let $P(g, u)$ be the minimum expected power required for the average mutual information rate over the next K channel blocks to be uW/NK bits per sec given that $G_1 = g$. This can be found by solving a finite horizon dynamic programming problem as in Sect. 3.2.2. Defined in this way $P(g, u)$ is not exactly a good power function, but it is strictly convex and increasing in u for all $g \neq 0$. As noted in Sect. 4.2.2, this is enough for most of the following results to apply.
2. The second possibility is that during each channel block U_n bits are encoded into a portion of a codeword; K channel blocks later, these bits are decoded at the receiver. For example if a convolutional code is used, then K could represent the number of channel blocks necessary for sufficient number of constraint lengths to be received. In this case, we still assume that the transmission power used

during a block is the minimum power so that the mutual information rate during that block is uW/N , *i.e.* this is still given by $P(g, u)$ as in (4.3). Thus the power used during a block depends only on the number of bits encoded during the block and the channel gain during the block. In this case, the only necessary modification to the discussion in Sect. 4.1 is that after leaving the buffer, the delay until a bit is decoded will now be $KN/W + D_P$ instead of $N/W + D_P$.

4.3 Fixed Number of Codewords

In this section we look at a different model of the situation in Fig. 4-1. For the models in Sect.'s 4.1 and 4.2, each codeword takes a fixed number of channel uses to transmit. The code rate is varied by varying the number of codewords used over this fixed length of time. In this section, we look at a model where one of a fixed number of codewords is chosen, but the length of time to transmit each codeword is variable. This can be thought of as a simple model of a system using a hybrid ARQ protocol as discussed in Sect. 2.2. In such a system the length of time to transmit a codeword depends on the number of re-transmission requests, where re-transmissions contain additional redundancy. We are again interested in minimizing both the long term average power and either probability of buffer overflow or average delay.

We still consider a discrete time model for the buffer in Fig. 4-1, where each time slot corresponds to N channel uses of a block fading channel. In this case we do not need to assume that $N \gg 1$, indeed we may assume that $N = 1$. In the following, a model for the situation in Fig. 4-1 is developed in which the buffer and encoder/decoder are replaced by a second buffer, whose occupancy corresponds to the reliability required by the data.

Assume that data arrives in fixed size packets of $\log M$ bits.⁸ As above we denote the number of bits that arrive between time $n - 1$ and n by A_n , where $\{A_n\}$ is still an ergodic Markov chain. From the fixed size packet assumption, at each time n , A_n will

⁸This assumption is made primarily for mathematical convenience; if we allowed an arbitrary number of bits to arrive at each time, we would have to deal with the situation where fewer than $\log M$ bits remained in the buffer.

be a multiple of $\log M$. Once again, we assume that $\{A_n\}$ is independent of the fading process and the additive noise. The arriving packets are placed into a transmission buffer of size \hat{L} packets. Periodically a packet is removed from the buffer and encoded into one of M codewords of infinite length. The transmitter then begins sending this codeword over the block fading channel. While transmitting the codeword, the transmitter can adjust the transmission energy by scaling the input symbol by an adjustable gain. Once the receiver can decode the message with acceptable probability of error, the transmitter stops transmitting the current codeword. The transmitter then proceeds to encode and transmit the next packet in the buffer.

The length of time to transmit a codeword is a random variable that depends on the channel gains and the transmission power used. This will be modeled using ideas from [TG95]⁹. Specifically, assume a random coding ensemble in which codewords are chosen from a Gaussian ensemble. Each input symbol is chosen independently from a $\mathcal{CN}(0, 1)$ distribution. We allow the transmitter to adjust the transmission energy at the start of each block. Let $\sqrt{P_i}$ be the gain used during the i th block, so that the transmitted signal for each channel use during the i th block appears to be chosen from a $\mathcal{CN}(0, P_i)$ distribution. As in the previous section, assume that the receiver knows the current gain, $\sqrt{P_n}$, used by the transmitter. As in Sect. 4.2.1, if a codeword is decoded after K blocks, there is the following random coding bound on the ensemble probability of error, for any $\rho \in (0, 1]$:

$$P_e \leq \exp \left(\rho \ln M - N \sum_{i=1}^K E_o(\rho, |g_i|^2 P_i) \right), \quad (4.18)$$

where

$$E_o(\rho, |g_i|^2 P_i) = \rho \ln \left(1 + \frac{|g_i|^2 P_i}{N_0 W (1 + \rho)} \right) \quad (4.19)$$

and $\{g_i\}$ is the sequence of channel gains. Suppose there is a maximal allowable average error probability of η . This error probability is achieved if the codeword is

⁹In [TG95] these ideas were used to model a multi-access communication situation.

decoded after K blocks where

$$N \sum_{i=1}^K E_o(\rho, |g_i|^2 P_i) \geq \rho \ln M - \ln \eta \quad (4.20)$$

for some fixed¹⁰ $\rho \in (0, 1]$. Thus once (4.20) is satisfied, the transmitter can stop transmitting the current codeword. Since the transmitter has perfect CSI, it will know when this occurs. Without perfect CSI, some form of feedback from the receiver is needed to notify the transmitter when to stop transmitting. As in [TG95], $(\rho \ln M - \ln \eta)$ can be considered the demand of a codeword once it enters the encoder and $NE_o(\rho, |g_i|^2 P_i)$ as the service given to that codeword in the i th time step.

Let \tilde{S}_n be $(\rho \ln M - \ln \eta)$ times the number of packets in the buffer at time n plus the remaining amount of “service” required by the current codeword. We make the approximation that when a codeword receives its service, the next codeword immediately begins service. Practically, one would wait to begin transmitting the next codeword until the next channel use. If the typical service time of a codeword is many channel uses this effect will be small. With this approximation and assuming that $\hat{L} \rightarrow \infty$, the process $\{\tilde{S}_n\}$ evolves according to¹¹:

$$\tilde{S}_{n+1} = \max\{\tilde{S}_n + \tilde{A}_{n+1} - \tilde{U}_n, \tilde{A}_{n+1}\} \quad (4.21)$$

where $\tilde{A}_n = (\rho \ln M - \ln \eta) \frac{A_n}{\log M}$ and $\tilde{U}_n = NE_o(\rho, |G_n|^2 P_n)$. We think of (4.21) as the dynamics of a new discrete time buffer with arrival process $\{\tilde{A}_n\}$ and departure process $\{\tilde{U}_n\}$. As stated above, the contents of this buffer corresponds to the amount of reliability or error exponent required by the packets in the original buffer plus the remaining reliability required by the codeword being transmitted. Note at any time n , $\lceil \frac{\tilde{S}_n}{(\rho \ln M - \ln \eta)} \rceil$ is the number of packets in the original buffer plus the current packet that is being transmitted. Thus the contents of the original buffer at any time n will be $\lceil \frac{\tilde{S}_n}{(\rho \ln M - \ln \eta)} \rceil - 1$.

¹⁰One would naturally like to some how choose the “optimum” $\rho \in (0, 1]$. For the Markov decision problem in the next section, this corresponds to the ρ which yields the minimum weighted combination of average delay and average power. Note varying ρ changes both the arrival process and the amount of energy needed; Such an optimization appears to be difficult to do analytically.

¹¹As in the previous section the max. in (4.21) is not really necessary

In the above case, it was assumed that $\hat{L} \rightarrow \infty$. When \hat{L} is finite, let $\tilde{L} = (\rho \ln M - \ln \eta)(\hat{L} + 1)$, and change (4.21) to

$$\tilde{S}_{n+1} = \min\{\max\{\tilde{S}_n + \tilde{A}_{n+1} - \tilde{U}_n, \tilde{A}_{n+1}\}, \tilde{L}\}. \quad (4.22)$$

This can again be considered the dynamics of a new discrete time buffer with arrival process $\{\tilde{A}_n\}$ and departure process $\{\tilde{U}_n\}$. In comparing this to the original buffer a further approximation is necessary. Specifically, when a overflow occurs in (4.22), only a portion of the arriving packet will be lost, while in the original system the entire packet would be lost. Thus $\lceil \frac{\tilde{S}_n}{(\rho \ln M - \ln \eta)} \rceil - 1$ will only be an upper bound on the buffer contents in the original system.

Again, we assume that at each time n , the transmitter can choose \tilde{U}_n based on the current channel state, G_n , buffer state, \tilde{S}_n , and source state \tilde{A}_n . Since $\tilde{U}_n = NE_o(\rho, |G_n|^2 P_n)$, a given choice of $\tilde{U}_n = u$ when $G_n = g$ requires $P_n = \frac{N_0 W(1+\rho)}{|g|^2} \left(e^{\frac{u}{N\rho}} - 1 \right)$. Motivated by this, define $P(g, u)$ to be:

$$P(g, u) = \frac{N_0 W(1+\rho)}{|g|^2} \left(e^{\frac{u}{N\rho}} - 1 \right). \quad (4.23)$$

As in the previous section this can be interpreted as the power cost for transmitting at rate u/N during a channel block when the fading state is g . Note $P(g, u)$ is a good power function as defined in Sect. 4.2.

In the above context we are interested in minimizing the long term average power as in 4.4 while either avoiding buffer overflows or keeping average delay small. As in Sect. 4.1, these problems can be formulated in a Markov decision framework with state space $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$, only now \mathcal{S} is the state space of the new buffer process, $\{\tilde{S}_n\}$. We again introduce an additional buffer cost given by either $b(s)$ or $b(s, a, u)$. This is given by (4.5), (4.6) or (4.7) for the probability of overflow case, or by (4.10) for the average delay case. The average power in (4.4) for a given policy, corresponds to the average power needed for the decoding rule discussed above. One could also look at other decoding rules as in [TG95], in particular the error and erasure decoding rule is sensible for the hybrid ARQ model.

4.4 Other possibilities

In this section we briefly mention some other possible variations of the situation in Fig. 4-1. We have a buffer cost term, $b(\cdot)$ corresponding to either average delay or probability of buffer overflow. This term could be modified to correspond to some other general quality of service measure. We have only considered the case of perfect transmitter CSI. Models with imperfect CSI could be considered. In these cases actual transmission rate could be modeled as a random variable depending on the channel realization. One way to model this is by taking a broadcast approach as discussed in Ch. 3. Finally the models in the preceding section could be combined in various ways. For example, the transmitter could vary both the number of codewords and the length of time over which they are transmitted.

4.5 Summary

To summarize, we have looked at several different models of the situation in Fig. 4-1. In each case we are interested in minimizing the long term average power while also reducing a buffer cost, which corresponds to either average delay or probability of buffer overflow. In each case we also assumed that there was “system constraint” which either limited the number of channel uses per codeword or limited the number of codewords. These problems were all viewed in a common Markov decision setting. In the next two chapters, we will analyze such problems.

CHAPTER 5

Optimal Power/Delay Policies

In the previous chapter several models of the situation in Fig. 4-1 were formulated. Each of these models was centered around a buffer control problem with many common characteristics. In this chapter we begin to analyze such problems. The analysis in this chapter will apply to any of the situations described in Ch. 4 that fit into the “generic” framework described below.

Let $\{S_n\}$ be the state of a discrete time buffer with dynamics

$$S_{n+1} = \min\{\max\{S_n + A_{n+1} - U_n, A_{n+1}\}, L\}, \quad (5.1)$$

where L may be infinite. This buffer is controlled by varying the transmission rate U_n based on the state $(S_n, G_n, A_n) \in \mathcal{S} \times \mathcal{G} \times \mathcal{A}$. The sequences $\{G_n\}$ and $\{A_n\}$ are independent and both are stationary ergodic Markov chains with uncontrollable dynamics. At each time n , a power cost of $P(G_n, U_n)$ is incurred for a given control choice U_n , where $P(\cdot, \cdot)$ is a good power function as in Sect. 4.2.1. Additionally at each time n a buffer cost $b(S_n)$ is incurred. In this chapter we assume that $b(s)$ depends only on s and is an increasing, convex function of s . Note this holds for $b(s)$ given by (4.7) or (4.10) but not by (4.5) or (4.6). In the following sections we place additional restrictions on this model as needed.

Recall that a control policy is a sequence of functions, $\{\mu_n\}$ with $\mu_n : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \mapsto \mathcal{U}$, where \mathcal{U} is the set of allowable control actions and $U_n = \mu_n(S_n, G_n, A_n)$ for all

times n . For a given control policy, the time average power used is given by¹

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \mathbb{E} \left(\sum_{n=1}^M P(G_n, \mu_n(S_n, G_n, A_n)) \right). \quad (5.2)$$

Likewise, the time average buffer cost is given by:

$$\limsup_{M \rightarrow \infty} \frac{1}{M} \mathbb{E} \left(\sum_{n=1}^M b(S_n) \right). \quad (5.3)$$

We are interested in minimizing each of these quantities. In general there is a trade-off between these two objectives, *i.e.*, both can not be minimized at the same time (except in the degenerate case where the arrival rate and channel state are fixed for all time). To understand this trade-off, we consider a weighted combination of these two criteria. Specifically for $\beta > 0$ we seek to find the policy $\{\mu_n\}$ which minimizes:

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \left(\sum_{n=1}^m \mathbb{E}(P(G_n, \mu_n(S_n, G_n, A_n)) + \beta b(S_n)) \right). \quad (5.4)$$

The problem of minimizing (5.4) over all policies $\{\mu_i\}$ is a Markov decision problem. Specifically it is an infinite horizon, average cost per stage problem or more simply an average cost problem. A policy which minimizes the above cost is referred to as an optimal policy. Such problems can be solved via dynamic programming techniques. For a given β , the solution to this problem gives the minimum weighted sum of the average power and average buffer cost. The constant β can be interpreted as a Lagrange multiplier associated with a constraint on the average buffer cost.

To begin, in the next section we give some background on Markov decision theory and dynamic programming. In Sect. 5.2 we show some structural characteristics of both the relative gain and the optimal policies for the above average cost problem. In both Sect. 5.1 and 5.2 we assume that the underlying state space is finite. In Sect. 5.3 we look at how the solution to (5.4) changes as β is varied.

¹The expectation in (5.2) and (5.3) is taken with respect to the joint distribution of the random variables involved. In general this value can depend on the initial state (S_0, G_0, A_0) ; as discussed in the next section, for the problem of interest this is not the case.

5.1 Some Markov Decision Theory

In this section we review some results of Markov decision theory which are used in the following. These are well known results; the reader is referred to [Ber95] or [Gal96] for more discussion and proofs. In this section and the next we restrict ourselves to problems with finite state and control spaces. Given the finite state space assumption, without loss of generality we can assume that $\mathcal{A} \subset \{0, \dots, L\}$, $\mathcal{S} = \{A_{min}, \dots, L\}$ and $\mathcal{U} = \{0, \dots, L\}$ where $A_{min} = \inf \mathcal{A}$. Also, we assume that $\mathcal{G} \subset \mathbb{C}$ with $|\mathcal{G}| < \infty$; thus we are restricting ourselves to models with narrow band fading.² By assuming finite state spaces, the mathematics are greatly simplified. For some of the models discussed in the previous chapter this is a natural assumption, while in other cases it is clearly an approximation. Let $Q_{a,a'} = \Pr(A_{n+1} = a' | A_n = a)$ and $Q_{g,g'} = \Pr(G_{n+1} = g' | G_n = g)$.

If a policy $\{\mu_i\}$ has the form $\mu_i = \mu$ for all i , it is called a stationary policy. We refer to such a stationary policy simply as the policy μ . For an average cost problem with a finite state and control space it is known that there always exists a stationary policy which is optimal. For the given average cost problem, we can consider the related α -discounted problem, where $\alpha \in (0, 1)$. In the α -discounted problem the future cost is discounted. In this case, one seeks to find the policy which minimizes:³

$$\lim_{m \rightarrow \infty} \mathbb{E} \left(\sum_{n=0}^m \alpha^n (P(G_n, \mu(S_n, G_n, A_n)) + \beta b(S_n)) \right). \quad (5.5)$$

A *Blackwell optimal* policy is a stationary policy μ which is optimal for all α -discounted problems with $\alpha \in (\bar{\alpha}, 1)$ where $0 < \bar{\alpha} < 1$. A Blackwell optimal policy exists for any finite state Markov decision problem, and a Blackwell optimal policy is also an optimal policy for the average cost problem.

All states in a Markov decision problem are said to *communicate* if for any two states (s, g, a) and (s', g', a') , there exists a stationary policy μ such that (s, g, a) will eventually be reached from (s', g', a') under this policy. From the assumption that

²Some of the following results clearly do not rely on this assumption and hold in other cases as well.

³In the finite state space case, the per stage cost $P(g, u) + \beta b(s)$ is bounded, and thus the limit in (5.5) exists.

$\{G_n\}$ and $\{A_n\}$ are independent and ergodic Markov chains, it follows that all states communicate in the Markov decision problem under consideration. Several results hold for finite state average cost problems with this property. First, the optimal cost in (5.4) does not depend on the initial state. We denote this optimal cost by J^* . Second, for every Blackwell optimal policy μ^* , there is a function $w : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \mapsto \mathbb{R}$ such that for all $(s, g, a) \in \mathcal{S} \times \mathcal{G} \times \mathcal{A}$,

$$\begin{aligned} & J^* + w(s, g, a) \\ &= P(g, \mu^*(s, g, a)) + \beta b(s) + \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a, a'} Q_{g, g'} w(f(s - \mu^*(s, g, a), a'), g', a') \\ &= \inf_{u \in \mathcal{U}} \left(P(g, u) + \beta b(s) + \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a a'} Q_{g g'} w(f(s - u, a'), g', a') \right). \end{aligned} \quad (5.6)$$

We have denoted $\min\{\max\{x + a, L\}, L\}$ by $f(x, a)$ so that the buffer dynamics are $S_{n+1} = f(S_n - U_n, A_{n+1})$. Equation (5.6) is Bellman's equation for this problem and w is called a relative gain vector. Finally, for finite state problems in which all states communicate, there always exists a solution to Bellman's equation which is a *unichain policy*, that is under such a policy the resulting state process $\{(S_n, G_n, A_n)\}$ is a Markov chain with a single recurrent class plus possibly one or more transient classes.

In the α -discounted problem (5.5), the optimal cost depends on the initial state. Let $J_\alpha^*(s, g, a)$ denote this cost when $S_0 = s$, $G_0 = g$, and $A_0 = a$. It can be shown that a stationary policy is optimal for the α -discounted problem if and only if it satisfies:

$$\begin{aligned} & J_\alpha^*(s, g, a) \\ &= P(g, \mu(s, g, a)) + \beta b(s) + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a, a'} Q_{g, g'} J_\alpha^*(f(s - \mu(s, g, a), a'), g', a') \\ &= \inf_{u \in \mathcal{U}} \left(P(g, u) + \beta b(s) + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a, a'} Q_{g, g'} J_\alpha^*(f(s - u, a'), g', a') \right). \end{aligned} \quad (5.7)$$

This is Bellman's equation for the discounted problem. The dynamic programming operator T is defined to be a map from functions on the state space to itself, *i.e.*,

$T : \mathbb{R}^{\mathcal{S} \times \mathcal{G} \times \mathcal{A}} \mapsto \mathbb{R}^{\mathcal{S} \times \mathcal{G} \times \mathcal{A}}$. For any such function J , TJ is defined to be the right hand side (5.7) with J_α^* replaced by J . Let T^k be the composition of k copies of T . Then, for any bounded function J defined on the state space it can be shown that:

$$\lim_{k \rightarrow \infty} T^k J = J_\alpha^*. \quad (5.8)$$

Thus the value of the above limit does not depend on which function J is initially chosen. Furthermore J_α^* is the unique bounded fixed point J of the equation $J = TJ$. Starting with an arbitrary function J and successively calculating TJ, T^2J, \dots is referred to as value iteration. This provides a method for calculating the solution to (5.7).

Finally we note that if all states communicate then the optimal cost of the discounted problem is related to the optimal cost of the average cost problem by:

$$J^* = \lim_{\alpha \rightarrow 1} (1 - \alpha) J_\alpha^*(s, g, a) \quad \forall (s, g, a) \in \mathcal{S} \times \mathcal{G} \times \mathcal{A}. \quad (5.9)$$

Furthermore for any fixed state (s^*, g^*, a^*) let

$$w(s, g, a) = \lim_{\alpha \rightarrow 1} (J_\alpha^*(s, g, a) - J_\alpha^*(s^*, g^*, a^*)). \quad (5.10)$$

Then w is a relative gain vector, *i.e.*, it satisfies Bellman's equation (5.6) for the average cost problem.

5.2 Structural results⁴

Our goal in this section is to prove several structural characteristics of both the optimal cost and optimal policy for the Markov decision problem described above. In this section we still assume that the state and control spaces are finite. For some of these results we will need to make some additional assumptions on the problem. First we discuss a required set of assumptions we will make about the channel process $\{G_n\}$ and the arrival process $\{A_n\}$; these rely on the notion of a stochastic order

⁴This section can be skipped without loss of continuity.

relation. Next we prove several characteristics about the optimal cost J_α^* of the discounted problem as well as the relative gain vector of the average cost problem. Some structural characteristics of the optimal policies are then developed.

5.2.1 Stochastic Order Relations

We are assuming that both the channel process $\{G_n\}$ and the arrival process $\{A_n\}$ are independent, stationary ergodic Markov chains, *i.e.*, each consists of a single class of states which is both recurrent and aperiodic. In order to prove some of the results in the following section, we place an additional restriction on these processes; we require that they be *stochastically monotone*. To define this property we first give a definition of a stochastic ordering of two random variables- a more detailed discussion of these ideas can be found in [Ros95].

Consider two real valued random variables X, Y defined on a common probability space. Then X is said to be *stochastically larger* than Y if $\Pr(X > a) \geq \Pr(Y > a)$ for all $a \in \mathbb{R}$. It can easily be shown that X is stochastically larger than Y if and only if $\mathbb{E}(f(X)) \geq \mathbb{E}(f(Y))$ for all non-decreasing functions $f : \mathbb{R} \mapsto \mathbb{R}$ [Ros95]. In particular, note if $f(x) = x$ and X is stochastically larger than Y , then $\mathbb{E}X \geq \mathbb{E}Y$. Likewise, it can be shown that X is stochastically larger than Y if and only if $\mathbb{E}(g(Y)) \geq \mathbb{E}(g(X))$ for all $g : \mathbb{R} \mapsto \mathbb{R}$ which are non-increasing.

A real valued Markov chain $\{X_n\}$ is defined to be *stochastically monotone* if $\Pr(X_n > a | X_{n-1} = x)$ is a non-decreasing function of x for all n and all a . In other words, X_n conditioned on $X_{n-1} = x$ is stochastically larger than X_n conditioned on $X_{n-1} = y$, for $x > y$. Thus if $\{X_n\}$ is stochastically monotone and f is non-decreasing, then $\mathbb{E}(f(X_n) | X_{n-1} = x)$ is a non-decreasing function of x .

At some places in the following we will assume that either the arrival process or the channel process is stochastically monotone, where we define a channel process $\{G_n\}$ to be stochastically monotone if $\{|G_n|\}$ is. Intuitively this means that the probability of a deep fade is no higher, given that the channel is good, than when it is already bad. Most realistic channel models satisfy this condition. Note that a memoryless process is always stochastically monotone. Thus a special case of the following is when the fading and arrival rate are memoryless.

5.2.2 Monotonicity of J_α^*

In this section we will prove several lemmas describing the structure of the optimal cost function, J_α^* , of the α -discounted problem. Specifically we will show that with certain assumptions $J_\alpha^*(s, g, a)$ is monotonic in each variable when the other variables are fixed. Specifically we show that $J_\alpha^*(s, g, a)$ is non-decreasing in s , and a and non-increasing in $|g|$. These results correspond to the intuition that it is less desirable for the buffer occupancy to be large, the channel to be in a deep fade or the arrival rate to be large. Note from (5.10) that the relative gain $w(s, g, a)$ of the average cost problem will also have these characteristics.

First we show monotonicity in the buffer occupancy.

Lemma 5.2.1 $J_\alpha^*(s, g, a)$ is non-decreasing in s for all $g \in \mathcal{G}$ and all $a \in \mathcal{A}$.

PROOF. We will prove this lemma by an induction argument. Specifically consider a bounded function J_0 defined on the state space, which is non-decreasing in s for all $g \in \mathcal{G}$ and $a \in \mathcal{A}$. For $k = 1, 2, \dots$ let $J_k(s, g, a)$ be the result of the k -th stage of the value iteration algorithm for the α -discounted problem starting with J_0 , i.e. $J_k = T^k J_0 = T J_{k-1}$. Thus

$$J_k(s, h, a) = \min_u \left(P(g, u) + \beta b(s) + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a, a'} Q_{g, g'} J_{k-1}(f(s - u, a'), g', a') \right) \quad (5.11)$$

From (5.8), it follows that $J_\alpha^* = \lim_{k \rightarrow \infty} J_k$. We show that if $J_{k-1}(s, g, a)$ is non-decreasing in s for all g and a , then the same must be true of $J_k(s, g, a)$. Thus by induction, $J_\alpha^* = \lim_{k \rightarrow \infty} J_k$ must satisfy the lemma. Let u be a control which achieves the minimum in (5.11) for a given state (s, g, a) , with $s \neq 0$. Now consider applying this same control in state $(s - 1, g, a)$. Then $P(g, u)$ will be the same. By assumption $b(s - 1) \leq b(s)$, and by the induction hypothesis the last term in (5.11) will be less than or equal to the corresponding cost in state (s, g, a) . Since $J_k(s - 1, g, a)$ is the minimum over all choices of u , then we must have $J_k(s - 1, g, a) \leq J_k(s, g, a)$ as desired. ■

Lemma 5.2.2 *If the fading process, $\{G_n\}$ is stochastically monotone, then $J_\alpha^*(s, g, a)$ is non-increasing in $|g|$. For those states where the optimal control is to transmit at some non-zero rate, J_α^* is strictly decreasing in $|g|$.*

PROOF. Again let $J_k = T^k J_0$. Assume that $J_{k-1}(s, g, a)$ is non-increasing in $|g|$ for all s and a . Since $\{G_n\}$ is stochastically monotone, it follows that

$$\sum_{g' \in \mathcal{G}} Q_{g, g'} J_{k-1}(f(s - u, a'), g', a')$$

is non-decreasing in $|g|$ for all s, u, a' . Using this it follows, as in lemma 5.2.1 that $J_k = T J_{k-1}$ is non-increasing in $|g|$ and thus by induction J_α^* will also be.

To prove the second statement, note that the optimal cost must satisfy Bellman's equation, *i.e.*, for all (s, g, a) ,

$$J_\alpha^*(s, g, a) = \min_u \left(P(g, u) + \beta b(s) + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{a, a'} Q_{g, g'} J_\alpha^*(f(s - u, a'), g', a') \right). \quad (5.12)$$

Only the first and third terms on the right hand side depend on g and each of these terms is non-increasing in $|g|$. Furthermore if $u > 0$, then $P(g, u)$ will be strictly decreasing in $|g|$. Now following a similar argument to the proof of lemma 5.2.1, consider two states: (s, g, a) and (s, \tilde{g}, a) with $|g| > |\tilde{g}|$. Assume that the optimal control for state (s, \tilde{g}, a) is $u > 0$. Consider applying this same control in (s, g, a) . Then this results in a value on the right hand side of Bellman's equation which is strictly less than $J_\alpha^*(s, \tilde{g}, a)$ and thus we have the desired result. ■

Lemma 5.2.3 *If the arrival process, $\{A_n\}$ is stochastically monotone, then $J_\alpha^*(s, g, a)$ is non-decreasing in a for all $s \in \mathcal{S}$ and all $g \in \mathcal{G}$.*

PROOF. Again we prove this by induction using value iteration. Let $J_k = T^k J_0$ and assume that $J_{k-1}(s, g, a)$ is non-decreasing in a . From this assumption and lemma 5.2.1 it follows that for a given u, g' and s , $J_{k-1}(f(s - u, a'), g', a')$ is a non-decreasing

function of a' . Thus, since $\{A_n\}$ is stochastically monotone,

$$\sum_{a' \in \mathcal{A}} Q_{a, a'} J_{k-1}(f(s - u, a'), g', a')$$

is non-decreasing in a . The remainder of the proof follows the same line of reasoning as in the previous two cases. ■

5.2.3 Structure of Optimal policies

We will now prove some structural characteristics of the optimal policy. Showing these results requires some additional restrictions. First we show that when the arrival process is memoryless, the optimal policy is always non-decreasing in the channel gain; in other words under the optimal policy one always transmits at the highest rate when the channel is best. Next we show that when no overflows occur, the optimal policy is non-decreasing in the buffer state.

Monotonicity in $|g|$ with memoryless fading.

When the fading is memoryless, then we have $Q_{g, g'} = \pi_G(g')$ for all g and g'

Lemma 5.2.4 *If $\{G_n\}$ is memoryless, then every Blackwell optimal policy $\mu^*(s, g, a)$ is non-decreasing in $|g|$.*

PROOF. Let $\mu^*(s, g, a)$ be a Blackwell optimal policy. We know that $\mu^*(s, g, a)$ must satisfy Bellman's equation for the α -discounted problem when α is close enough to 1, *i.e.*, we have for all (s, g, a) :

$$\begin{aligned} J_\alpha^*(s, g, a) &= (TJ_\alpha^*)(s, g, a) = P(g, \mu^*(s, g, a)) + \beta b(s) + \\ &\quad \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} \pi_G(g') Q_{a, a'} J_\alpha^*(f(s - \mu^*(s, g, a), a'), g', a') \end{aligned}$$

To simplify notation, define

$$\hat{J}_\alpha(s, u) \triangleq \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} \pi_G(g') Q_{a, a'} J_\alpha^*(f(s - u, a'), g', a').$$

Note that, by the memoryless assumption, $\hat{J}_\alpha(s, u)$ does not depend on g .

To establish a contradiction, assume that there exist two states (s, g, a) and (s, g', a) with $|g| < |g'|$ such that $\mu^*(s, g, a) > \mu^*(s, g', a)$. Consider using $\mu^*(s, g, a)$ in state (s, g', a) , then from Bellman's equation (5.7) we have:

$$P(g, \mu^*(s, g, a)) + \hat{J}_\alpha(s, \mu^*(s, g, a)) \leq P(g, \mu^*(s, g', a)) + \hat{J}_\alpha(s, \mu^*(s, g', a)).$$

Similarly, we have

$$P(g', \mu^*(s, g', a)) + \hat{J}_\alpha(s, \mu^*(s, g', a)) \leq P(g', \mu^*(s, g, a)) + \hat{J}_\alpha(s, \mu^*(s, g, a))$$

These inequalities imply:

$$P(g, \mu^*(s, g, a)) - P(g, \mu^*(s, g', a)) \leq P(g', \mu^*(s, g, a)) - P(g', \mu^*(s, g', a))$$

Recall that since P is a good power function, $P(g, u) = \frac{1}{|g|^2}A(u)$ where $A(u)$ is an increasing function of u . The above inequality can then be written as:

$$\frac{1}{|g|^2} (A(\mu^*(s, g, a)) - A(\mu^*(s, g', a))) \leq \frac{1}{|g'|^2} (A(\mu^*(s, g, a)) - A(\mu^*(s, g', a)))$$

Since $|g| < |g'|$, and $\mu^*(s, g, a) > \mu^*(s, g', a)$ this inequality can only be true if $A(u)$ is decreasing, which gives us the desired contradiction. ■

Note that if the fading had memory, then the term \hat{J} defined above would depend on g and the above arguments would not apply.

Monotonicity in s with no overflows.

In this section we consider the case where buffer overflows do not occur. For example assume that $|g| > 0$ for all $g \in \mathcal{G}$, $a \ll L$ for all $a \in \mathcal{A}$ and $b(s)$ is given by (4.7). Then by choosing β large enough, the optimal policy will never allow the buffer to reach state L . In this case we show that the optimal policy is always non-decreasing in the buffer state s . The reason for assuming no overflows is to avoid having to deal with the edge effects which occur when the buffer overflows. We first show a convexity property of the optimal α -discounted cost, $J_\alpha^*(s, g, a)$ in s .

A function $f : \mathcal{S} \mapsto \mathbb{R}$ is defined to be *convex* if for all $s \in \{1, \dots, L-1\}$,

$$f(s+1) + f(s-1) \geq 2f(s). \quad (5.13)$$

Note that if $f : \mathbb{R} \mapsto \mathbb{R}$ is convex, then its restriction to \mathcal{S} will also be. The following lemma provides alternative characterizations of convexity.

Lemma 5.2.5 *Let $f : \mathcal{S} \mapsto \mathbb{R}$ be a function defined on $\mathcal{S} = \{0, \dots, L\}$; then the following are equivalent:*

- I.** f is convex.
- II.** $f(s+1) - f(s)$ is non-decreasing in s .
- III.** $f(s) + f(t) \geq f(\lceil \frac{s+t}{2} \rceil) + f(\lfloor \frac{s+t}{2} \rfloor)$ for all $s, t \in \mathcal{S}$.

PROOF. That **III** implies **I** is straightforward. We first show that **I** implies **II**. Then we show that **II** implies **III**.

Assume that **I**, is true. From (5.13) it follows that

$$f(s+1) - f(s) \geq f(s) - f(s-1)$$

for all $s \in \{1, \dots, L-1\}$. Iterating this, **II** follows.

Assume that **II** is true, we will show that **III** must be true by an induction argument. First note that **III** is true with equality for all $s, t \in \mathcal{S}$ with $|s-t| \leq 1$. Assume that **III** is true for all $s, t \in \mathcal{S}$ with $|s-t| \leq d$ for some $d \in \{1, \dots, L\}$, we then show that **III** must be true for any s, t with $|s-t| = d+1$.

Assume $\tilde{s}, \tilde{t} \in \mathcal{S}$ with $\tilde{s} = \tilde{t} + (d+1)$. From **II**,

$$f(\tilde{s}) - f(\tilde{s}-1) \geq f(\tilde{t}+1) - f(\tilde{t})$$

and thus

$$f(\tilde{s}) + f(\tilde{t}) \geq f(\tilde{t}+1) + f(\tilde{s}-1)$$

Note $|\tilde{s} - 1 - (\tilde{t} + 1)| = d - 1$ and thus by the induction hypothesis,

$$f(\tilde{t} + 1) + f(\tilde{s} - 1) \geq f\left(\left\lceil \frac{s+t}{2} \right\rceil\right) + f\left(\left\lfloor \frac{s+t}{2} \right\rfloor\right).$$

Therefore combining the previous two inequalities we have that **III** holds for all s, t such that $|s - t| \leq d + 1$. ■

Lemma 5.2.6 *If no overflows occur, then for all $g \in \mathcal{G}$, $a \in \mathcal{A}$, $J_\alpha^*(s, g, a)$ is convex in s .*

PROOF. Let $J_k = T^k J_0$ and where $J_0(s, g, a)$ is any function which is convex in s for all g, a . Once again we prove the lemma by induction. Assume that J_{k-1} has the desired convexity property; we show that J_k also has this property.

For any $s \in \{1, \dots, L-1\}$, let u_{s+1} be the optimal control choice for $J_k(s+1, g, a)$, *i.e.*,

$$\begin{aligned} J_k(s+1, g, a) = & P(g, u_{s+1}) + \beta b(s+1) \\ & + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{g, g'} Q_{a, a'} J_{k-1}(f(s+1 - u_{s+1}, a'), g', a'). \end{aligned}$$

Likewise let u_{s-1} be the optimal control choice for $J_k(s-1, g, a)$. Then

$$\begin{aligned}
& J_k(s+1, g, a) + J_k(s-1, g, a) \\
&= P(g, u_{s+1}) + P(g, u_{s-1}) + \beta b(s+1) + \beta b(s-1) \\
&\quad + \alpha \sum_{g \in \mathcal{G}, a \in \mathcal{A}} Q_{g, g'} Q_{a, a'} (J_{k-1}(f(s+1 - u_{s+1}, a'), g', a') + J_{k-1}(f(s-1 - u_{s-1}, a'), g', a')) \\
&\geq P(g, \lfloor \frac{u_{s+1} + u_{s-1}}{2} \rfloor) + P(g, \lceil \frac{u_{s+1} + u_{s-1}}{2} \rceil) + 2\beta b(s) \\
&\quad + \sum_{g \in \mathcal{G}, a \in \mathcal{A}} Q_{g, g'} Q_{a, a'} \left(J_{k-1} \left(\left\lfloor \frac{f(s+1 - u_{s+1}, a') + f(s-1 - u_{s-1}, a')}{2} \right\rfloor, g', a' \right) \right. \\
&\quad \left. + J_{k-1} \left(\left\lceil \frac{f(s+1 - u_{s+1}, a') + f(s-1 - u_{s-1}, a')}{2} \right\rceil, g', a' \right) \right)
\end{aligned}$$

where the last inequality follows from the convexity of P , b and J_{k-1} and the previous lemma. Recall that $f(x, a) = \min\{\max\{x + a, L\}, L\}$; assuming that $x \geq 0$ and no overflows occur, $f(x, a) = x + a$. Thus

$$\begin{aligned}
\left\lfloor \frac{f(s+1 - u_{s+1}, a') + f(s-1 - u_{s-1}, a')}{2} \right\rfloor &= s + a' - \left\lceil \frac{u_{s+1} + u_{s-1}}{2} \right\rceil \\
&= f(s - \left\lceil \frac{u_{s+1} + u_{s-1}}{2} \right\rceil, a')
\end{aligned}$$

and similarly

$$\left\lceil \frac{f(s+1 - u_{s+1}, a') + f(s-1 - u_{s-1}, a')}{2} \right\rceil = f(s - \left\lfloor \frac{u_{s+1} + u_{s-1}}{2} \right\rfloor, a')$$

Combining this with the above inequalities we have

$$\begin{aligned}
& J_k(s+1, g, a) + J_k(s-1, g, a) \\
&\geq P(g, \lfloor \frac{u_{s+1} + u_{s-1}}{2} \rfloor) + P(g, \lceil \frac{u_{s+1} + u_{s-1}}{2} \rceil) + 2\beta b(s) \\
&\quad + \sum_{g \in \mathcal{G}, a \in \mathcal{A}} Q_{g, g'} Q_{a, a'} \left(J_{k-1}(f(s - \lfloor \frac{u_{s+1} + u_{s-1}}{2} \rfloor, a'), g', a') \right. \\
&\quad \left. + J_{k-1}(f(s - \lceil \frac{u_{s+1} + u_{s-1}}{2} \rceil, a'), g', a') \right) \\
&\geq 2J_k(s, g, a)
\end{aligned}$$

Thus we have the desired convexity of J_k . ■

Lemma 5.2.7 *Assuming that no overflows occur, there exists a Blackwell optimal policy $\mu^*(s, g, a)$ which is non-decreasing in s for all $a \in \mathcal{A}$ and all $g \in \mathcal{G}$.*

PROOF. Let $\mu^*(s, g, a)$ be a Blackwell optimal policy. We know that $\mu^*(s, g, a)$ must satisfy Bellman's equation for the α -discounted problem when α is close enough to 1, *i.e.*, we have for all (s, g, a) :

$$\begin{aligned} J_\alpha^*(s, g, a) &= (TJ_\alpha^*)(s, g, a) \\ &= P(g, \mu^*(s, g, a)) + \beta b(s) \\ &\quad + \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{g, g'} Q_{a, a'} J_\alpha^*(f(s - \mu^*(s, g, a), a'), g', a') \end{aligned}$$

To establish a contradiction assume the lemma is not true. Then given any Blackwell optimal policy there must exist two states (s, g, a) and $(s + 1, g, a)$ such that $\mu^*(s, g, a) > \mu^*(s + 1, g, a)$. For this value of g and a , let

$$\hat{J}_\alpha(s - u) \triangleq \alpha \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{g, g'} Q_{a, a'} J_\alpha^*(f(s - u, a'), g', a'). \quad (5.14)$$

Consider using $\mu^*(s + 1, g, a)$ in state (s, g, a) , then using Bellman's equation (5.7) for $J^*(s, g, a)$ we have

$$\begin{aligned} &P(g, \mu^*(s, g, a)) + \hat{J}_\alpha(s - \mu^*(s, g, a)) \\ &\leq P(g, \mu^*(s + 1, g, a)) + \hat{J}_\alpha(s - \mu^*(s + 1, g, a)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &P(g, \mu^*(s + 1, g, a)) + \hat{J}_\alpha(s + 1 - \mu^*(s + 1, g, a)) \\ &\leq P(g, \mu^*(s, g, a)) + \hat{J}_\alpha(s + 1 - \mu^*(s, g, a)). \end{aligned}$$

We now argue that one of these inequalities can be assumed to be strict. If not then we can form a new Blackwell optimal policy $\tilde{\mu}^*$ by setting $\tilde{\mu}^*(s + 1, g, a) = \mu^*(s, g, a)$

and setting $\tilde{\mu}^* = \mu^*$ for every other state. If after doing this, there are no states $(\tilde{s}, \tilde{g}, \tilde{a})$ and $(\tilde{s} + 1, \tilde{g}, \tilde{a})$ such that $\mu^*(\tilde{s}, \tilde{g}, \tilde{a}) > \mu^*(\tilde{s} + 1, \tilde{g}, \tilde{a})$, then the lemma is proved. Otherwise, we repeat the above procedure until a pair of states is found in which one of the above inequalities is strict.

Combining these inequalities we get

$$\begin{aligned} & \hat{J}_\alpha(s + 1 - \mu^*(s + 1, g, a)) - \hat{J}_\alpha(s - \mu^*(s + 1, g, a)) \\ & < \hat{J}_\alpha(s + 1 - \mu^*(s, g, a)) + \hat{J}_\alpha(s - \mu^*(s, g, a)). \end{aligned} \quad (5.15)$$

Note from the previous lemma, $J_\alpha^*(s, g, a)$ is convex in s . Therefore $J_\alpha^*(s + 1, g, a) - J_\alpha^*(s, g, a)$ is a non-decreasing function of s . Thus, from (5.14) it follows that $\hat{J}_\alpha(x + 1) - \hat{J}_\alpha(x)$ is non-decreasing in x . From (5.15), it follows that $s - \mu^*(s + 1, g, a) < s - \mu^*(s, g, a)$ but this contradicts the assumption that $\mu^*(s + 1, g, a) < \mu^*(s, g, a)$. ■

Policies with overflows

In this section we look at some characteristics of policies for which overflows occur. Assume that under a given policy, $\Pr(S_{n+1} = L | (S_n, G_n, A_n) = (s, g, a)) = 1$ for some state (s, g, a) . In other words, from state (s, g, a) the buffer will be full in the next time step with probability one. Denote the set of all states (s, g, a) with this property by Ω_{of} . Recall that, in this chapter, we assume that $b(s)$ only depends on s , and thus does not depend on the number of bits lost due to overflow. This implies that the optimal control for any state $(s, g, a) \in \Omega_{of}$ is $\mu^*(s, g, a) = 0$. For an α -discounted problem, there exists an optimal policy with the characteristic that if $(s, g, a) \in \Omega_{of}$ then any state (s', g, a) such that $s \leq s'$ will also be in Ω_{of} . To see this note that since it is optimal to transmit nothing in state (s, g, a) , we have for all controls $u \in \mathcal{U}$:

$$\begin{aligned} & P(g, u) + \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{g, g'} Q_{a, a'} J_\alpha^*(f(s - u, a'), g', a') \\ & \geq P(g, 0) + \sum_{g' \in \mathcal{G}, a' \in \mathcal{A}} Q_{g, g'} Q_{a, a'} J_\alpha^*(L, g', a') \end{aligned} \quad (5.16)$$

From lemma 5.2.1, $J_\alpha^*(f(s - u, a'), g', a')$ is non-decreasing in s and therefore the left-hand side of (5.16) is non-decreasing in s . Thus it is also optimal to transmit

nothing in state (s', g, a) . Similarly if $\{G_n\}$ and $\{A_n\}$ are stochastically monotone and $(s, g, a) \in \Omega_{of}$, then any state (s', g', a') such that $s \leq s'$, $|g| \geq |g'|$, and $a \leq a'$ will also be in Ω_{of} . This follows from lemma 5.2.2 and lemma 5.2.3.

Consider the case where the arrival process is constant, *i.e.*, $A_n = \bar{A}$ for all n . In this case if the buffer overflows with some positive probability then Ω_{of} must be non-empty. Furthermore the only way that the buffer overflows is if it first enters some state in Ω_{of} . Finally consider the limiting case where the buffer size is $\bar{A} + 1$ and the arrival rate is again a constant. In this case the buffer state space can be assumed to be $\{\bar{A}, \bar{A} + 1\}$. Using the above results we can give a fairly complete characterization of the optimal policy in this case. Specifically when the buffer state is \bar{A} , the channel state is partitioned into a set $\mathcal{G}_{\bar{A}}$ and its complement $\mathcal{G}_{\bar{A}}^C$, such that for every $g \in \mathcal{G}_{\bar{A}}$ the optimal policy transmits at rate \bar{A} and every $g \in \mathcal{G}_{\bar{A}}^C$ the optimal policy is to transmit nothing. Similarly for the buffer state $\bar{A} + 1$, the channel states are partitioned into two sets $\mathcal{G}_{\bar{A}+1}$ and $\mathcal{G}_{\bar{A}+1}^C$ where again for every state $g \in \mathcal{G}_{\bar{A}+1}$, the optimal policy is to transmit at rate $\bar{A} + 1$ and for every $g \in \mathcal{G}_{\bar{A}+1}^C$ the optimal policy is to transmit nothing. From the above discussion $\mathcal{G}_{\bar{A}+1} \subset \mathcal{G}_{\bar{A}}$. Additionally if the fading process is stochastically monotone, then there exists some state $g_{\bar{A}} \in \mathcal{G}$ such that $\mathcal{G}_{\bar{A}} = \{g \in \mathcal{G} : |g| > |g_{\bar{A}}|\}$ and similarly there exists a state $g_{\bar{A}+1}$ corresponding to the set $\mathcal{G}_{\bar{A}+1}$. Restricting attention to the set of states in $\mathcal{G}_{\bar{A}}$ note that the power allocation used over these states will correspond to inverting the channel (*cf.* (3.33)).

5.2.4 More general state spaces

In the analysis of this section we only considered problems with finite state spaces. The reason for the finite state assumption was to simplify the arguments; it is not necessary for many of these results to hold. In this section we comment briefly on how these results can be extended to more general state spaces.

First consider the case, where $\mathcal{S} = \{0, 1, 2, \dots\}$ and both \mathcal{G} and \mathcal{A} are finite. Then by using the “approximating sequence” techniques in [Sen99], the above results can be generalized to countable buffer state spaces. The idea here is to consider a sequence of problems with finite buffers of size L , and let $L \rightarrow \infty$. Under suitable assumptions, the relative gains, optimal costs and optimal policy of the finite state space problems will converge to the corresponding quantities for the problem with countable buffer

state space.

For a general average cost problem (without a finite state space), additional assumptions are needed to assure the existence of a stationary policy which is optimal. In [HLL96] some such assumptions are provided for general state spaces. Typically such problems are approached by considering a sequence of α -discounted problems and letting $\alpha \rightarrow 1$. With suitable assumptions, it can be argued that the optimal policy of the discounted problems converge to an optimal policy for the average cost problems. Such assumptions typically involve some type of compactness of the underlying state space and some degree of continuity for the the optimal policy and the optimal discounted cost.

5.3 Optimum power/delay curve

For a given choice of β , the minimum of (5.4) over all policies gives the minimum of the average power plus β times the average buffer cost. Assume that the stationary policy μ^* minimizes (5.4) for a given β . Let \bar{P}^{μ^*} and \bar{b}^{μ^*} be the corresponding average power and average buffer cost, as given in (5.2) and (5.3) respectively. Then \bar{P}^{μ^*} must be the minimum average power over all policies for which the average buffer cost under the policy is less than or equal to \bar{b}^{μ^*} . Let \mathcal{B} be the subset of \mathbb{R}^+ such that for every $B \in \mathcal{B}$ there exists some policy μ (with finite average power) for which the average buffer cost is less than or equal to B (note \mathcal{B} is clearly a convex set). For any $B \in \mathcal{B}$, define $P^*(B)$ to be the minimum average power such that the average buffer cost is less than B . Thus, by the above argument, $P^*(\bar{b}^{\mu^*}) = \bar{P}^{\mu^*}$. Since the buffer cost is typically related to some measure of delay, we refer to $P^*(B)$ as the (*optimum*) *power/delay curve*. In this section we will examine some properties of this curve. First we prove the following proposition about the structure of $P^*(B)$ under the assumption that the buffer state space \mathcal{S} and control space \mathcal{U} are \mathbb{R}^+ . In particular, this means that no overflows occur.

Proposition 5.3.1 *If $\mathcal{S} = \mathcal{U} = \mathbb{R}^+$ then the optimum power/delay curve, $P^*(B)$, is a non-increasing, convex function of $B \in \mathcal{B}$. Except for the degenerate case where channel and arrival processes are both constant, it is a decreasing and strictly convex function of B .*

PROOF. That $P^*(B)$ is non-increasing is obvious. We show that it is convex. Let B^1 and B^2 be two average buffer costs in \mathcal{B} with corresponding values $P^*(B^1)$ and $P^*(B^2)$. We want to show that for any $\lambda \in [0, 1]$,

$$P^*(\lambda B^1 + (1 - \lambda)B^2) \leq \lambda P^*(B^1) + (1 - \lambda)P^*(B^2). \quad (5.17)$$

We will prove this using sample path arguments. Let $\{G_n(\omega)\}_{n=1}^\infty$ and $\{A_n(\omega)\}_{n=1}^\infty$ be a given sample path of channel states and arrival states. Let $\{U_n^1(\omega)\}$ be a sequence of control actions corresponding to the policy which attains $P^*(B^1)$. Let $\{S_n^1(\omega)\}$ be the corresponding sequence of buffer states. Likewise define $\{U_n^2(\omega)\}$ and $\{S_n^2(\omega)\}$ corresponding to $P^*(B^2)$. As noted previously we can assume that $U_n^i(\omega) \leq S_n^i(\omega)$ for $i = 1, 2$, for all ω , and for all n . Now consider the new sequence of control actions, $\{U_n^\lambda(\omega)\}$, where for all n ,

$$U_n^\lambda(\omega) = \lambda U_n^1(\omega) + (1 - \lambda)U_n^2(\omega).$$

Let $\{S_n^\lambda(\omega)\}$ be the sequence of buffer states using this policy. Assume at time $n = 0$, $S_0^\lambda(\omega) = S_0^1(\omega) = S_0^2(\omega) = 0$ for all sample paths, ω . By a simple recursion, it follows⁵ that for all n , $S_n^\lambda(\omega) = \lambda S_n^1(\omega) + (1 - \lambda)S_n^2(\omega)$. Thus,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E} b(S_n^\lambda(\omega)) &= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E} b(\lambda S_n^1(\omega) + (1 - \lambda)S_n^2(\omega)) \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E} (\lambda b(S_n^1(\omega)) + (1 - \lambda)b(S_n^2(\omega))) \\ &= \lambda B^1 + (1 - \lambda)B^2, \end{aligned} \quad (5.18)$$

where the expectation is taken over all sample paths. The inequality in the second line above follows from the assumption that $b(s)$ is convex in s . From the convexity of $P(h, u)$ in u , we have for all n

$$P(G_n(\omega), U_n^\lambda(\omega)) \leq \lambda P(G_n(\omega), U_n^1(\omega)) + (1 - \lambda)P(G_n(\omega), U_n^2(\omega)).$$

⁵Note this is where the assumption that no overflows occur is used.

Again, summing and taking expectations we have

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E}P(G_n(\omega), U_n^\lambda(\omega)) \leq \lambda P^*(B^1) + (1 - \lambda)P^*(B^2). \quad (5.19)$$

Thus we must have $P^*(\lambda B_1 + (1 - \lambda)B_2) \leq \lambda P^*(B_1) + (1 - \lambda)P^*(B_2)$ as desired.

The final statement in the proposition will follow from the results in the next section. ■.

Note that varying β and finding the optimal policy for each value can provide different points on the power/delay curve. It is natural to then ask if all values of $P^*(B)$ can be found in this way, with an appropriate choice of β . One way of viewing this problem is as a *multi-objective optimization* problem [Saw85]. By this we mean an optimization problem with a vector valued objective function $c : X \mapsto \mathbb{R}^n$. In our case c has two components corresponding to the average delay and average power. For such problems, a feasible solution, x is defined to be *Pareto optimal* if there exists no other feasible \hat{x} such that $c(\hat{x}) < c(x)$, where the inequality is to be interpreted component-wise. It can be seen that the points $\{(P^*(B), B) : B \in \mathcal{B}\}$ are a subset of the Pareto optimal solutions for this problem.⁶ In general one can not find every Pareto optimal solution by considering minimization of scalar objectives $k'f$ where $k \in \mathbb{R}^n$. If $P^*(B)$ is strictly convex as in the above proposition, it follows that every point on $P^*(B)$ (and thus every interesting Pareto optimal solution) can be found by solving the minimization (5.4) for some choice of β .

5.3.1 Lower bound on average buffer cost

Next we lower bound the average buffer cost over all policies μ . Since $b(s)$ is non-decreasing with the buffer state space, $b(s)$ will be minimized if the transmitter sets $U_n = S_n = A_n$ for all n . With such a policy, the transmitter empties the entire buffer at each stage. This results in

$$\bar{b}^\mu = \mathbb{E}_A b(A),$$

⁶Assume that $\{P^*(B) : B \in \mathcal{B}\}$ is not the entire set of Pareto optimal solutions, then for any remaining Pareto optimal point (\tilde{P}, \tilde{B}) it must be that $P^*(\tilde{B}) \leq \tilde{P}$. Thus these other Pareto optimal solutions are not very interesting for us.

where A is a random variable with distribution π_A , the steady-state distribution of $\{A_n\}$. This bound is tight if (1) $b(s)$ is strictly increasing and (2) $\inf\{|g| : g \in \mathcal{G}\} > 0$. If the second requirement is not satisfied then the above policy will require infinite transmission power. When these requirements are satisfied the power required for the above policy is given by

$$\mathbb{E}_{G,A}P(G, A)$$

where $G \sim \pi_G$ and $A \sim \pi_A$. In this case $\mathcal{B} = \{B : B \geq \mathbb{E}_A b(A)\}$ and $P^*(\mathbb{E}_A b(A)) = \mathbb{E}_{G,A}P(G, A)$. When $A_n = \bar{A}$ for all n , then $\mathbb{E}_{G,A}P(G, A) = \mathbb{E}_G P(G, \bar{A})$; we denote this quantity by $\mathcal{P}_d(\bar{A})$. For the model in Sect. 4.1 this corresponds to the minimum average power such that the channel will have a delay-limited capacity of $\bar{A}W/N$. For channels whose delay-limited capacity is zero (*i.e.*, requirement (2) does not hold), $\mathcal{P}_d(\bar{A})$ must then be infinite for any $\bar{A} > 0$. Finally note that for $b(s)$ given by (4.10), $\mathbb{E}_A b(A) = 1$, this is the minimum delay any bit can experience in the discrete time model.

5.3.2 Lower Bound on Average Power

In this section we look at a lower bound on the average power required by any policy. Clearly one can minimize the average power by transmitting nothing, in which case the average power will be zero. If the buffer is finite, then the average buffer cost for such a policy will simply be $b(L)$. Thus $P^*(B) = 0$ for all $B \geq b(L)$. Now assume that the buffer is infinite and that $b(s) \rightarrow \infty$ as $s \rightarrow \infty$. In this case we can ask what is the minimum average power needed over all policies such that the average buffer cost is finite.

Since there are no overflows, the long term average transmission rate must be \bar{A} to keep the average buffer cost finite. Define $\mathcal{P}_a(\bar{A})$ to be the solution to

$$\begin{aligned} & \underset{\Psi: \mathcal{G} \rightarrow \mathbb{R}^+}{\text{minimize}} && \mathbb{E}P(G, \Psi(G)) \\ & \text{subject to :} && \mathbb{E}(\Psi(G)) \geq \bar{A} \end{aligned} \tag{5.20}$$

We have restricted Ψ to be only a function of the channel state G in this optimization. For the model of Sect. 4.1 this corresponds to the minimum power required so that

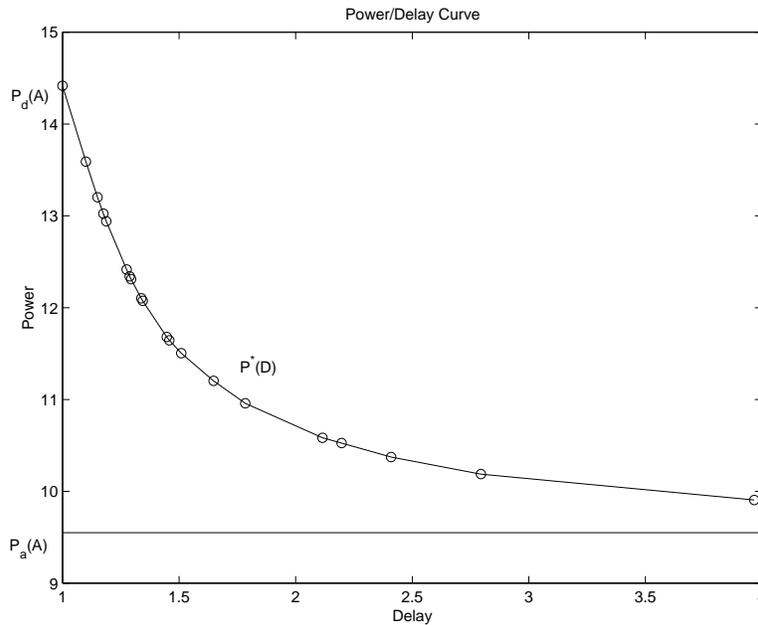


Figure 5-1: Example of power/delay curve.

the average capacity of the channel is $\bar{A}W/N$ (*cf.* Sect. 3.2.3). The quantity $\mathcal{P}_a(\bar{A})$ is the minimum average power needed to transmit at average rate \bar{A} with no other constraints. Thus $\mathcal{P}_a(\bar{A})$ is a lower bound to $P^*(B)$ for all $B \in \mathcal{B}$. If both the channel and arrival processes are constant, then $\mathcal{P}_a(\bar{A}) = \mathcal{P}_d(\bar{A})$; in this case, the power delay curve is a horizontal line. When the the channel and arrival processes are not both constant, $\mathcal{P}_a(\bar{A}) < \mathcal{P}_d(\bar{A})$. In the next chapter we will consider this bound in more detail.

5.3.3 An example

We conclude this section with an example. Figure 5-1 shows the power/delay curve for a channel with memoryless fading and two states ($|\mathcal{G}| = 2$); in one state $|g|^2 = 0.3$ and in the other state $|g|^2 = 0.9$. The sequence of channel states is i.i.d. and each state is equally likely. The arrival process has a constant rate of $\bar{A} = 5$ and the power needed to transmit u bits is given by $P(g, u) = \frac{10}{|g|^2}(e^{u/10} - 1)$. To calculate

the optimal policy, we discretize the buffer state space and allowable control actions. Using dynamic programming techniques, $P^*(D)$ can be obtained computationally (within a small error margin) for various choices of β ; the computed values of $P^*(D)$ are indicated in the figure. For this example $\mathcal{P}_d(\bar{A}) = 14.42$ and $\mathcal{P}_a(\bar{A}) = 9.55$. $\mathcal{P}_a(\bar{A})$ is indicated by a horizontal line in the figure.

5.4 Summary

In this chapter we began to analyze the buffer control problems formulated in Ch. 4. Specifically we considered the average cost Markov decision problem with per stage cost $P(h, u) + \beta b(s)$. In the first part of the chapter we demonstrated several characteristics of the relative gain and optimal policies for such problems. In the second part of the chapter we considered the behavior of the optimal solution as β varies.

CHAPTER 6

Asymptotic Analysis

In this chapter, we analyze several asymptotic versions of the Markov decision problems from Ch. 4. For example, assume that the buffer cost corresponds to probability of overflow (*i.e.*, $b(s, a, u)$ is given by (4.5)). For this case we consider the optimal solution for a given β as the buffer size L goes to infinity. One can often explicitly find the optimal cost of the Markov decision problem in these asymptotic regimes. Our approach to such problems is closely related to that in [Tse94], where buffer control problems for variable rate lossy compression are studied. The underlying mathematical structure of the problem in [Tse94] is very similar to the type of problem we are interested in here. First we will consider the case mentioned above where the buffer cost corresponds to probability of overflow and we let the buffer size go to infinity. In this case we find the limiting value of the optimal cost and bound the rate at which this limit is approached. We will also give a simple buffer management scheme which exhibits convergence rates near these bounds. We will then discuss a similar set of results when the buffer cost corresponds to average delay instead of overflow probability or maximum delay. In this case we consider the asymptotic performance as the average delay grows. Thus these results characterize the tail of the power/delay curve, $P^*(D)$ as $D \rightarrow \infty$. First we need to establish some notation and preliminary results.

6.1 Preliminaries

We again consider the generic buffer control problem as outlined at the start of Ch. 5. However, in this chapter we make several different assumptions which are discussed in the following. First, we allow the buffer cost term to depend on a and u as in (4.5). Next, we assume that the arrival process, $\{A_n\}$ is memoryless. This simplifies much of the following analysis. Also, in this chapter we assume that the buffer state space is either $[0, L]$ or all of \mathbb{R}^+ . Likewise, we assume that the control choice can be any real value. Note that such an assumption is appropriate for the model from section 4.3, since the buffer state represents the amount of “error exponent” to be transmitted, and there is no reason to constrain this quantity to be an integer. For a model as in section 4.1, this assumption is not quite as natural, since the buffer size represents the number of bits to be transmitted. One could think of this real valued buffer as an approximation in this case. If the typical number of bits per codeword is large, such an approximation is reasonable. We will also assume that the channel state space \mathcal{G} is a compact subset of \mathbb{C} and that the arrival state space \mathcal{A} is a compact subset of \mathbb{R}^+ . Let $A_{min} = \inf \mathcal{A}$ and $A_{max} = \sup \mathcal{A}$ (by the compactness assumption such quantities exist and are in \mathcal{A}). Let P be a good power function as in Sect. 4.2. Recall in (5.20) we defined $\mathcal{P}_a(\bar{A})$ to be the solution to the following optimization problem:

$$\begin{aligned} & \underset{\Psi: \mathcal{G} \rightarrow \mathbb{R}^+}{\text{minimize}} \quad \mathbb{E}P(G, \Psi(G)) \\ & \text{subject to : } \mathbb{E}(\Psi(G)) \geq \bar{A} \end{aligned} \tag{6.1}$$

where Ψ is a rate allocation, denoting the rate transmitted as a function of the channel state. The quantity $\mathcal{P}_a(\bar{A})$ is the minimum average power needed to transmit at average rate \bar{A} . Assume that P is given by (4.3) and thus corresponds to transmitting at capacity. In this case the above problem is the inverse of the optimization for finding the channel’s average capacity as in Sect. 3.2.3. Namely for a given rate \bar{A} , the capacity of this channel with average power constraint $\mathcal{P}_a(\bar{A})$ will be \bar{A} . This is achieved with a water-filling rate and power allocation (*cf.* (3.35)). Let $\Psi^{\bar{A}}$ be this rate allocation. This rate allocation is the almost surely unique solution to the above problem and has the following characteristics: (i) it is a function only of $|g|$, (ii) it is

a continuous and non-decreasing function of $|g|$ for all $\bar{A} > 0$, and (iii) for any $g \in \mathcal{G}$, $\Psi^{\bar{A}}(g)$ is a continuous and non-decreasing function of \bar{A} . For any other good power function, the optimal rate allocation will also have these characteristics. This can be seen by introducing Lagrange multipliers and solving the above nonlinear program.

Next we prove some results about of $\mathcal{P}_a(\bar{A})$.

Theorem 6.1.1 *If P is a good power function then \mathcal{P}_a is monotonically increasing and strictly convex.*

PROOF. First we show that \mathcal{P} is strictly convex. Let A^1 and A^2 be two distinct average rates and let Ψ^1 and Ψ^2 be two optimal rate allocations such that for $j = 1, 2$:

$$\mathcal{P}(A^j) = \mathbb{E}P(G, \Psi^j(G)) \quad (6.2)$$

where $G \sim \pi_G$. For any $\lambda \in [0, 1]$, define $\Psi^\lambda = \lambda\Psi^1 + (1 - \lambda)\Psi^2$. Then

$$\mathbb{E}\Psi^\lambda(G) = (\lambda)A^1 + (1 - \lambda)A^2 \quad (6.3)$$

Since $P(g, u)$ is a good power function and hence strictly convex in u we have:

$$P(g, \Psi^\lambda(g)) < \lambda P(g, \Psi^1(g)) + (1 - \lambda)P(g, \Psi^2(g)) \quad (6.4)$$

Combining the above yields:

$$\lambda\mathcal{P}(A^1) + (1 - \lambda)\mathcal{P}(A^2) > \mathbb{E}P(G, \Psi^\lambda(G)) \geq \mathcal{P}(\lambda A^1 + (1 - \lambda)A^2) \quad (6.5)$$

The last inequality follows from the definition of \mathcal{P} . This shows that \mathcal{P} is strictly convex.

Next we show that \mathcal{P} is increasing. Let A^1 and A^2 be two average rates such that $A^1 < A^2$. Let Ψ^2 be an optimal rate allocation corresponding to A^2 as above and let $\alpha = A^1/A^2$. Form a new allocation, Ψ^α by setting $\Psi^\alpha = \alpha\Psi^2$. Then

$$\mathbb{E}\Psi^\alpha(G) = \alpha A^2 = A^1 \quad (6.6)$$

Since $P(g, u)$ is a good power function it is decreasing in u , and so

$$P(g, \Psi^\alpha(g)) \leq P(g, \Psi^2(g)) \text{ for all } g \in \mathcal{G} \quad (6.7)$$

with equality only in those state g for which $\Psi^2(g) = 0$. Thus we have:

$$\mathcal{P}(A^1) \leq \mathbb{E}P(G, \Psi^\alpha(G)) < \mathcal{P}(A^2) \quad (6.8)$$

i.e. \mathcal{P} is decreasing. ■

Now we will define some additional notation that will be useful. Since the arrival process is assumed to be memoryless stationary policy μ will only depend on the buffer state and the channel state. That is $\mu : \mathcal{S} \times \mathcal{G} \mapsto \mathbb{R}$, where $\mu(s, g)$ denotes the number of bits transmitted in state (s, g) . In the following we only consider stationary policies. We show that, at least asymptotically, such a policy is optimal so there is no loss in making such an assumption.¹

For a given policy μ the sequence of combined buffer, channel and arrival states $\{(S_n, G_n, A_n)\}$ form a Markov chain.² We will assume in the following that under every policy μ , this Markov chain is ergodic; in other words it has a unique steady-state distribution $\pi_{(\mathcal{S}, \mathcal{G}, \mathcal{A})}^\mu$ which represents the asymptotic distribution of (S_n, G_n, A_n) under any initial distribution³. For a given steady-state distribution, let π_S^μ be the marginal distribution on the buffer state space; in other words, $\pi_S^\mu = \pi_{(\mathcal{S}, \mathcal{G}, \mathcal{A})}^\mu \circ T^{-1}$, where T is the natural projection from $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$ onto \mathcal{S} . Let $S^{\mu L}$, G , A be random variables whose joint distribution is the steady-state distribution of (S_n, G_n, A_n) under policy μ .

Conditioned on the buffer being in state $s \in \mathcal{S}$ we will denote the average power

¹If the state space of the average cost problem was finite, then as in the previous chapter, we know that there always exists a stationary policy which is optimal. As noted in Sect. 5.2.4, for a problem with an infinite state space, this is not necessarily true without additional assumptions.

²Since the state space of this process is uncountable, there is some question as to whether we should call $\{(S_n, G_n, A_n)\}$ a Markov chain. Some authors reserve the term Markov chain for discrete time Markov processes whose state space is either countable or finite, while others allow more general state spaces as in this case.

³The existence of stationary measures and ergodicity for Markov chains with general state spaces is discussed in [Fel57] and [Doo90]. If the stochastic transition kernel for the Markov chain satisfies some mild regularity conditions, then a stationary measure will exist.

used in this state by $\bar{P}^\mu(s)$, *i.e.*, $\bar{P}^\mu(s) = \mathbb{E}(P(G, \mu(S, G)|S = s)$. The time average power under policy μ will be denoted by \bar{P}^μ , this is given by (5.2). From the assumed ergodicity,

$$\bar{P}^\mu = \mathbb{E}_S \bar{P}^\mu(S) = \int_{\mathcal{S}} \bar{P}^\mu(s) d\pi_S^\mu(s) \quad (6.9)$$

Similarly \bar{b}^μ is defined to be the average buffer cost under policy μ as in (5.3). Again, by the assumed ergodicity we have

$$\bar{b}^\mu = \mathbb{E}b(S, A, \mu(S, G))$$

When it is clear which control strategy we are referring to, we will drop the μ from the above notation.

6.2 Probability of overflow

In this section we consider the case where the buffer cost corresponds to probability of overflow or, with a constant arrival rate, to the probability that a maximum delay constraint is violated. In Ch. 4 several different cost functions were proposed for this case; these are given in (4.5)-(4.7). The results in this section will apply to any of these formulations. Let p_{of}^μ denote the steady-state probability that the buffer overflows under policy μ . Thus when $b(s, a, u)$ is given by (4.5), $\bar{b}^\mu = p_{of}^\mu$; note that this is a lower bound on the average cost for the either of the other cost functions in (4.6) or (4.7). When $b(s)$ is given by (4.7), $\bar{b}^\mu = \pi_S^\mu(L)$; multiplying this by A_{max} gives an upper bound each of the possible cost functions.

The total average cost in (5.4) corresponding to a policy μ is $\bar{P}^\mu + \beta \bar{b}^\mu$. We study this cost as $L \rightarrow \infty$. To be more precise, consider a sequence of buffer state spaces $\mathcal{S}_1, \mathcal{S}_2, \dots$, where $\mathcal{S}_L = [1, L]$ is a buffer of size L . Let μ_L be a control policy when the buffer state space is \mathcal{S}_L . Then $\bar{P}^{\mu_L} + \beta \bar{b}^{\mu_L}$ is the corresponding average cost. We are interested in what can be said about the sequence of average costs as $L \rightarrow \infty$.

For large enough β , as $L \rightarrow \infty$, one would expect that under a good policy \bar{b}^{μ_L} would go to zero. Given that the buffer does not overflow, then $\mathcal{P}_a(\bar{A})$ is the minimum average power required to transmit at a rate of \bar{A} bits per time unit. This

suggests that $\mathcal{P}_a(\bar{A})$ is a reasonable guess for the limit of $\bar{P}^{\mu L} + \beta \bar{b}^{\mu L}$ under optimal policies. For β large enough it can be shown that $\mathcal{P}_a(\bar{A})$ is a lower bound on the optimal cost for every value of L . To see this note that if the buffer overflows with probability $p_{of}^{\mu L}$, then at most $p_{of}^{\mu L} A_{max}$ bits are lost on average due to overflow. Thus the average transmission rate must be greater than or equal to $\bar{A} - p_{of}^{\mu L} A_{max}$. So we have $\bar{P}^{\mu L} \geq \mathcal{P}_a(\bar{A} - p_{of}^{\mu L} A_{max})$. A lower bound on the average cost of the Markov decision problem (for any of the above choices for $b(\cdot)$) is then

$$\mathcal{P}_a(\bar{A} - p_{of}^{\mu L} A_{max}) + \beta p_{of}^{\mu L}. \quad (6.10)$$

If we assume that \mathcal{P}_a is differentiable⁴ at \bar{A} , then, since \mathcal{P}_a is convex, it is lower bounded by the first two terms of its Taylor series about \bar{A} . Thus we have:

$$\mathcal{P}_a(\bar{A} - p_{of}^{\mu L} A_{max}) \geq \mathcal{P}_a(\bar{A}) + \mathcal{P}'_a(\bar{A})(-p_{of}^{\mu L} A_{max}). \quad (6.11)$$

If $\beta > A_{max} \mathcal{P}'_a(\bar{A})$, it follows that $\mathcal{P}_a(\bar{A})$ is a lower bound on the average cost of the Markov decision problem for every value of L .

In the following we will show that this bound is indeed achievable in the limit as $L \rightarrow \infty$. A more interesting question will be to look at the rate at which this limit is approached. Such an approach is motivated by work in [Tse94] on variable rate lossy compression. We will now take a few moments to discuss this work and its relationship to our problem.

6.2.1 Variable Rate Lossy Compression

In variable rate lossy compression one is interested in compressing blocks of real valued data. Each block of data is first quantized and then the resulting reproduction vector is losslessly encoded using a fixed to variable length code. In [Tse94] the following version of this problem is considered. The data is generated by a source which is modulated by a Markov chain. The statistics of the source depend on the state of the Markov chain. After quantizing and encoding, the resulting codeword is placed

⁴Since \mathcal{P}_a is convex it is differentiable almost everywhere with respect to Lebesgue measure. At those points where it is not differentiable a similar statement can be made using a sub-gradient.

into a buffer. The data in this buffer is then transmitted over a link with a fixed rate. Additionally, based on the buffer size and the state of the source, the quantizer is allowed to change. When a coarser quantizer is used, on average fewer bits will enter the buffer but at the expense of more distortion. One can formulate a Markov decision problem in this framework, where the average cost consists of a weighted sum of the average distortion and the probability that the buffer overflows.

By identifying distortion with power and the state of the source with the state of the channel, one can see similarities between this problem and ours. In both cases one is interested in minimizing a function which is decreasing in the buffer drift, while keeping the buffer from overflowing. The development in the following two sections is patterned after that in chapter 2 of [Tse94]; in these sections we prove similar results for the problem at hand. First we give bounds on the rate of convergence of the cost. Then we show that a sequence of simple policies has a convergence rate near this bound.

6.2.2 A Bound on the Rate of Convergence

If β is chosen large enough as above, then the only way we could get $\bar{P}^{\mu_L} + \beta \bar{b}^{\mu_L} \rightarrow \mathcal{P}_a(\bar{A})$ is if both $\bar{P}^{\mu_L} \rightarrow \mathcal{P}_a(\bar{A})$ and $\bar{b}^{\mu_L} \rightarrow 0$. In this section we deal with each of these terms separately. Assume that $b(s, a, u)$ is given by (4.5). Given that $p_{of}^{\mu_L} \rightarrow 0$, we bound the rate at which $\bar{P}^{\mu_L} \rightarrow \mathcal{P}(\bar{A})$. Since the cost in (4.5) is a lower bound for each of the other cost, this bound also applies for these costs.

The notation used in this section is the same as in the previous sections with the addition of the superscript μ_L to denote that this quantity corresponds to the case where the buffer state space is \mathcal{S}_L and the control policy μ_L is used. For example, $S_n^{\mu_L}$ will denote the buffer size at time n under policy μ_L .

In this section, we restrict ourselves to *admissible* sequences of policies, these are defined next.

Definition: A sequence of stationary policies $\{\mu_L\}$ is *admissible* if it satisfies the following conditions:

1. Under every policy μ_L , $(S_n^{\mu_L}, G_n, A_n)$ is an ergodic Markov chain.

2. There exists an $\epsilon > 0$, a $\delta > 0$ and an $M > 0$ such that for all $L > M$ and for all $s \in \mathcal{S}_L$

$$\Pr(A - \mu_L(S^{\mu_L}, G) > \delta | S^{\mu_L} = s) > \epsilon$$

where⁵ S^{μ_L} , G , A are random variables whose joint distribution is the steady-state distribution of (S_n, G_n, A_n) under policy μ_L .

We have already discussed the first assumption. The second assumption means that for large enough buffers, there is a positive steady-state probability that the transmission rate will be less than the number of bits that arrive in the next time step no matter what the buffer occupancy is. If $\Pr(G = 0) > \epsilon$ and $A_{min} > \delta$ then this assumption must be satisfied by any policy that uses finite power. If this is not the case, then this is a restriction on the allowable policies.

We also assume in the following that at $x = \bar{A}$, the first and second derivatives of $\mathcal{P}_a(x)$ exist and are non-zero. Recall, $\mathcal{P}_a(x)$ is a strictly convex and increasing function of \bar{A} . For such a function, the first and second derivatives of $\mathcal{P}_a(x)$ exist and are non-zero at every point except for a set with measure zero⁶. Thus, this is not a very restrictive assumption.

The following notation is used to compare the rate of growth of two sequences $\{a_n\}$ and $\{b_n\}$:

- $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$,
- $a_n = O(b_n)$ if $\limsup_{n \rightarrow \infty} \frac{|a_n|}{|b_n|} < \infty$,
- $a_n = \Omega(b_n)$ if $b_n = O(a_n)$,
- $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$

We are now ready to give the bound on the rate of convergence of the average power and the overflow probability. We state it in the following theorem.

⁵This notation will be used through out this section.

⁶This follows from Lebesgue's theorem which states that a monotonic function is differentiable almost everywhere [RN55].

Theorem 6.2.1 *For any admissible sequence of policies, $\{\mu_L\}$, the rate of convergence of the average power required and the buffer fullness probability have the following relationship: if $p_{of}^{\mu_L} = o(1/L^2)$, then $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}) = \Omega(1/L^2)$.*

Note this theorem says that for each choice of β , the total cost, $\bar{P}^{\mu_L} + \beta p_{of}^{\mu_L}$ can not converge to $\mathcal{P}_a(\bar{A})$ faster than $1/L^2$. Before proving this theorem, we first prove the following two lemmas. For $i \in \mathcal{S}_L$, let $\Delta^{\mu_L}(i) = \mathbb{E}(A - \mu_L(S^{\mu_L}, G) | S^{\mu_L} = i)$. This is the drift of the buffer in state i ignoring any overflows. Since $\{A_n\}$ is memoryless, we then have $\Delta^{\mu_L}(i) = \bar{A} - \mathbb{E}(\mu_L(S^{\mu_L}, G) | S^{\mu_L} = i)$.

Lemma 6.2.2 *For any stationary policy μ ,*

$$0 \leq \int_{\mathcal{S}} \Delta^{\mu}(s) d\pi_S^{\mu}(s) \leq A_{max} p_{of}^{\mu}$$

Recall we are interested in sequences of policies for which $p_{of}^{\mu_L} \rightarrow 0$. This lemma implies that the drift averaged over the buffer state space must converge to zero for such a sequence of policies.

PROOF. To prove this lemma we will use a decomposition argument that will be used several more times. Let us define two new processes E_n and F_n which are related to the buffer state process S_n . Let $F_n = A_n - U_{n-1}$; this represents the net change in the buffer size between time $n - 1$ and time n disregarding any overflows. Let E_n be the cumulative number of bits that have been lost due to buffer overflow up till time n , i.e., $E_n = \sum_{i=0}^{n-1} [S_i + A_{i+1} - U_i - L]^+$. Assuming that the buffer is empty at $n = 0$, we then have:

$$S_n = \sum_{m=1}^n F_m - E_n \tag{6.12}$$

And thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} S_n = \lim_{n \rightarrow \infty} \left(\mathbb{E} \frac{1}{n} \sum_{m=1}^n F_m - \mathbb{E} \frac{1}{n} E_n \right)$$

We consider each of these terms separately. Since S_n is bounded, we have $\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} S_n = 0$. We can think of F_n as a reward gained at time n by the Markov chain $\{S_n, G_n, A_n\}$.

Since we are assuming that the Markov chain $\{S_n, G_n, A_n\}$ is ergodic it follows that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=1}^n F_m &= \mathbb{E}(A - \mu(S, G)) \\ &= \int_{\mathcal{S}} \Delta^\mu(s) d\pi_S^\mu(s) \end{aligned} \quad (6.13)$$

Finally we consider the E_n term. This term is clearly non-negative and there is a one-to-one correspondence between when E_n increases and when the buffer overflows. In these cases E_n increases by at most A_{max} bits. Thus we have that

$$0 \leq \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} E_n \leq A_{max} p_{of}^\mu \quad (6.14)$$

Combining these results yields the desired relationship. ■

Lemma 6.2.3 *Let M , ϵ and δ be as in the definition of admissibility. For every admissible sequence of buffer control schemes $\{\mu_L\}$, it holds that for all $L > M$ there exists a $s_L \in \mathcal{S}_L$ such that*

$$\int_{s > s_L} \Delta^{\mu_L}(s) d\pi_S^{\mu_L}(s) \leq -\frac{\epsilon \delta^2}{4L} + A_{max} p_{of}^{\mu_L}.$$

In Theorem 6.2.1, we are interested in a sequence of policies such that $p_{of}^{\mu_L} = o(1/L^2)$. This lemma implies that for large enough L , the average drift over the “tail end” of the buffer will be negative for such a sequence of policies.

PROOF. Assume $L > M$. Without loss of generality we can assume that $m = 2L/\delta$ is an integer. Consider partitioning \mathcal{S}_L into the following m segments of length $\delta/2$: $[0, \delta/2], (\delta/2, \delta], \dots, ((m-1)\delta/2, L]$. Let $((c-1)\delta/2, c\delta/2]$ be one of these segments which has the maximum probability with respect to $\pi_S^{\mu_L}$. Therefore,

$$\pi_S^{\mu_L}(((c-1)\delta/2, c\delta/2]) \geq \frac{1}{m} = \frac{\delta}{2L}. \quad (6.15)$$

Let $s_L = c\delta/2$. Now we define a new process \hat{S}_n by

$$\hat{S}_n = \max\{S_n, s_L\}. \quad (6.16)$$

This process represents the projection of original buffer process S_n on the set $[s_L, L]$.

Let F_n and E_n be defined as in the decomposition of S_n in (6.12). We now want to come up with a related decomposition of \hat{S}_n . Let \hat{F}_n be defined by

$$\hat{F}_n = \begin{cases} F_n & \text{if } F_n \geq 0 \text{ and } \hat{S}_{n-1} > s_L, \\ \max(0, F_n + S_{n-1} - s_L) & \text{if } F_n \geq 0 \text{ and } \hat{S}_{n-1} = s_L, \\ \hat{S}_n - \hat{S}_{n-1} & \text{if } F_n < 0. \end{cases} \quad (6.17)$$

Thus \hat{F}_n is the net change in \hat{S}_n between times $n-1$ and n , ignoring overflows. The desired decomposition of \hat{S}_n is then

$$\hat{S}_n = \sum_{m=1}^n \hat{F}_m - E_n. \quad (6.18)$$

As in Lemma 6.2.2 we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{S}_n &= 0 \\ \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=1}^n \hat{F}_m &= \int_{\mathcal{S}} \lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s) d\pi_S^{\mu_L}(s) \\ \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} E_n &\leq A_{max} p_{of}^{\mu_L} \end{aligned}$$

Where $\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s)$ is the steady-state expected value of \hat{F}_n conditioned on $S_n = s$.

Using these in the decomposition of \hat{S}_n yields:

$$\int_{\mathcal{S}} \lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s) d\pi_S^{\mu_L}(s) \leq A_{max} p_{of}^{\mu_L} \quad (6.19)$$

Now we want to relate $\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s)$ to changes in the original process.

We consider three cases:

1. First when $S_{l-1} > s_L$, the upward changes in both chains are equivalent, while the size of the downward transitions in the \hat{S} chain may be smaller, so that

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s) \geq \lim_{l \rightarrow \infty} \mathbb{E}(F_l | S_{l-1} = s) = \Delta^{\mu_L}(s), \quad \forall s > s_L. \quad (6.20)$$

2. Next assume that $S_{l-1} \in ((c-1)\delta/2, c\delta/2]$ (recall that $s_L = c\delta/2$). In this case \hat{F}_l is non-negative and $\hat{F}_l \geq F_l - \delta/2$. Thus, for all $s \in ((c-1)\delta/2, c\delta/2]$

$$\mathbb{E}(\hat{F}_l | S_{l-1} = s) \geq \delta/2 \Pr(\hat{F}_l > \delta/2 | S_{l-1} = s) \quad (6.21)$$

$$\geq \delta/2 \Pr(F_l > \delta | S_{l-1} = s). \quad (6.22)$$

The inequality (6.21) follows from Markov's inequality. Taking the limit and using the assumption that μ is admissible we have

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s) \geq \lim_{l \rightarrow \infty} \delta/2 \Pr(F_l > \delta | S_{l-1} = s) \geq \frac{\delta\epsilon}{2}. \quad (6.23)$$

3. Finally when $S_{l-1} \leq (c-1)\delta/2$ we have

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1} = s) \geq 0 \quad (6.24)$$

Now combining all these into (6.19) we get:

$$\int_{((c-1)\delta/2, c\delta/2]} \frac{\epsilon\delta}{2} d\pi_S^{\mu_L}(s) + \int_{s > s_L} \Delta^{\mu_L}(s) d\pi_S^{\mu_L}(s) \leq A_{max} p_{of}^{\mu_L}. \quad (6.25)$$

Using (6.15) we have

$$\int_{((c-1)\delta/2, c\delta/2]} \frac{\epsilon\delta}{2} d\pi_S^{\mu_L}(s) \geq \frac{\epsilon\delta^2}{4L}. \quad (6.26)$$

Substituting this into (6.25) yields the desired relationship. ■

Combining the results of the previous two lemmas we get the following corollary.

Corollary 6.2.4 *Let M , ϵ and δ be as in the definition of admissibility. For every admissible sequence of buffer control schemes $\{\mu_L\}$, it holds that for all $L > M$*

$$\int_{s \leq s_L} \Delta^{\mu_L}(s) d\pi_S^{\mu_L}(s) \geq \frac{\epsilon\delta^2}{4L} - A_{max} p_{of}^{\mu_L},$$

where s_L is as defined in lemma 6.2.3.

Thus for a sequence of policies with $p_{of}^{\mu_L} = o(1/L^2)$, if L is large enough, the average drift over the initial part of the buffer will be non-negative.

We are now ready to prove Theorem 6.2.1. To prove this theorem we will first relate $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A})$ to a function of the drifts. Then we will use the previous two lemmas to relate this function to $p_{of}^{\mu_L}$.

PROOF. Assume we have a sequence of admissible buffer control schemes, $\{\mu_L\}$, such that $p_{of}^{\mu_L} = o(1/L^2)$. For a given L , the average rate we are transmitting at when $S_n = s$ is $\bar{A} - \Delta^{\mu_L}(s)$. Therefore a lower bound on the average power used conditioned on $S_n = s$ is $\mathcal{P}_a(\bar{A} - \Delta^{\mu_L}(s))$. Averaging over the buffer state space yields

$$\bar{P}^{\mu_L} \geq \int_S \mathcal{P}_a(\bar{A} - \Delta^{\mu_L}(s)) d\pi_S^{\mu_L}(s) \quad (6.27)$$

Via a first order Taylor expansion of $\mathcal{P}_a(x)$ around $x = \bar{A}$, $\mathcal{P}_a(x)$ can be written as:

$$\mathcal{P}_a(x) = \mathcal{P}_a(\bar{A}) + (x - \bar{A})\mathcal{P}'_a(\bar{A}) + Q(x - \bar{A}) \quad (6.28)$$

The remainder term $Q(x)$ has the following properties which all follow from the strict convexity and monotonicity of $\mathcal{P}_a(x)$:

1. $Q(x)$ is convex,
2. For $x \neq 0$, $Q(x) > 0$ and $Q(0) = 0$,
3. $Q'(x) > 0$ for $x > 0$, $Q'(x) < 0$ for $x < 0$, and $Q'(0) = 0$.

Combining (6.27) and (6.28), we now have:

$$\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}) \geq \mathcal{P}'_a(\bar{A}) \int_{\mathcal{S}} (-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s) + \int_{\mathcal{S}} Q(-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s). \quad (6.29)$$

To prove the theorem we will show that (6.29) is $\Omega(1/L^2)$; thus $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A})$ must be also. We can bound the first term in (6.29) by noting that $\mathcal{P}'_a(\bar{A}) > 0$ (since \mathcal{P}_a is increasing) and then using Lemma 6.2.2 to get

$$-\mathcal{P}'(\bar{A}) A_{max} p_{of}^{\mu_L} \leq \mathcal{P}'(\bar{A}) \int_{\mathcal{S}} (-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s) \leq 0 \quad (6.30)$$

From this it follows that

$$|\mathcal{P}'(\bar{A}) \int_{\mathcal{S}} (-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s)| = o(1/L^2). \quad (6.31)$$

The second term in (6.29) requires a little more work. Assume that $L > M$ so that Lemma 6.2.3 applies. Let s_L be defined as in that lemma and let $V_L = (s_L, L]$. Since Q is non-negative, we have

$$\int_{\mathcal{S}} Q(-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s) \geq \int_{V_L} Q(-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}^{\mu_L}(s). \quad (6.32)$$

The right hand side can further be bounded as

$$\begin{aligned} \int_{V_L} Q(-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}}(s) &= \int_{V_L} Q(-\Delta^{\mu_L}(s)) d\pi_{\mathcal{S}^L}^{\mu_L}(s) + (1 - \pi_{\mathcal{S}^L}^{\mu_L}(V_L))Q(0) \\ &\geq Q\left(\int_{V_L} -\Delta^{\mu_L}(s) d\pi_{\mathcal{S}}^{\mu_L}(s)\right). \end{aligned} \quad (6.33)$$

Here we have used that $Q(0) = 0$ and the convexity of Q . From Lemma 6.2.3 we have for $L > M$:

$$\int_{V_L} -\Delta^{\mu_L}(s) d\pi_{\mathcal{S}}^{\mu_L}(s) \geq \frac{\epsilon\delta^2}{4L} - A_{max} p_{of}^{\mu_L} \quad (6.34)$$

Again recall that $p_{of}^{\mu L} = o(\frac{1}{L^2})$; thus there exists constants $B > 0$ and $M'' \geq M$ such that for $L > M''$ we have:

$$\int_{V_L} -\Delta^{\mu L}(s) d\pi_S^{\mu L}(s) \geq \frac{B}{L} > 0 \quad (6.35)$$

Using the fact that $Q(x)$ is increasing for $x > 0$, we have, for $L > M''$,

$$Q\left(\int_{V_L} -\Delta^{\mu L}(s) d\pi_S^{\mu L}(s)\right) \geq Q\left(\frac{B}{L}\right). \quad (6.36)$$

Now, by taking the second order Taylor series of $Q(x)$ around $x = 0$ and noting that $Q(0) = Q'(0) = 0$ and $Q''(0) > 0$ we see that

$$Q\left(\frac{B}{L}\right) = \frac{1}{2} \left(\frac{B}{L}\right)^2 Q''(0) + o(1/L^2) = \Omega\left(\frac{1}{L^2}\right) \quad (6.37)$$

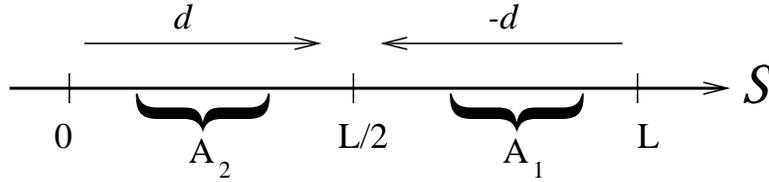
Using (6.32), (6.33), (6.36), and (6.37) we have:

$$\int_S Q(-\Delta^{\mu L}(s)) d\pi_S^{\mu L}(s) = \Omega\left(\frac{1}{L^2}\right) \quad (6.38)$$

Thus we have a bound for the second term in (6.29). Combining this with our bound for the first term in (6.31), we see that $\bar{P}^{\mu L} - \mathcal{P}(\bar{R}) = \Omega(1/L^2)$ and the proof is complete. ■

6.2.3 A Nearly Optimal Simple Policy

In this section we will show that when the fading and arrival processes are memoryless one can achieve performance near the bound in Thm. 6.2.1 by using a simple control strategy. This control strategy involves dividing the buffer in half. When the buffer is more than half full, one control policy will be used, and when the buffer is less than half full, another policy is used. Conditioned on the buffer occupancy being in a given half, the control policy will only be a function of the channel state. The policy used in the upper half of the buffer will have a negative drift and in the lower half, the policy will have a positive drift. Thus such a policy tends to regulate the buffer

Figure 6-1: A simple policy with drift v .

to be half full. Some intuition as to why this may be desirable is provided by the following arguments. Clearly we don't want the buffer to get too full; the fuller the buffer, the more likely an overflow. On the other hand keeping the buffer too empty is not desirable when it comes to minimizing the power cost. To see this note that to minimize the average power one would like to use a policy which transmits at a higher rate when the channel is good and a lower rate when the channel is bad. But, if the the buffer is too empty then there may not be enough bits stored up to take advantage of a sequence of good channel states.

We emphasize that in this section in addition to the arrival process being memoryless we are also assuming that the channel processes is; *i.e.*, both $\{G_n\}$ and $\{A_n\}$ are sequences of i.i.d. random variables. In this section, we assume that $b(s)$ is given by (4.7) so that $\bar{b}^\mu = \pi_S^\mu(L)$. As noted above, multiplying this by A_{max} gives an upper bound on any of the other costs. From this observation it follows that the results in this section hold for any of these other costs.

In the following we will first describe the type of policy we will consider in more detail. Our goal is to prove that nearly optimal convergence rates can be attained with such a policy. We will first prove several lemmas which will aid us in proving this claim. These arguments are based on similar results in [Tse94].

Simple Policy: Consider partitioning the buffer state space into two disjoint sets $[L/2, L]$ and $[0, L/2)$. Let $\psi^1 : \mathcal{G} \mapsto \mathbb{R}^+$ and $\psi^2 : \mathcal{G} \mapsto \mathbb{R}^+$ denote two rate allocations which are only functions of the channel state. For a given $v > 0$, we will define a

simple policy with drift v to be a policy μ with the form:⁷

$$\mu(s, g) = \begin{cases} \psi^1(g) & \text{if } s \in [L/2, L] \\ \psi^2(g) & \text{if } s \in [0, L/2) \end{cases}$$

where $\mathbb{E}_G(\psi^1(G)) = \bar{A} + v$ and $\mathbb{E}_G(\psi^2(G)) = \bar{A} - v$. Thus the drift in any state $s \leq L/2$ will be $-v$ and the drift in any state $s > L/2$ will be v . This is illustrated in Figure 6-1. At times, we will refer to this as the simple policy (ψ^1, ψ^2)

Since the channel and arrival process are memoryless the random variables $\{A_{n+1} - \psi^1(G_n)\}$ are i.i.d. and likewise the random variables $\{A_{n+1} - \psi^2(G_n)\}$ are i.i.d. Thus, when using a simple policy, while the buffer occupancy stays in a given half the buffer process, $\{S_n\}$ will be a random walk (ignoring any edge effects). In analyzing the overflow probability, we will use some results about random walks which are summarized next.

Random Walks: We will state some basic results concerning random walks; we refer to [Gal96] for proofs and further discussion. Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_1| < \infty$. The stochastic process, $\{S_n\}_{n=0}^\infty$ is a *random walk* starting at x if $S_0 = x$ and for $n \geq 1$ $S_n = S_{n-1} + X_n$. We will be interested in the first time a random walk crosses one of two thresholds, a and b . Specifically for a given $a > x$ and $b < x$, let $N = \inf\{n : S_n \geq a \text{ or } S_n \leq b\}$. Wald's identity gives the following relationship between the expected time till a threshold is crossed and the expected value of the random walk at that time:

$$\mathbb{E}(S_N) - x = \mathbb{E}(X_1)\mathbb{E}(N) \tag{6.39}$$

We now state some asymptotic rates related to this problem. We focus on the case where $\mathbb{E}X_1 < 0$ and we let $a \rightarrow \infty$.⁸ Let $\gamma(r) = \ln \mathbb{E}(e^{X_1 r})$ be the semi-invariant moment generating function of X_1 . We will assume that $\gamma(r)$ is finite in an open interval I containing 0 and that it has a root for some $r^* > 0$. From the convexity of

⁷Of course if at some time n , $S_n < L/2$ and $\psi^1(G_n) > S_n$ then the transmitter will only transmit at rate S_n . Note in this case the actual power used will be less than $P(G_n, \psi^1(G_n))$

⁸Of course similar results hold if $\mathbb{E}X_1 > 0$

$\gamma(r)$ it follows that this will be the unique positive root of $\gamma(r)$. Then the following are true:

$$\Pr(S_N \geq a) = \Theta(e^{-r^*(a-x)}) \quad (6.40)$$

$$\Pr(S_N \leq b) = \Theta(1) \quad (6.41)$$

Using these relationships we can prove the following lemma which gives the rate of convergence of the fullness probability for a sequence of simple policies.

Lemma 6.2.5 *Let $\{\mu_L\} = \{(\psi_L^1, \psi_L^2)\}$ be a sequence of simple policies with drifts $\{v_L\}$, where $v_L \rightarrow 0$ as $L \rightarrow \infty$. Let $M^* = \sup_L \sup_{s \in \mathcal{S}_L, g \in \mathcal{G}, a \in \mathcal{A}} \{|a - \mu_L(s, g)|\}$ denote the maximum change in the buffer size in one time step for any policy. We assume that for L large enough, $M^* \ll L$. Let $r^*(v_L)$ be the unique positive root of the semi-invariant moment generating function of $A - \psi_L^1(G)$. Then we have:*

$$\pi_S^{\mu_L}(L) = o\left(\exp\left(-\frac{1}{2}r^*(v_L)L\right)\right)$$

Before proving this lemma, we give a sketch of some of the general ideas behind the proof. Consider a renewal process, where renewals occur every time that $S_n = L$. Let T^L denote the average inter-renewal time. Then we have $\pi_S^{\mu_L}(L) = 1/T^L$. We will try to estimate T^L . Figure 6-2 illustrates a typical sample path of S_n between renewals. As shown in this figure, once the buffer is in state L it will typically return to the region near the middle of the buffer and spend a lot of time near the middle before finally returning to L . Suppose we look at the buffer process once it first enters $[L/2, L/2 + M^*]$ and try to find the expected time for the buffer to reach L from this point. Starting in $[L/2, L/2 + M^*]$ the buffer process can be modeled as a random walk until it crosses either L or $L/2$. From Wald's identity the time until this happens is $\approx \frac{1}{v_L}$ and the probability it crosses L first will be $\approx \exp(-\frac{L}{2}r^*(v_L))$. If it first crosses $L/2$, it will eventually return to $[L/2, L/2 + M^*]$ again. Once S_n enters $[L/2, L/2 + M^*]$ again, it can be modeled by another random walk and the same argument holds. Thus we can make the following approximation, we can think of this as a series of independent trials, where each trial has a probability of success of $\exp(-\frac{L}{2}r^*(v_L))$. Thus the expected number of trials until a success is $\approx \exp(\frac{L}{2}r^*(v_L))$.

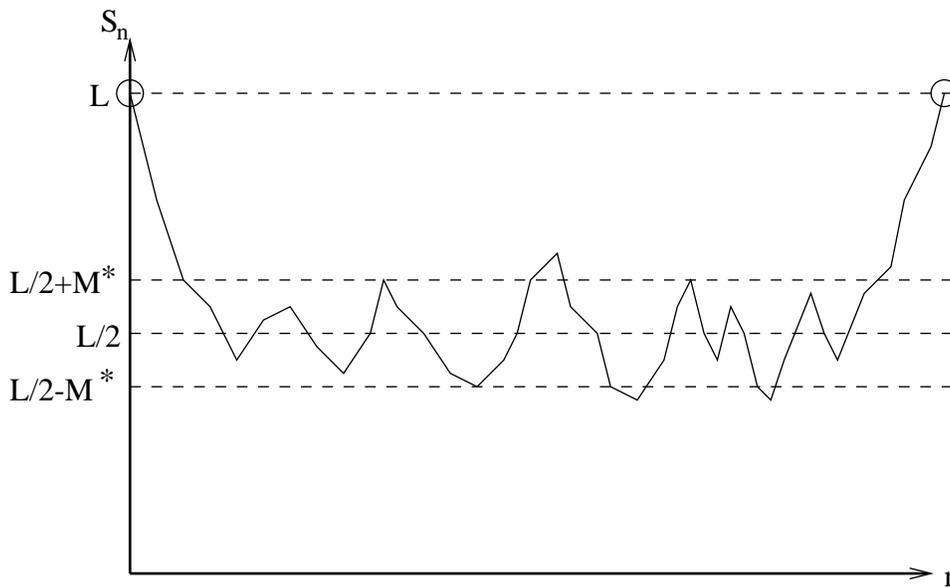


Figure 6-2: Sample path of S_n between renewals events.

Ignoring the time to get back if the buffer crosses $L/2$, the average length of each trial is $\approx \frac{1}{v_L}$. Thus the expected total time until a success will be $\approx \frac{\exp(\frac{L}{2}r^*(v_L))}{v_L}$. Using this as an approximation for T^L , we get that $\pi_S^{\mu^L}(L) \approx v_L \exp(-\frac{L}{2}r^*(v_L))$ which is $o(\exp(-\frac{L}{2}r^*(v_L)))$ as desired.

In the following proof we will show that the above argument can be made precise.

PROOF. The proof of this lemma will make use of the fact that while the buffer process stays in $[L/2, L]$ it behaves as a random walk. We will now explicitly define this random walk. Define $\{W_n^i\}$ to be a random walk starting at $i \in [L/2, L]$, where $W_n^i = W_{n-1}^i + (A_n - \psi_L^1(G_{n-1}))$ and $W_0^i = i$. Also let us define the stopping time $N^i = \inf\{n > 0 : W_n^i \geq L \text{ or } W_n^i < L/2\}$. Thus if $S_0 = i$, we will have $S_n = W_n^i$ for all $n < N^i$ and N^i will be the first time that either the buffer becomes full or the buffer occupancy becomes less than $L/2$. Finally let η^i denote the distribution of the

random variable W_{Ni}^i . Note that,

$$\eta^i \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) = 1 - \eta^i ([L, \infty)). \quad (6.42)$$

Now we are ready to prove the lemma. Assume that L is large enough so that $M^* \ll L$. As discussed above we consider a renewal process where renewals occur every time $S_n = L$, and we let T^L denote the expected inter-renewal time. Denote the residual life of the renewal process at time n by V_n . We can then use the random walk $\{W_n^L\}$ to write T^L as:

$$T^L = \mathbb{E}(N^L) + \int_{[\frac{L}{2} - M^*, \frac{L}{2})} \mathbb{E}(V_{N^L} | S_{N^L} = i) d\eta^L(i) \quad (6.43)$$

$$\geq \int_{[\frac{L}{2} - M^*, \frac{L}{2})} \mathbb{E}(V_{N^L} | S_{N^L} = i) d\eta^L(i) \quad (6.44)$$

$$\geq \left(\inf_{\{0 < i \leq M^*\}} \mathbb{E}(V_n | S_n = \frac{L}{2} - i) \right) \eta^L \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) \quad (6.45)$$

The expected residual time given that the buffer state is $s \in [0, L/2)$ is the sum of the expected time to re-enter $[L/2, L]$ plus the expected residual time once the process re-enters $[L/2, L]$. Let Q_n be the length of time from time n until S_n first enters $[L/2, L]$ again. Then we have for all $k \in (0, M^*]$,

$$\mathbb{E}(V_n | S_n = L/2 - k) = \mathbb{E}(Q_n | S_n = L/2 - k) + \mathbb{E}(V_{n+Q_n} | S_n = L/2 - k) \quad (6.46)$$

$$\geq \mathbb{E}(V_{n+Q_n} | S_n = L/2 - k) \quad (6.47)$$

$$\geq \inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(V_n | S_n = L/2 + i) \quad (6.48)$$

As in (6.43), the expected residual time given the buffer is in state $L/2 + i$ for

$i \in [0, M^*]$ can be expressed using the random walk $\{W_n^{\frac{L}{2}+i}\}$ as

$$\mathbb{E}(V_n | S_n = L/2 + i) = \mathbb{E}(N^{\frac{L}{2}+i}) + \int_{[\frac{L}{2}-M^*, \frac{L}{2})} \mathbb{E}(V_{N^{\frac{L}{2}+i}} | S_{N^{\frac{L}{2}+i}} = j) d\eta^{\frac{L}{2}+i}(j) \quad (6.49)$$

$$\geq \mathbb{E}(N^{\frac{L}{2}+i}) + \left(\inf_{\{0 < j \leq M^*\}} \mathbb{E}(V_n | S_n = \frac{L}{2} - j) \right) \eta^{\frac{L}{2}+i} \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) \quad (6.50)$$

Minimizing over i yields:

$$\begin{aligned} \inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(V_n | S_n = L/2 + i) &\geq \inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i}) \\ &+ \left(\inf_{\{0 < j \leq M^*\}} \mathbb{E}(V_n | S_n = \frac{L}{2} - j) \right) \inf_{\{0 \leq i \leq M^*\}} \eta^{\frac{L}{2}+i} \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) \end{aligned} \quad (6.51)$$

Substituting this into (6.46), we have:

$$\begin{aligned} \mathbb{E}(V_n | S_n = L/2 - k) &\geq \inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i}) \\ &+ \left(\inf_{\{0 < j \leq M^*\}} \mathbb{E}(V_n | S_n = \frac{L}{2} - j) \right) \inf_{\{0 \leq i \leq M^*\}} \eta^{\frac{L}{2}+i} \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) \end{aligned} \quad (6.52)$$

We can again minimize over $k \in (0, M^*]$; doing this, then using (6.42) and simplifying the resulting expression yields:

$$\inf_{\{0 < i \leq M^*\}} \mathbb{E}(V_n | S_n = L/2 - i) \geq \frac{\inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i})}{\sup_{\{0 \leq i \leq M^*\}} \eta^{\frac{L}{2}+i}([L, \infty))} \quad (6.53)$$

Finally we can substitute this back into (6.43) to get:

$$T^L \geq \frac{\inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i})}{\sup_{\{0 \leq i \leq M^*\}} \eta^{\frac{L}{2}+i}([L, \infty))} \eta^L \left(\left[\frac{L}{2} - M^*, \frac{L}{2} \right) \right) \quad (6.54)$$

Now we can bound each of these terms. First note

$$\inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i}) \geq 1 \quad (6.55)$$

Thus for L large enough,

$$\inf_{\{0 \leq i \leq M^*\}} \mathbb{E}(N^{\frac{L}{2}+i}) \geq \frac{1}{v_L} \quad (6.56)$$

The remaining terms can be bounded using the results we stated about random walks.

Using (6.41) we have:

$$\eta^L(\lfloor \frac{L}{2} - M^*, \frac{L}{2} \rfloor) = \Theta(1), \quad (6.57)$$

and from (6.40) we have:

$$\sup_{\{0 \leq i \leq M^*\}} \eta^{\frac{L}{2}+i}([L, \infty)) = \Theta(\exp(-r^*(v_L)\frac{L}{2})). \quad (6.58)$$

Thus we have that $T^L = \Omega\left(\frac{\exp(r^*(v_L)L/2)}{v_L}\right)$ and therefore

$$\begin{aligned} \pi_S^{\mu L}(L) &= O(v_L \exp(-r^*(v_L)L/2)) \\ &= o(\exp(-r^*(v_L)L/2)) \end{aligned}$$

as desired.⁹ ■

Recall, at the start of this chapter $\Psi^x : \mathcal{G} \mapsto \mathbb{R}^+$ was defined to be the optimal solution to (6.1), *i.e.*, the (a.s.) unique rate allocation with average rate x and average power $\mathcal{P}_a(x)$. Also recall, when $P(g, u)$ corresponds to transmitting at capacity, Ψ^x corresponds to a water-filling rate allocation. A simple policy (ψ^1, ψ^2) with drift v , is defined to be *optimal* if $\psi^1 = \Psi^{\bar{A}+v}$ and $\psi^2 = \Psi^{\bar{A}-v}$. We will show that using a sequence of optimal simple policies can achieve nearly the optimal convergence rate in Th. 6.2.1. Before proving this we prove Lemma 6.2.6 below. This lemma help us to relate changes in v_L to the changes in $r^*(v_L)$ when an optimal simple policy is used.

Lemma 6.2.6 *Let $r^*(v)$ denote the unique nonzero root of the semi-invariant moment generating function of $A - \Psi^{\bar{A}+v}(G)$ (for $v \neq 0$). Assume that for all v in a*

⁹By a slightly more elaborate proof one can strengthen the above result to show that $\pi_S^{\mu L}(L) = \Theta(v_L \exp(-r^*(v_L)L/2))$, but this will not be useful for us here.

neighborhood of 0, that $\frac{d^2}{dv^2}\mathbb{E}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))}$ exists and that¹⁰

$$\frac{d^2}{dv^2}\mathbb{E}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))} = \mathbb{E}\frac{d^2}{dv^2}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))}.$$

Then,

$$\left.\frac{dr^*(v)}{dv}\right|_{v=0} = \frac{2}{\text{Var}(A - \Psi^{\bar{A}}(G))}.$$

PROOF. From the definition of $r^*(v)$ we have:

$$\mathbb{E}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))} = 1 \tag{6.59}$$

Differentiating this equation twice with respect to v , and using the above assumption, we have, for all v in a neighborhood of 0,

$$\mathbb{E}\frac{d^2}{dv^2}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))} = 0.$$

Letting $S(v) = \frac{dr^*(v)}{dv}$ then,

$$\begin{aligned} \mathbb{E}\frac{d^2}{dv^2}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))} &= \mathbb{E}e^{r^*(v)(A-\Psi^{\bar{A}+v}(G))} \left\{ \left((A - \Psi^{\bar{A}+v}(G))S(v) \right. \right. \\ &\quad \left. \left. - r^*(v) \left(\frac{d}{dv}\Psi^{\bar{A}+v}(G) \right) \right)^2 + (A - \Psi^{\bar{A}+v}(G)) \left(\frac{d}{dv}S(v) \right) \right. \\ &\quad \left. - 2S(v) \left(\frac{d}{dv}\Psi^{\bar{A}+v}(G) \right) - r^*(v) \left(\frac{d^2}{dv^2}\Psi^{\bar{A}+v}(G) \right) \right\} \\ &= 0 \end{aligned}$$

¹⁰As an example of when these assumptions will hold, assume that $|\mathcal{A}| < \infty$ and $|\mathcal{G}| < \infty$. In this case if the second derivative of $\Psi^{\bar{A}+v}(g)$ with respect to v exists and is continuous at $v = 0$ for all g , then the above assumptions hold. When $P(g, u)$ corresponds to transmitting at capacity, this will be true for all but a finite number of values of \bar{A} . These values correspond to those rates \bar{A} for which the “water level” $\frac{1}{\lambda}$ in some state g is exactly equal to $\frac{N_g}{|g|^2}$ (cf. (3.19)).

Next we evaluate this at $v = 0$. In doing this, note that for $v = 0$, the random variable $A - \Psi^{\bar{A}}(G)$ is zero mean, and thus $r^*(0) = 0$. Additionally note that since

$$\mathbb{E}\Psi^{\bar{A}+v}(G) = \bar{A} + v$$

then $\frac{d}{dv}\Psi^{\bar{A}+v}(G) = 1$ and $\frac{d^2}{dv^2}\Psi^{\bar{A}+v}(G) = 0$. Thus we have

$$S(0)^2\text{Var}(A - \Psi^{\bar{A}}(G)) - 2S(0) = 0. \quad (6.60)$$

This equation has two roots, corresponding to the two roots of $\ln(\mathbb{E}e^{r(A - \Psi^{\bar{A}}(G))}) = 0$. The root $S(0) = 0$ corresponds to the root of the log moment generating function that is always at zero, and the root at $\frac{2}{\text{Var}(A - \Psi^{\bar{A}}(G))}$ corresponds to the non-zero root, as desired. ■

We are now ready to prove that there exists a sequence of simple policies with nearly optimal convergence rates. We state this in the following proposition:

Proposition 6.2.7 *For any $K \geq 2$, there exists a sequence of simple policies $\{\mu_L\}$, such that $\pi_S^{\mu_L}(L) = o(1/L^K)$ and $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}) = O(\frac{\ln^2 L}{L^2})$.*

PROOF. Let $\{\mu_L\}$ be a sequence of optimal simple policies $(\Psi^{\bar{A}+v_L}, \Psi^{\bar{A}-v_L})$ with associated drifts $v_L = \frac{K\text{Var}(\Psi^{\bar{A}}(G))\ln L}{L}$, for some $K \geq 2$. Such a sequence of policies satisfies the assumptions of Lemma 6.2.5, and thus we have:

$$\pi_S^{\mu_L}(L) = o(\exp(-r^*(v_L)L/2)) \quad (6.61)$$

Taking the first two terms of the Taylor series of $r^*(v_L)$ around $v_L = 0$ we have:

$$r^*(v_L) = r^*(0) + \left. \frac{dr^*(v_L)}{dv_L} \right|_{v_L=0} \cdot v_L + O(v_L^2) \quad (6.62)$$

Using Lemma 6.2.6 and the fact that $r^*(0) = 0$ this simplifies to:

$$r^*(v_L) = \frac{2v_L}{\text{Var}(\Psi^{\bar{A}}(G))} + O(v_L^2) \quad (6.63)$$

Using the given choice of v_L we have:

$$r^*(v_L) = \frac{2K \ln L}{L} + O\left(\left(\frac{\ln L}{L}\right)^2\right) \quad (6.64)$$

Substituting this into (6.61) along with our choice of v_L yields:

$$\pi_S^{\mu_L}(L) = o\left(\exp\left(-K \ln L + O\left(\frac{(\ln L)^2}{L}\right)\right)\right) \quad (6.65)$$

$$= o\left(\frac{1}{L^K}\right) \quad (6.66)$$

Thus we have the desired rate of convergence of the overflow probability. Next we consider the rate of convergence of $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A})$ using this same sequence of simple policies. Since we are using optimal simple policies we have:

$$\bar{P}^{\mu_L} \leq \pi_S^{\mu_L}([L/2, L])\mathcal{P}_a(\bar{A} + v_L) + \pi_S^{\mu_L}([0, L/2])\mathcal{P}_a(\bar{A} - v_L) \quad (6.67)$$

Here the inequality arises due to the edge effects that can occur when the buffer is too empty. Expanding $\mathcal{P}_a(x)$ by its first order Taylor series around $x = \bar{A}$ yields:

$$\begin{aligned} \bar{P}^{\mu_L} &\leq \pi_S^{\mu_L}([L/2, L])(\mathcal{P}_a(\bar{A}) + \mathcal{P}'(\bar{A})(v_L) + O(v_L^2)) \\ &\quad + \pi_S^{\mu_L}([0, L/2])(\mathcal{P}_a(\bar{A}) + \mathcal{P}'(\bar{A})(-v_L) + O(v_L^2)) \\ &= \mathcal{P}_a(\bar{A}) + \mathcal{P}'(\bar{A}) (\pi_S^{\mu_L}([L/2, L])v_L - \pi_S^{\mu_L}([0, L/2])v_L) + O(v_L^2) \end{aligned} \quad (6.68)$$

From Lemma 6.2.2 we have that

$$\pi_S^{\mu_L}([L/2, L])v_L - \pi_S^{\mu_L}([0, L/2])v_L \leq \bar{A}\pi_S^{\mu_L}(L) \quad (6.69)$$

and using (6.65) we have:

$$\pi_S^{\mu_L}([L/2, L])v_L - \pi_S^{\mu_L}([0, L/2])v_L = o\left(\frac{1}{L^K}\right). \quad (6.70)$$

Substituting this into (6.68) along with our choice for v_L yields:

$$\begin{aligned}\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}) &\leq O\left(\frac{1}{L^K}\right) + O\left(\frac{(\ln L)^2}{L^2}\right) \\ &= O\left(\frac{(\ln L)^2}{L^2}\right)\end{aligned}$$

Thus we have the desired rates of convergence of the average powers and of the overflow probability, proving this proposition. ■

We conclude this section with several comments. First as a means of comparison suppose we use the same power control policy for every state in the buffer. In this case, it can be shown that if the overflow probability decays faster than $\frac{1}{L}$, then the average power must decay to the optimal power at a rate slower than $\frac{1}{L}$. Thus adjusting the power allocation based on the queue size enables one to essentially square the rates of convergence. The above simple policy achieves most of this gain.

In Ch. 4 we noted that for the receiver to reliably receive the transmitted signal, it must know the transmission rate and power used by the transmitter. When the transmission policy depends on the buffer state at the transmitter, some overhead may be required to convey this information to the receiver (unless the arrival rate is constant). For the simple policies defined above this overhead is only 1 bit, indicating in which half of the buffer the current buffer state lies.

Next note that this proposition seems to suggest that by choosing K large, one can get the very fast convergence of the overflow probability and still essentially the same convergence rate of the power. The issue here is that the larger K is, the larger L will need to be before the asymptotic convergence rate on the power is meaningful.

Finally consider the same sequence of simple policies as in Prop. 6.2.7 but with $K \in (0, 2)$. Following through the proof in this case yields $\pi^{\mu_L}(L) = o(\frac{1}{L^K})$ and $\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}) = O(\frac{1}{L^K})$.

6.3 Average Buffer Delay

In this section we consider the case where the buffer is infinite and the buffer cost is given by (4.10), *i.e.*, $\mathcal{S} = [0, \infty)$ and $b(s) = s/\bar{A}$. In this case the time average

buffer cost, \bar{b}^μ , for a given policy μ corresponds to the time average buffer delay. Additionally, assuming that $\{(S_n^\mu, G_n, A_n)\}$ is ergodic,

$$\bar{b}^\mu = \frac{\mathbb{E}S^\mu}{A}$$

which by Little's law is equal to the expected steady-state delay in the buffer. We denote this quantity by \bar{D}^μ to emphasize that we are considering the average delay case. In this section we look at the asymptotic performance attainable by any sequence of policies $\{\mu_k\}$ such that $\bar{D}^{\mu_k} \rightarrow \infty$, *i.e.*, a sequence of policies with increasing average delay. For such a sequence of policies we are interested in what can be said about the corresponding sequence of average powers, \bar{P}^{μ_k} .

From Sect. 5.3.2, since no buffer overflows occur, $\mathcal{P}_a(\bar{A})$ is a lower bound on \bar{P}^μ for any policy μ such that $\bar{D}^\mu < \infty$. Furthermore, from the strict convexity of $\mathcal{P}_a(\bar{A})$, if $\bar{P}^\mu = \mathcal{P}_a(\bar{A})$, then it must be that $\mu(s, g, a) = \Psi^{\bar{A}}(g)$ for all $(s, g, a) \in \mathcal{S} \times \mathcal{G} \times \mathcal{A}$ except for possibly a set with measure zero. With such a policy, the average drift in each buffer state will be zero, and thus $\bar{D}^\mu = \infty$. Thus we have shown that $\mathcal{P}_a(\bar{A})$ is strictly less than \bar{P}^μ for all policies with finite average delay. Clearly one can find a sequence of policies $\{\mu_k\}$ with $\bar{D}^{\mu_k} \rightarrow \infty$, such that $\bar{P}^{\mu_k} \rightarrow \mathcal{P}_a(\bar{A})$. We examine the rate at which this limit is approached. Specifically, we show that $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = \Theta((\frac{1}{\bar{D}^{\mu_k}})^2)$. To do this we follow a similar approach as we did in the previous section. First, we bound the rates at which these quantities can converge; then we give a sequence of simple schemes which show this bound is tight. Before doing this we make a few comments on how this section relates to some of our prior work.

In the previous section we looked at the asymptotic behavior of the buffer control problem as $L \rightarrow \infty$ with $b(s)$ given by (4.7). For a constant arrival rate, this corresponded to the probability a maximum delay constraint was violated. Thus letting $L \rightarrow \infty$ corresponds to loosening this maximum delay constraint. In this section, we consider a sequence of problems where each problem has an average delay requirement. Again we are interested in the performance as the delay requirement is loosened. In the context of a Markov decision problem with per stage cost $P(g, u) + \beta b(s)$, the analysis in this section corresponds to studying a sequence of problems with a corresponding sequence $\{\beta_k\}$ where $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. In other words, as k grows the

average delay term is given less weight in the total cost. The average cost of the k th problem will be $\bar{P}^{\mu_k} + \beta_k \bar{D}^{\mu_k}$; in this section we study the behavior of the costs \bar{P}^{μ_k} and \bar{D}^{μ_k} . This analysis can also be viewed in the context of the optimum power/delay curve, $P^*(B)$, defined in Sect. 5.3. Recall $P^*(B)$ corresponds to the minimum average power required so that the average buffer cost is no more than B . The results in this section characterize the “tail” behavior of $P^*(B)$ as $B \rightarrow \infty$ for the case where $b(s)$ is given by (4.10). Specifically it follows that $P^*(B)$ converges to $\mathcal{P}_a(\bar{A})$ at rate $\Theta(\frac{1}{B^2})$ (cf. Fig. 5-1).

6.3.1 A Bound on the Rate of Convergence

We bound the rate at which $\bar{P}^{\mu_k} \rightarrow \mathcal{P}_a(\bar{A})$ for any sequence of policies $\{\mu_k\}$ such that $\bar{D}^{\mu_k} \rightarrow \infty$. We are still assuming that the arrival process is memoryless, but in this section we do allow the channel process to have memory. As in Sect. 6.2, we only consider admissible sequences of policies. For the average delay case these policies are defined next.

Definition: A sequence of buffer control policies $\{\mu_k\}$ is *admissible* if it satisfies the following assumptions:

1. For all k , $\bar{D}^{\mu_k} < \infty$, and $\lim_{k \rightarrow \infty} \bar{D}^{\mu_k} = \infty$.
2. Under each policy μ_k , $\{(S_n^{\mu_k}, G_n, A_n)\}$ forms an ergodic Markov chain.
3. There exists an $\epsilon > 0$, a $\delta > 0$ and a $M > 0$ such that for all $k > M$ and for all $s \leq 2\mathbb{E}(S^{\mu_k})$,

$$\Pr(A_{-\mu_k}(S^{\mu_k}, G) > \delta | S^{\mu_k} = s) > \epsilon$$

where S^{μ_k} , G , A are random variables whose joint distribution is the steady state distribution of (S_n, G_n, A_n) under policy μ_k .

We are interested in sequences of policies for which $\bar{D}^{\mu_k} \rightarrow \infty$. The first assumption says a sequence of policies is admissible only if the average delay of these policies has the desired behavior. The second assumption is the same as that made in Sect.

6.2. The third assumption is also similar to one made in Sect. 6.2. Here this assumption means that for large k and any buffer state $s < 2\mathbb{E}(S^{\mu_k})$, there is a positive steady state probability that the next buffer state is bigger than $s + \delta$. Once again if $\Pr(G = 0) > \epsilon$ and $A_{min} > \delta$ then this must be satisfied. We still assume that at $x = \bar{A}$, the first and second derivatives of $\mathcal{P}_a(x)$ exist and are non-zero.

As in the probability of overflow case (*cf.* Lemma 6.2.3), for any admissible sequence of policies $\{\mu_k\}$ the average drift over the tail of the buffer will be negative (for k large enough). This is stated in the following lemma.

Lemma 6.3.1 *Let M , δ and ϵ be as given in the definition of an admissible sequence. For any admissible sequence of buffer control schemes $\{\mu_k\}$, for all $k > M$, there exists an $s_k \in \mathcal{S}$ such that*

$$\int_{s > s_k} \Delta^{\mu_k}(s) d\pi_S^{\mu_k}(s) \leq \frac{-\epsilon\delta^2}{16\mathbb{E}(S^{\mu_k})}$$

PROOF.¹¹ Let M , δ and ϵ be as in the definition of admissibility and assume that $k > M$. Let $F_n = A_n - U_{n-1}$; this represents the net change in the buffer occupancy between time $n - 1$ and n . Thus, assuming the buffer is empty at time 0, we have¹²

$$S_n = \sum_{m=1}^n F_m. \quad (6.71)$$

By assumption as $n \rightarrow \infty$, $(S_n^{\mu_k}, G_n, A_n)$ reaches steady-state. Thus the Markov inequality implies

$$\lim_{n \rightarrow \infty} \Pr(S_n \geq 2\mathbb{E}(S^{\mu_k})) \leq \frac{1}{2}, \quad (6.72)$$

and so

$$\lim_{n \rightarrow \infty} \Pr(S_n < 2\mathbb{E}(S^{\mu_k})) > \frac{1}{2}. \quad (6.73)$$

Let $m = 4\mathbb{E}(S^{\mu_k})/\delta$ where we can take δ to divide $2\mathbb{E}(S^{\mu_k})$. Consider partitioning $[0, 2\mathbb{E}(S^{\mu_k})]$ into the following m segments: $[0, \delta/2), [\delta/2, \delta), \dots, [(m-1)\delta/2, 2\mathbb{E}(S^{\mu_k})]$,

¹¹The proof of this lemma follows a similar line of reasoning to Lemma 6.2.3. We refer the reader to the proof of Lemma 6.2.3 for a more detailed discussion.

¹²Note, since the buffer is infinite, we do not have to consider overflows as in Lemma 6.2.3.

where each segment has a length of $\delta/2$. Let $[(c-1)\delta/2, c\delta/2)$ be one of these segments which has the maximal probability with respect to π^{μ_k} , so that

$$\pi_S^{\mu_k}([(c-1)\delta/2, c\delta/2)) \geq \frac{1}{2m} = \frac{\delta}{8\mathbb{E}(S^{\mu_k})}. \quad (6.74)$$

Let $s_k = c\delta/2$ and define the process $\{\hat{S}_n\}$ by

$$\hat{S}_n = \max\{S_n, s_k\}. \quad (6.75)$$

Thus \hat{S}_n is equal to S_n restricted to $[s_k, \infty)$. Let $\hat{F}_n = \hat{S}_n - \hat{S}_{n-1}$ be the net change in \hat{S}_n , so that

$$\hat{S}_n = \sum_{m=1}^n \hat{F}_m. \quad (6.76)$$

Thus

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{S}_n = \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=1}^n \hat{F}_m. \quad (6.77)$$

By assumption $\bar{D}^{\mu_k} < \infty$; therefore $\lim_{n \rightarrow \infty} \mathbb{E} S_n < \infty$. Furthermore, $\hat{S}_n \leq S_n + s_k$ for all n and so, $\mathbb{E}(\hat{S}_n) \leq \mathbb{E}(S_n) + s_k < \infty$. Thus,

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{S}_n = 0 \quad (6.78)$$

As in Lemma 6.2.3, the quantity \hat{F}_n can be considered a reward gained at time $n-1$ by the ergodic Markov chain $\{(S_n, G_n, A_n)\}$. Thus we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=1}^n \hat{F}_m = \int_S \lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s) d\pi_S^\mu(s) \text{ a.s.} \quad (6.79)$$

Here $\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s)$ is the steady-state expected value of \hat{F} conditioned on $S = s$. Using (6.78), (6.79), and (6.77) yields:

$$\int_S \lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s) d\pi_S^\mu(s) = 0 \quad (6.80)$$

Next we relate this to changes in the original process by considering three cases:

1. First when $S_{l-1} \geq s_k$, then $\hat{F}_l \geq F_l$ and thus

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s) \geq \lim_{l \rightarrow \infty} \mathbb{E}(F_l | S_{l-1}=s) = \Delta^\mu(s), \quad \forall s \geq s_k. \quad (6.81)$$

2. Next when $(c-1)\delta/2 \leq S_{l-1} < c\delta/2 = s_k$, \hat{F}_l is nonnegative and $\hat{F}_l \geq F_l - \delta/2$. Thus,

$$\mathbb{E}(\hat{F}_l | S_{l-1}=s) \geq \delta/2 \Pr(\hat{F}_l > \delta/2 | S_{l-1}=s) \quad (6.82)$$

$$\geq \delta/2 \Pr(F_l > \delta | S_{l-1}=s). \quad (6.83)$$

Here (6.83) follows from the Markov inequality. Next taking the limit and using that μ is admissible we have: $\forall s \in [(c-1)\delta/2, c\delta/2)$,

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s) \geq \lim_{l \rightarrow \infty} \delta/2 \Pr(F_l > \delta | S_{l-1}=s) \geq \frac{\epsilon\delta}{2}. \quad (6.84)$$

3. Finally, when $S_{l-1} < (c-1)\delta/2$, \hat{F}_l is also non-negative, and thus

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{F}_l | S_{l-1}=s) \geq 0, \quad \forall s < (c-1)\delta/2 \quad (6.85)$$

Combining (6.81), (6.84), and (6.85) into (6.80) yields:

$$\int_{((c-1)\delta/2, c\delta/2]} \frac{\epsilon\delta}{2} d\pi_S^\mu(s) + \int_{s>s_k} \Delta^\mu(s) d\pi_S^\mu(s) \leq 0. \quad (6.86)$$

The first term can be bounded using (6.74):

$$\int_{((c-1)\delta/2, c\delta/2]} \frac{\epsilon\delta}{2} d\pi_S^\mu(s) \geq \frac{\epsilon\delta^2}{16\mathbb{E}(S^{\mu_k})} \quad (6.87)$$

Substituting this into (6.86) yields the desired result. \blacksquare

We are now ready to prove the following bound on the rate of convergence.

Theorem 6.3.2 *Any admissible sequence of buffer control policies μ_k must satisfy*

$$\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = \Omega((1/\bar{D}^{\mu_k})^2).$$

The proof of this will follow a similar line of reasoning to the proof of Thm. 6.2.1. We refer the reader to that proof for a more detailed discussion. Recall, in the proof of Thm. 6.2.1, Lem. 6.2.2 and Lem. 6.2.3 were used. Here, in place of Lem. 6.2.2 we will use the fact that for any policy μ which has $\mathbb{E}S < \infty$,

$$\int_{\mathcal{S}} \Delta^\mu(s) d\pi_S^\mu(s) = 0. \quad (6.88)$$

This follows since the buffer size is infinite and thus no bits are lost due to overflow. In place of Lem. 6.2.3 we use Lem. 6.3.1.

PROOF. Let $\{\mu_k\}$ be a sequence of admissible policies. For the k th policy, the average transmission rate conditioned on being in state s is $\mathbb{E}(\mu_k(S^{\mu_k}, G)|S^{\mu_k}=s) = \bar{A} - \Delta^{\mu_k}(s)$. Thus the average power used when the buffer is in state s is lower bounded by $\mathcal{P}_a(\bar{A} - \Delta^{\mu_k}(s))$. Averaging over the buffer state space we have:

$$\bar{P}^{\mu_k} \geq \int_{\mathcal{S}} \mathcal{P}_a(\bar{A} - \Delta^{\mu_k}(s)) d\pi_S(s) \quad (6.89)$$

Via a first order Taylor expansion around $x = \bar{A}$, $\mathcal{P}_a(x)$ can be written as:

$$\mathcal{P}_a(x) = \mathcal{P}_a(\bar{A}) + \mathcal{P}'_a(\bar{A})(x - \bar{A}) + Q(x - \bar{A}) \quad (6.90)$$

where the remainder term $Q(x)$ has the following properties: (i) $Q(x)$ is strictly convex, (ii) for $x \neq 0$, $Q(x) > 0$ and $Q(0) = 0$, and (iii) $Q'(x) > 0$ for $x > 0$, $Q'(x) < 0$ for $x < 0$, and $Q'(0) = 0$. These all follow from the strict convexity and

monotonicity of \mathcal{P}_a . Substituting this into (6.89) yields:

$$\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) \geq \mathcal{P}'_a(\bar{A}) \int_{\mathcal{S}} (-\Delta^{\mu_k}(s)) d\pi_{\mathcal{S}}(s) + \int_{\mathcal{S}} Q(-\Delta^{\mu_k}(s)) d\pi_{\mathcal{S}}(s) \quad (6.91)$$

$$= \int_{\mathcal{S}} Q(-\Delta^{\mu_k}(s)) d\pi_{\mathcal{S}}(s), \quad (6.92)$$

where we have used (6.88). Let s_k be as defined in Lem. 6.3.1 and assume that $k > M$ so that the lemma applies. Then we have

$$\begin{aligned} & \bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) \\ & \geq \int_{s > s_k} Q(-\Delta^{\mu_k}(s)) d\pi_{\mathcal{S}}(s) \\ & = \int_{s > s_k} Q(-\Delta^{\mu_k}(s)) d\pi_{\mathcal{S}}(s) + \pi_{\mathcal{S}}([0, s_k])Q(0) \\ & \geq Q\left(\int_{s > s_k} -\Delta^{\mu_k}(s) d\pi_{\mathcal{S}}(s)\right) \\ & \geq Q\left(\frac{\epsilon\delta^2}{16\mathbb{E}S^{\mu_k}}\right) \end{aligned} \quad (6.93)$$

We have again used the convexity and monotonicity properties of Q along with the result of Lem. 6.3.1. Finally, expanding Q in a Taylor series around 0, and using that $Q(0) = Q'(0) = 0$ we have:

$$\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) \geq \frac{1}{2}Q''(0) \left(\frac{\epsilon\delta^2}{16\mathbb{E}S^{\mu_k}}\right)^2 + o\left(\left(\frac{\epsilon\delta^2}{16\mathbb{E}S^{\mu_k}}\right)^2\right). \quad (6.94)$$

That $Q''(0)$ exists and is non-zero follows from the assumption that the second derivative of $\mathcal{P}_a(x)$ exists and is non zero at $x = \bar{A}$. Thus we have $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = \Omega\left(\left(\frac{1}{\mathbb{E}(S^{\mu_k})}\right)^2\right)$. Using Little's law, this gives us $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = \Omega\left((1/\bar{D}^{\mu_k})^2\right)$ as desired. ■

6.3.2 An Optimal Sequence of Simple Policies

As in Sect. 6.2.3, we now look at the performance of a sequence of simple control strategies. The strategies we are interested in, again involve dividing the buffer into two parts. Intuitively, such strategies are desirable for the same reasons as in the case of a maximum delay constraint. That is, it is not desirable for the buffer be too full, since this will make the average delay large; it is also not desirable for the buffer to be too empty, because there may not be enough bits to take advantage of a good channel state.

In this case, we show that when the fading and arrival process are memoryless, a sequence of simple policies can exactly achieve the bound given in Thm. 6.3.2. Thus this bound is tight when the fading and arrival processes are memoryless. Note this is stronger than what was proved in Sect. 6.2.3; in that section, the sequence of simple policies only had a nearly optimal convergence rate.

First we describe the type of simple policies considered for the average delay case. Then we show that these policies can achieve the optimal convergence rate. Again since the arrival process is memoryless, a policy is only a function of S_n and G_n .

Simple Policy: For a given $v \in (0, \bar{A})$, partition the buffer state space into two disjoint sets: $[1/v, \infty)$ and $[0, 1/v]$. Let $\psi^1 : \mathcal{G} \mapsto \mathbb{R}^+$ and $\psi^2 : \mathcal{G} \mapsto \mathbb{R}^+$ be two rate allocations which are only functions of the channel state. We define a *simple policy with drift v* , to be a policy μ with the form:¹³

$$\mu(s, g) = \begin{cases} \psi^1(g) & \text{if } s \in [1/v, \infty) \\ \psi^2(g) & \text{if } s \in [0, 1/v] \end{cases}$$

where $\mathbb{E}_G(\psi^1(G)) = \bar{A} + v$ and $\mathbb{E}_G(\psi^2(G)) = \bar{A} - v$. We refer to a simple policy as optimal if $\psi^1 = \Psi^{\bar{A}+v}$ and $\psi^2 = \Psi^{\bar{A}-v}$, where Ψ^x is still the rate allocation with average rate x which achieves $\mathcal{P}_a(x)$.

While the buffer occupancy either stays less than $1/v$ or stays greater than $1/v$,

¹³More generally, we could partition the buffer into the sets $[0, K/v)$ and $[K/v, \infty)$ where $K > 0$. These sets could then be used in the definition of a simple policy. The following results still hold with such a generalization.

the buffer process will again be a random walk. The analysis in this section will use another result about random walks which is summarized next.

More on random walks. We want to give another result concerning random walks. In this case we are interested in bounding the steady-state probability that a random walk restricted to the positive axis exceeds a given value.

Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables with $-\infty < \mathbb{E}X_1 < 0$. Assume that $\gamma(r) = \ln \mathbb{E}(e^{X_1 r})$ has a root $r^* > 0$. Let $\{S_n\}$ be the process defined by $S_0 = 0$, and for $n \geq 1$, $S_n = (S_{n-1} + X_n)^+$. Thus S_n is a random walk restricted to the positive axis. Assume that $S_n \rightarrow S$ in distribution, where S is a random variable with the steady-state distribution. Then we have:

$$\lim_{n \rightarrow \infty} \Pr(S_n > i) = \Pr(S > i) \leq e^{-r^* i} \quad \forall i > 0. \quad (6.95)$$

When S_n represents the waiting time in a G/G/1 queue, this result is known as Kingman's bound. We refer to [Gal96] for a proof.

Using this result, we prove the following lemma which gives a bound on the average delay for an optimal simple policy.

Lemma 6.3.3 *For an optimal simple policy, μ , with drift v , the average delay satisfies:*

$$\bar{D}^\mu \leq \frac{1/v}{\bar{A}} + \frac{e^{r^*(v)\eta(v)}}{\bar{A}r^*(v)} \quad (6.96)$$

where $\eta(v)$ is a non-negative function such that $\eta(v) \rightarrow 0$ as $v \rightarrow 0$ and $r^*(v)$ is the unique positive root of $\gamma(r) = \ln(\mathbb{E}_{A,G}[e^{(A-\Psi^{\bar{A}+v(G)})r}])$.

There are two main ideas in the following proof. First, Little's law is used to relate the average delay to the average buffer occupancy. Second, (6.95) is used to bound the probability that the buffer occupancy is large.

PROOF. From Little's law we have:

$$\bar{D}^\mu = \frac{\mathbb{E}(S)}{\bar{A}}, \quad (6.97)$$

where $\mathbb{E}(S)$ is the expected buffer occupancy in steady-state. This can be written as the integral of the complimentary distribution function of S , *i.e.*

$$\mathbb{E}(S) = \int_0^\infty \Pr(S > s) ds. \quad (6.98)$$

Upper bounding $\Pr(S > s)$ by 1 for $s \leq 1/v$, yields:

$$\mathbb{E}(S) \leq 1/v + \int_0^\infty \Pr(S > s + 1/v) ds. \quad (6.99)$$

For all $v \geq 0$, let

$$\eta(v) = \sup\{\Psi^{\bar{A}+v}(g) - \Psi^{\bar{A}-v}(g) : g \in \mathcal{G}\}.$$

We show that $\eta(v)$ is non-negative and converges to zero as $v \rightarrow 0$. As noted at the start of this chapter, $\Psi^x(g)$ is a continuous function of $|g|$ for all $x \geq 0$. Recall \mathcal{G} is assumed to be compact, thus $\Psi^x(g)$ will be bounded for all x . Therefore, $\eta(v)$ is also bounded. Likewise, since $\Psi^x(g)$ is non-decreasing in x for all g , $\eta(v)$ will be non-negative. Finally, for all g , $\Psi^x(g)$ is continuous in x ; therefore, for all g , $\{\Psi^{\bar{A}+v}(g) - \Psi^{\bar{A}-v}(g)\}$ converges monotonically to 0 as $v \rightarrow 0$. Thus, by Dini's theorem [Dud89], $\lim_{v \rightarrow 0} \eta(v) = 0$.

Next we bound $\Pr(S > s + 1/v)$. Consider a second buffer process $\{\check{S}_n\}$ defined as follows. This second process only uses the policy $\Psi^{\bar{A}+v}$ and is restricted to stay in $[1/v, \infty)$ for all time. Specifically, let $\check{U}_n = \Psi^{\bar{A}+v}(H_n)$ and let $\check{S}_{n+1} = \max\{\check{S}_n + A_{n+1} - \check{U}_n, A_{n+1}, 1/v\}$. We assume that this buffer process and the original buffer process observe the same sequence of channel and source states. Furthermore assume that at time 0, $\check{S}_0 = \max\{S_0, 1/v\}$. We claim that for all $n \geq 0$, $\check{S}_n \geq S_n - \eta(v)$. This will be shown by induction on n . By assumption $\check{S}_0 \geq S_0 \geq S_0 - \eta(v)$. Assume at time n , $\check{S}_n \geq S_n - \eta(v)$, we will show that this holds for time $n + 1$. Consider the following two cases:

Case 1: $S_n > 1/v$. In this case $\check{U}_n = U_n$, and thus,

$$\begin{aligned}\check{S}_{n+1} &\geq \max\{\check{S}_n - \check{U}_n + A_{n+1}, A_{n+1}\} \\ &\geq \max\{S_n - \eta(v) - U_n + A_{n+1}, A_{n+1}\} \\ &\geq \max\{S_n - U_n + A_{n+1}, A_{n+1}\} - \eta(v) \\ &= S_{n+1} - \eta(v)\end{aligned}$$

Case 2: $S_n \leq 1/v$. In this case $\check{S}_n \geq 1/v \geq S_n$ and $\check{U}_n \leq U_n + \eta(v)$. Thus

$$\begin{aligned}\check{S}_{n+1} &\geq \max\{\check{S}_n - \check{U}_n + A_{n+1}, A_{n+1}\} \\ &\geq \max\{S_n - (U_n + \eta(v)) + A_{n+1}, A_{n+1}\} \\ &\geq \max\{S_n - U_n + A_{n+1}, A_{n+1}\} - \eta(v) \\ &= S_{n+1} - \eta(v).\end{aligned}$$

Thus we have $\check{S}_n \geq S_n - \eta(v)$ for all $n \geq 0$. From this it follows that for all $n \geq 0$ and all s , $\Pr(S_n > 1/v + s) \leq \Pr(\check{S}_n > 1/v + s - \eta(v))$. Letting $n \rightarrow \infty$ we have

$$\Pr(S > 1/v + s) \leq \Pr(\check{S} > 1/v + s - \eta(v))$$

where S and \check{S} are random variables with the steady-state distributions for the respective processes. Note, the process $\{\check{S}_n\}$ is a random walk restricted to $[1/v, \infty)$. Therefore using (6.95) we have

$$\Pr(\check{S} > 1/v + s - \eta(v)) \leq e^{-r^*(v)(s-\eta(v))}$$

and thus,

$$\Pr(S > 1/v + s) \leq e^{-r^*(v)(s-\eta(v))}.$$

Substituting this into (6.99) and carrying out the integration yields:

$$\mathbb{E}(S) \leq 1/v + \int_0^\infty e^{-r^*(v)(s-\eta(v))} ds \quad (6.100)$$

$$= 1/v + \frac{e^{r^*(v)\eta(v)}}{r^*(v)} \quad (6.101)$$

Finally, substituting this into (6.97) gives the desired result. ■

Now we prove that any sequence of optimal simple policies whose drifts converge to zero will achieve the bound in Thm. 6.3.2.

Theorem 6.3.4 *Let $\{\mu_k\}$ be a sequence of optimal simple policies with drifts $\{v_k\}$, such that $v_k \rightarrow 0$ as $k \rightarrow \infty$. Then we have $\bar{P}^{\mu_k} - \mathcal{P}(\bar{A}) = O((\frac{1}{D^{\mu_k}})^2)$.*

PROOF. Let $\{\mu_k\}$ be a sequence of optimal simple policies as stated in the theorem. We show that $\bar{D}^{\mu_k} = O(\frac{1}{v_k})$ and $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = O((v_k)^2)$. The desired result then follows directly. First we show that $D^{\mu_k} = O(\frac{1}{v_k})$.

From Lem. 6.3.3 we have:

$$\bar{D}^{\mu_k} \leq \frac{1/v_k}{\bar{A}} + \frac{e^{r^*(v_k)\eta(v_k)}}{\bar{A}r^*(v_k)} \quad (6.102)$$

The first term on the right hand side is clearly $O(1/v_k)$. We focus on the second term. To bound the growth of this term, first note that Lemma 6.2.6 still applies. Thus we have:

$$\left. \frac{dr^*(v)}{dv} \right|_{v=0} = \frac{2}{\text{Var}(A - \Psi^{\bar{A}}(G))}. \quad (6.103)$$

Taking the Taylor series of $r^*(v)$ around $v = 0$ and using this lemma we have

$$r^*(v) = 0 + \Lambda v + o(|v|) \quad (6.104)$$

where $\Lambda = \frac{2}{\text{Var}(A - \Psi^{\bar{A}}(H))}$. From Lem. 6.19 we have that $\eta(v) \rightarrow 0$. Thus, it follows that

$$r^*(v)\eta(v) = \Lambda\eta(v)v + o(|v|).$$

With these expansions we have

$$\frac{e^{r^*(v_k)\eta(v_k)}}{\bar{A}r^*(v_k)} = \frac{e^{\Lambda\eta(v_k)v_k+o(v_k)}}{\bar{A}(\Lambda v_k + o(v_k))}. \quad (6.105)$$

Now since:

$$\lim_{k \rightarrow \infty} \frac{v_k e^{\Lambda\eta(v_k)v_k+o(v_k)}}{\bar{A}(\Lambda v_k + o(v_k))} = \frac{1}{\bar{A}\Lambda} \quad (6.106)$$

it follows that:

$$\frac{e^{r^*(v_k)\eta(v_k)}}{\bar{A}r^*(v_k)} = O(1/v_k) \quad (6.107)$$

and therefore $\bar{D}^{\mu_k} = O(1/v_k)$ as desired.

Next we show that $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = O((v_k)^2)$. This follows from the same type of argument as in Prop. 6.2.7. For the simple policy μ_k , the average power is

$$\bar{P}^{\mu_k} = \pi_S^{\mu_k}((1/v_k, \infty))\mathcal{P}_a(\bar{A} + v_k) + \pi_S^{\mu_k}([0, 1/v_k])\mathcal{P}_a(\bar{A} - v_k) \quad (6.108)$$

Taking the Taylor series of $\mathcal{P}(x)$ around $x = \bar{A}$ we have

$$\bar{P}^{\mu_k} = \mathcal{P}_a(\bar{A}) + \mathcal{P}'_a(\bar{A})(\pi_S^{\mu_k}((1/v, \infty))v_k - \pi_S^{\mu_k}([0, 1/v])v_k) + O((v_k)^2) \quad (6.109)$$

Now $\pi_S^{\mu_k}((1/v, \infty))v_k - \pi_S^{\mu_k}([0, 1/v])v_k$ will be less than or equal to the average drift due to edge effects when the buffer is near empty. Thus $\pi_S^{\mu_k}((1/v, \infty))v_k - \pi_S^{\mu_k}([0, 1/v])v_k \leq 0$ and therefore $\bar{P}^{\mu_k} - \mathcal{P}_a(\bar{A}) = O((v_k)^2)$ as desired. ■

6.3.3 Summary and Further Directions

In this chapter we considered two asymptotic versions of the buffer control problem from Ch. 4. First we examined the case where the buffer cost corresponded to the probability of buffer overflow. In this case we analyzed the problem as the buffer size L gets large. Next we analyzed the case where the buffer cost corresponded to average delay. We analyzed the problem as the average delay grew. In both cases our analysis consisted of the following two parts:

1. The rate at which the average power converged to $\mathcal{P}_a(\bar{A})$ was bounded given

that the average buffer cost was changing at a given rate and the arrival process was memoryless.

2. A sequence of simple policies was demonstrated to have either the optimal or the nearly optimal convergence rate when the fading and arrival process were memoryless.

Note when bounding the convergence rate, we did not assume memoryless fading.

To conclude this chapter we discuss several directions in which this work can be extended.

Finite Buffer and average delay: Suppose that we have a finite buffer and are interested, first, in avoiding buffer overflows and, second, in keeping the average delay small. A simple policy as in Section 6.2 is asymptotically good for avoiding buffer overflows and minimizing the power. Recall, these simple policies divide the buffer in half and use different policies in each half. One can get the same asymptotic performance by dividing the buffer into unequal portions. By putting the threshold closer to the front of the buffer, the average delay is reduced. This can be shown using similar arguments to those in Sect. 6.3.

Fading and Arrivals with Memory: As noted above, we have only proved that a simple policy is optimal or nearly optimal for the case where the fading process is memoryless. Suppose this process has memory and a simple policy is used. In this case, conditioned on the buffer staying in a given region, the buffer process will no longer be a random walk, but a Markov modulated random walk. Large deviation bounds similar to those for random walks exist for Markov modulated random walks [Gal96]. Using these bounds, the proofs in Sect. 6.2.3 and Sect. 6.3.2 can probably be extended to the case where the fading processes have memory. Throughout this chapter we have assumed memoryless arrivals. It is likely that the results in this chapter can also be extended to the case where the arrival process has memory.

CHAPTER 7

Multiple Users

In the previous chapters we have considered a single user communicating over a fading channel. The main problem examined was how to allocate resources over time in order to optimally trade-off the needed power with some buffer cost. In addition to fading, another inherent difficulty in wireless communications is that there are generally many users who must share the available communication resources. With multiple users, the problem becomes how to allocate resources both over time and between the users. We look at such problems in this chapter. Again the goal will be to understand the trade-offs between minimizing the required power and also minimizing some buffer cost for each user.

In a wireless network, if multiple users transmit at the same time, their respective signals may interfere with each other. A common way to avoid such situations is to divide the available resources into orthogonal channels, for example by TDMA or FDMA. Different users are then assigned to distinct channels so that when multiple users transmit, they will not interfere with each other. If we assume that such an assignment is done as part of a higher layer protocol, then each user can effectively be treated as a single user in a fading channel and the previous results apply. With a large number of users, there may not be enough channels to assign one to each user. To overcome this, in many systems, these channels are reused by users who are spatially far enough apart so that the interference between their signals is weak. For example in FDMA cellular systems, the same frequency slot is re-used in non-adjacent cells. In such situations the interfering users are often viewed as part of the additive

noise. In this case, once again, each user can be treated as a single user in a fading channel.

In many cases, a fixed assignment of users to orthogonal channels may not be desirable. When users are transmitting bursty data it may be desirable to dynamically allocate resources between users. Performing such a dynamic allocation is often simpler if the resources are not divided into orthogonal channels; this is one advantage of a CDMA system, such as IS-95. Also, assigning users to orthogonal channels is generally not optimal from a minimum power perspective. In a military setting, LPI considerations may make other approaches, such as spread spectrum, more desirable. Finally in a non-cellular architecture, such as a packet-radio network, it is difficult to ensure that users with the same channel stay spatially separated. If we do not separate users in the above fashion, then the single user model of the previous chapter is not applicable. Instead, multiple user or network models must be considered. These situations will be addressed in the current chapter.

While considering multiple-user models, we still assume that routing and some scheduling decisions are done at a higher network layer. This is done for two reasons - to keep the model manageable and because this is typically the way a practical network would be designed and operated. Specifically, we assume that we are given a set of transmitters, a set of receivers, and a statistical description of the traffic each transmitter has to send to each receiver. This traffic description consists of a stochastic model of the arrival process and any delay constraints for this link. We will assume that this traffic corresponds to “link” flows in the network. These flows are determined by a higher layer routing protocol and depend on the network architecture and the end-to-end traffic requirements of each user. Architecturally it may make sense to do some scheduling at a higher layer. For example, in a wireless environment, a user can not both send and receive data in a single frequency band at the same time. The transmitted power will be much greater than any received power. Due to the time-varying nature of the channel, one can not build a good enough echo-canceller to remove the transmitted power. Once again, this problem can be avoided by assigning each user one channel in which to transmit and an orthogonal channel in which to receive; here each of these channels may be shared by many other users. If we assume that this scheduling is done by a higher layer protocol, then the set of

traffic requirements given above corresponds to the traffic requirement for one such channel; in this case, the set of transmitters and set of receivers will be disjoint sets.

As an example of the above situation consider a conventional cellular system. The only routing decision in such systems is assigning users to base stations. For each base-station, the up-link traffic from the users to the base-station is assigned to one orthogonal channel and down-link traffic to another. For example, in IS-95 systems, these “channels” are disjoint 1.32 MHz segments of bandwidth, which are separated by a guard-band of 45MHz. In a packet-radio network, these routing and scheduling decisions can be much more involved [Kas99].

We look at the problem of minimizing the total power needed to satisfy the given traffic requirements. As in the single user case, if the traffic requirements consist only of required long term average rates, this problem is equivalent to characterizing the capacity region of the network of users. The capacity region of a general network is unsolved, but many special cases of such networks have been well studied. Examples include the multiple-access channel and the degraded broadcast channel. In this chapter we primarily consider the multiple-access channel. In Sect. 7.1 we formulate a buffer control model for a multiple-access situation. In Sect. 7.2 we show that the asymptotic analysis from Sect. 6.2 can be extended to the multiple-access model. In Sect. 7.3 other multi-user situations are briefly discussed.

7.1 Buffer Control Model for Multiple-Access Channel

The multiple-access channel models the situation in which there are multiple transmitters communicating to a single receiver. This is an appropriate model for the reverse link in a cellular network if the out of cell interference is considered part of the additive noise. It may also be an appropriate model in a packet-radio network if the higher layer scheduling algorithm insures that simultaneous transmissions to other receivers contribute little interference. In this section we consider extending the buffer control models from Chapter 4 to this situation. First we discuss this channel model in more detail as well as some of the known capacity results for this channel.

Then we discuss a buffer model for this situation and formulate a Markov decision problem to illustrate the trade-off between average power and some buffer cost such as average delay or probability of overflow.

7.1.1 Channel Model

We consider a Gaussian multiple-access channel with M users communicating to a single receiver. In this chapter, we will only consider a narrow-band, block fading model of the multiple-access channel. Extensions to wide-band, block memoryless channels follow as in the single user case. We consider a discrete-time, baseband model of this channel. As in Sect. 2.1, we assume each channel use represents a complex sample of a continuous time channel with samples taken at rate W . The transmitted signal of each user is multiplied by a time-varying gain which models the fading. Over each block of N channel uses the channel gain of each user is fixed.¹ For $m \in \mathbb{Z}$, $n \in \{1, \dots, N\}$, let $X_{m,n}^i$ represent the i th user's transmitted signal during the n th channel use of the m th block.² The received signal $Y_{m,n}$ is then given by:

$$Y_{m,n} = \sum_{i=1}^M G_m^i X_{m,n}^i + Z_{m,n} \quad (7.1)$$

where G_m^i represents the fading experienced by user i during the m th channel block and $\{Z_{m,n} : m \in \mathbb{Z}, n \in \{1, \dots, N\}\}$ is a set of i.i.d. circularly symmetric Gaussian random variables with zero mean and $\mathbb{E}Z_{m,n}Z_{m,n}^* = N_0$.

The channel can again be represented as a vector input/vector output channel (*cf.* (2.5))

$$\mathbf{Y}_m = \sum_{i=1}^M G_m^i \mathbf{X}_m^i + \mathbf{Z}_m \quad (7.2)$$

where $\mathbf{Y}_m = [Y_{m,1}, \dots, Y_{m,N}]^T$, $\mathbf{X}_m^i = [X_{m,1}^i, \dots, X_{m,N}^i]^T$ and $\mathbf{Z}_m = [Z_{m,1}, \dots, Z_{m,N}]^T$. The model in (7.2) is to be thought of as a discrete time model where samples occur at

¹As in the single user case, we assume that time is measured at the receiver.

²Throughout this chapter we will use superscripts to denote different users and subscripts to denote time samples. To avoid confusion, the i th power of x will be denoted $(x)^i$.

rate W/N . We assume that for each i , $\{G_m^i\}$ takes values in the state space $\mathcal{G}^i \subset \mathbb{C}$. Let $\mathbf{G}_m = (G_m^1, \dots, G_m^M)$ be a random vector representing the joint fading state during the m th block. We assume that $\{\mathbf{G}_m\}$ is a stationary ergodic Markov chain with state space $\prod_i \mathcal{G}^i \subset \mathbb{C}^M$ which determines the joint fading state of the M users during the m th block. We do not assume that the components of \mathbf{G}_m are independent or identically distributed. For users separated by many wavelengths, independence would be a reasonable assumption, but it is not necessary for the following arguments. Conditioned on \mathbf{G}_{m-1} , \mathbf{G}_m is assumed to be independent of each user's input signal and the additive noise prior to time m . Let $\pi_{\mathbf{G}}$ denote the steady-state distribution of $\{\mathbf{G}_m\}$. This channel will be referred to as a *block fading, multiple-access channel*.

We assume that the receiver has perfect channel state information, *i.e.*, at time n it knows \mathbf{G}_n . Assume that each user i is subject to an average power constraint of \bar{P}^i , *i.e.*,

$$\lim_{K \rightarrow \infty} \frac{1}{KN} \sum_{k=1}^K \mathbb{E}((\mathbf{X}_k^i)^\dagger \mathbf{X}_k^i) \leq \bar{P}^i/W$$

where \bar{P}^i represents the average power used in the continuous time channel. If the transmitters have no CSI, then the capacity region of the channel is the set of rate vectors (R^1, \dots, R^M) which satisfy:

$$\sum_{i \in Q} R^i \leq \mathbb{E}_{\mathbf{G}} C \left(\sum_{i \in S} |G^i|^2 \bar{P}^i \right) \quad \forall Q \subset \{1, \dots, M\} \quad (7.3)$$

where $\mathbf{G} = (G^1, \dots, G^M)$ is a random variable with distribution $\pi_{\mathbf{G}}$, and R^i is the rate of user i in bits per second. As in the previous chapters, $C(x) = W \log(1 + \frac{x}{N_o W})$. The set of rate vectors satisfying (7.3) form a bounded polyhedron in \mathbb{R}^M . The subset of these rate vectors for which the constraint corresponding to $Q = \{1, \dots, M\}$ is tight is called the *dominant face* of the capacity region. Rate vectors on the dominant face have the property that, given any rate vector \mathbf{x} in the capacity region, there exists a rate vector \mathbf{y} on the dominant face such that $\mathbf{y} \geq \mathbf{x}$. The dominant face itself will be an $M - 1$ dimensional bounded polyhedron.

With multiple users, there is a new issue to be considered with regard to the transmitters' state information. Specifically, does a transmitter have knowledge only of its

own channel state or does it have knowledge of the joint channel state. If the channel state is learned via a pilot signal, the former assumption would be appropriate. If the channel state is learned via feedback from the receiver, the latter may be appropriate. This may require more overhead on the feedback link. We refer to this second case as *complete* state information. We will focus on this latter case; it appears to be more tractable and it is an upper bound over what is attainable with only local knowledge.

With perfect, complete state information, the transmitter can again adjust its power and rate based on the joint fading state. A joint power allocation for the M users is a map $\mathbf{P} : \prod_i \mathcal{G}^i \mapsto \mathbb{R}^M$, where $\mathbf{P}(\mathbf{g}) = (P^1(\mathbf{g}), \dots, P^M(\mathbf{g}))$ and $P^i(\mathbf{g})$ is the average power used by the i th user when the joint fading state is \mathbf{g} . For a given power allocation, let $\mathcal{C}^M(\mathbf{P})$ be the set of rates (R^1, \dots, R^M) such that

$$\sum_{i \in Q} R^i \leq \mathbb{E}_{\mathbf{G}} C \left(\sum_{i \in Q} |G^i|^2 P^i(\mathbf{G}) \right) \quad \forall Q \subset \{1, \dots, M\} \quad (7.4)$$

Assume that user i has an average power constraint of \bar{P}^i . Let Λ denote the set of all power allocations which satisfy these average power constraints, *i.e.* $\Lambda = \{\mathbf{P} : \mathbb{E}_{\mathbf{G}}(P^i(\mathbf{G})) \leq \bar{P}^i\}$. In [TH98] it is shown that the capacity region of the channel with complete state information is given by:

$$\mathcal{C}^M = \bigcup_{\mathbf{P} \in \Lambda} \mathcal{C}^M(\mathbf{P}). \quad (7.5)$$

This capacity region is not a polyhedron, but the union of the polyhedrons $\mathcal{C}^M(\mathbf{P})$. Likewise, assume we restrict Λ to be the set of power allocations such that $P^i(\mathbf{g}) = P^i(g^i)$ for all i and which satisfy the average power constraint. In this case (7.5) is the capacity region of the channel with only local state information.

As in the single user case, to achieve a rate vector near the boundary of either the capacity region in (7.3) or in (7.5), usually requires the use of codewords long enough to average over a typical fading realization³. When delay constraints limit

³Another interesting way of approaching any rate in these capacity regions is if the number of users get large [HT95]. In this case one can average over the different users, instead of over the fading realization. We will not consider this here.

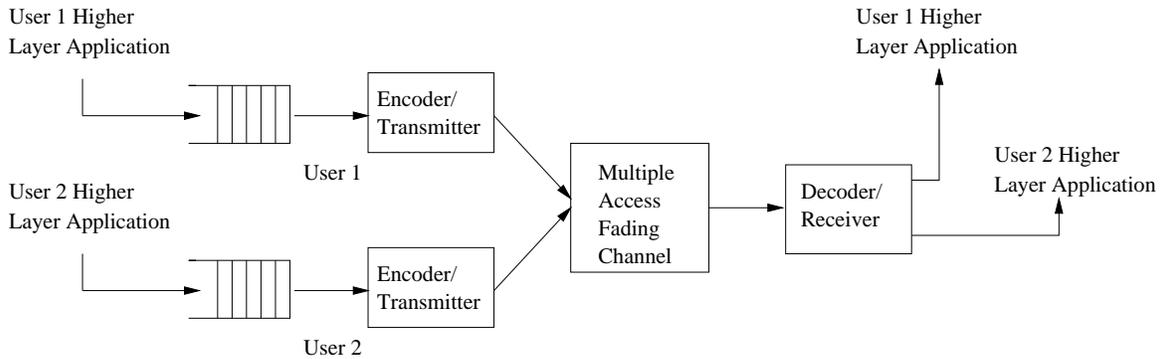


Figure 7-1: Model for 2 user multiple-access system.

the codeword lengths, once again these capacity regions may not give a meaningful indication of the achievable performance. In such cases, capacity vs. outage and delay-limited capacity have been considered for the block fading, multiple-access channel. For example, the delay-limited capacity of such channels has been looked at in [HT98]. Both delay-limited capacity and capacity vs. outage for the multiple-access case are natural extensions of the ideas in Ch. 3. As in the single user case, the usefulness of these concepts strongly depends on the time scale of the fading and the quality of service required by the users. For more general classes of service requirements and fading time scales, we consider a generalization of the single user buffer model from Ch. 4.

7.1.2 Buffer model

In this section we generalize the single user buffer model to the multiple-access case. We only consider a model with two users; the generalization to more than two users follows directly. The situation we wish to model is illustrated in Fig. 7-1. Each user is receiving data from some higher layer protocol. The data is put into a transmission buffer, each user having its own buffer. Each user can remove some data from its own buffer, encode it and transmit it over a block fading multiple-access channel as defined in the previous section. After sufficient delay this data can be decoded and sent to corresponding peer processes at the receiver.

Once again, we consider a discrete time model for this system where the time between two adjacent samples corresponds to one block of N channel uses of the block fading channel and thus N/W seconds. Between time $n - 1$ and n assume that A_n^i bits arrive from user i 's application and are placed into user i 's buffer which has size L^i . Let U_n^i be the number of bits taken out of user i 's buffer at the start of the n th time slot and let S_n^i denote the occupancy of the buffer at the start of this block (before the U_n^i bits are removed). Thus for $i = 1, 2$, user i 's buffer dynamics are given by:⁴

$$S_{n+1}^i = \min\{\max\{S_n^i + A_n^i - U_n^i, A_n^i\}, L^i\}. \quad (7.6)$$

We refer back to Fig. 4-2 for an illustration of this.

Let $\mathbf{A}_n = (A_n^1, A_n^2)$ be the joint arrival process. In this chapter, as in Ch. 6 we assume that the arrivals are memoryless⁵ with state space $\mathcal{A}^1 \times \mathcal{A}^2$, that is $\{\mathbf{A}_n\}$ is an i.i.d. sequence with $A_n^i \in \mathcal{A}^i$ for all n and $i = 1, 2$. Assume that for $i = 1, 2$ \mathcal{A}^i is a compact subset of \mathbb{R}^+ . Let $A_{\max}^i = \sup \mathcal{A}^i$, $A_{\min}^i = \inf \mathcal{A}^i$, and let $(\bar{A}^1, \bar{A}^2) = \mathbb{E}\mathbf{A}_n$.

We are interested in situations where the transmitter can adjust the transmission rate and power based on its knowledge of the current state. We will model this system so that the state at time n is the 4-tuple $(S_n^1, S_n^2, G_n^1, G_n^2)$ consisting of both users' buffer occupancies as well as both users' channel states (since we are assuming memoryless arrivals, we don't need to include (A_n^1, A_n^2) in the state). Assume that each user as well as the receiver have perfect and complete CSI. Also assume that each user has complete knowledge of the joint buffer state. In other words, each transmitter knows both its own channel and buffer state as well as those of the other user. Again, this is only one of several assumptions that may be appropriate depending on the over all architecture and clearly it is an idealized assumption. Note that with a constant arrival rate and perfect channel state information, each transmitter could calculate the other's buffer state for all time, by simply knowing the other user's transmission policy. Without this assumption, gaining this knowledge would require each user to forward its buffer state to the receiver and then for the receiver to relay

⁴As in the single user case, if $U_n^i \leq S_n^i$ for all n then the max is not needed.

⁵While we assume that $\{\mathbf{A}_n\}$ is memoryless, we do not need to assume that the processes $\{A_n^1\}$ and $\{A_n^2\}$ are independent of each other. The reason for the memoryless assumption is primarily to simplify notation not due to any mathematical necessity.

this information to the other user. Equivalently, we can assume that the receiver has complete state information, makes the control decisions and forwards this information to the transmitters. This corresponds to centralized control by the receiver. These issues are discussed more in Sect. 7.2.5 below.

In the above context, we again want to analyze the trade-off between the average power needed and some other cost corresponding to delay or probability of buffer overflow. In the next section, we discuss possible “power costs” for such problems. Then in Sect. 7.1.4, we consider possible “buffer costs” and formulate a Markov decision problem.

7.1.3 Power Cost

Recall that in the single user case, if u bits are transmitted when the channel state is g , then the user incurs a cost of $P(g, u)$, which represents the amount of power required for the user to reliably transmit those bits in the given channel state. We initially considered $P(g, u)$ to be the amount of power required for the mutual information between the transmitted message and the received message over the block to be greater than u . This corresponds to requiring uW/N to be less than the capacity of the Gaussian channel with gain g and average power constraint P . We subsequently generalized to other choices of $P(g, u)$ which had similar characteristics. We follow the same program for the multiple-access case.

For a given joint channel state (g^1, g^2) , suppose that for $i = 1, 2$, user i transmits u^i bits in a given block using average power P^i . As in the single user case, assume that (P^1, P^2) must be chosen so that the pair $(u^1W/N, u^2W/N)$ lie in the multiple-access capacity region of the complex Gaussian channel with channel gains (g^1, g^2) and average power constrains $(P^1/W, P^2/W)$. In other words, (u^1, u^2) must satisfy

$$\sum_{i \in Q} u^i W/N \leq C \left(\sum_{i \in Q} |g^i|^2 P^i \right) \quad \forall Q \subset \{1, 2\}. \quad (7.7)$$

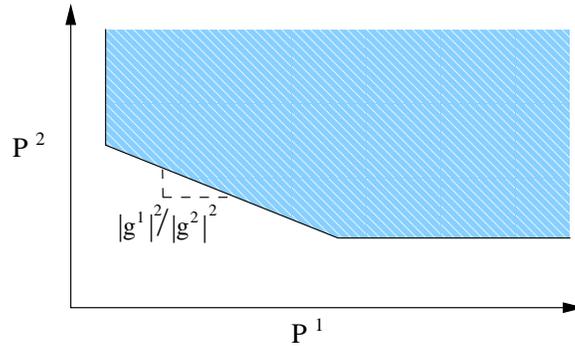


Figure 7-2: Set of all (P^1, P^2) that satisfy the constraints in (7.8).

These constraints can equivalently be written as:

$$\sum_{i \in Q} |g^i|^2 P^i \geq C^{-1} \left(\sum_{i \in Q} u^i W/N \right) \quad \forall Q \subset \{1, 2\}. \quad (7.8)$$

From this it is obvious that the set of all (P^1, P^2) which satisfy these constraints is an unbounded polyhedron⁶— an example is shown Fig. 7-2. We will refer to the edge of this polyhedron corresponding to the constraint

$$|g^1|^2 P^1 + |g^2|^2 P^2 = C^{-1}((u^1 + u^2)W/N)$$

as the *minimum power face* of this polyhedron. This face has the characteristic that for any pair (P^1, P^2) within the polyhedron, there exists a $(\tilde{P}^1, \tilde{P}^2)$ on the minimum power face, such that $\tilde{P}^i \leq P^i$ for $i = 1, 2$. Note that for any (P^1, P^2) on the minimum power face, (u^1, u^2) will lie on the dominant face of the corresponding capacity region. Also if (P^1, P^2) is at one of the corner points of this polyhedron, then (u^1, u^2) will lie at a corner point of the corresponding capacity region.

In the single user case, we set $P(g, u)$ to be the minimum power such that uW/N is less than the corresponding capacity. With two users, any point on the minimum

⁶As shown in [TH98], the set of *received* powers that satisfy these constraints is a contra-polymatroid. If $h^1 \neq h^2$, then the set of transmitted powers will not be a contra-polymatroid.

power face could be considered a “minimum” pair of powers. In the following we choose the pair which minimizes a convex combination of the two user’s powers; this minimum convex combination of the powers will be denoted by $P(g^1, g^2, u^1, u^2)$. That is for some $\lambda \in [0, 1]$, let

$$P(g^1, g^2, u^1, u^2) = \min_{P^1, P^2} \lambda P^1 + (1 - \lambda) P^2$$

$$\text{subject to: } \sum_{i \in Q} |g^i|^2 P^i \geq C^{-1} \left(\sum_{i \in Q} u^i W/N \right) \quad \forall Q \subset \{1, 2\}. \quad (7.9)$$

Notice, when $\lambda = 1/2$, $P(g^1, g^2, u^1, u^2)$ is proportional to the minimum sum power required for user i to transmit reliably at rate $u^i W/N$. Since (7.9) is a linear program, a solution (P^1, P^2) can always be found in which (P^1, P^2) are at a corner point of the polyhedron (7.8). Thus (u^1, u^2) will lie on a corner point of the dominant face of the resulting capacity region; such a rate pair can be achieved via stripping and without the need for joint decoding or rate-splitting. Also, notice that the corner point in which the solution lies depends only on λ , g^1 , and g^2 . In particular, if $\lambda = 1/2$, then the solution is always to decode the user with the better channel first. From the perspective of the user with the better channel, this requires more power than the other corner point. For an arbitrary λ , user 1 will be decoded first only when $|g^1|^2 \geq \frac{\lambda}{1-\lambda} |g^2|^2$, thus as λ increases, user 2 will be decoded first more frequently.

Notice that the right hand side of each of the constraints in (7.9) is convex and increasing in the pair (u^1, u^2) . From this observation the following proposition follows directly.

Proposition 7.1.1 $P(g^1, g^2, u^1, u^2)$ is strictly convex in (u^1, u^2) , and if $u^1 < \tilde{u}^1$ and $u^2 < \tilde{u}^2$ then $P(g^1, g^2, u^1, u^2) < P(g^1, g^2, \tilde{u}^1, \tilde{u}^2)$.

Extending the results of Ch. 6 will rely only on these characteristics. Thus the following derivations also apply to other definitions of P with these same characteristics. Therefore, any function P which satisfies Prop. 7.1.1 is defined to be a *good power function* for the multiple-access case. Next we discuss some other possible candidates for $P(g^1, g^2, u^1, u^2)$.

The following is one example of another power function which is also good. This example corresponds to *dynamic allocation* of bandwidth between users. Specifically, assume that during each block of N channel uses a $\zeta^1 \in [0, 1]$ is chosen. One user is allowed to transmit using a bandwidth of $\zeta^1 W$ and the other user is allowed to transmit using the disjoint bandwidth of $(1 - \zeta^1)W$. Let $P(g^1, g^2, u^1, u^2)$ be the minimum convex combination of the power needed for the expected mutual information between user i and the receiver to be $u^i W/N$ in this situation. In other words, for some $\gamma \in (0, 1)$

$$\begin{aligned}
 P(g^1, g^2, u^1, u^2) &= \min_{P^1, P^2, \zeta^1} \gamma P^1 + (1 - \gamma) P^2 \\
 \text{subject to:} \quad & u^i W/N \leq \zeta^i W \log \left(1 + \frac{|g^i|^2 P^i}{\zeta N_0 W} \right) \quad \forall i \in \{1, 2\} \\
 & \zeta^1 + \zeta^2 = 1, \quad \zeta^i \in [0, 1]
 \end{aligned} \tag{7.10}$$

Note that the quantities of the left-hand side of the constraints in (7.10) are strictly concave and increasing functions of P^1 and P^2 . From this it again follows that P is a good power function. This formulation also holds if instead we assume that the available time-slot is dynamically allocated between users. Recall if ζ^1 is fixed for all time slots, then, as stated in the introduction, the problem decouples into two single user problems.

In the formulation in (7.10) we are still assuming that the mutual information during a slot is a good indication of the rate at which data can be reliably transmitted - as we have argued previously, this will be true provided the number of degrees of freedom available to each user is large. When ζ^1 is small enough, then the number of degrees of freedom available to the first user will also be small. This seems to indicate a problem with the above formulation. One way to avoid this is to assume that instead of sharing the available bandwidth during each block, user 1 is allowed to transmit in a block with probability ζ and otherwise user 2 is allowed to transmit. Thus when ever a user is transmitting, he can use the entire slot, and therefore has all of the degrees of freedom available. In this case, u^1 and u^2 are the expected number of bits transmitted by each user under such a randomized policy. The results in the following section can be modified to hold with this generalization.

One common situation that does not result in a good power function is the following. Suppose that each user is constrained to use a single user Gaussian codebook. Furthermore, assume the the receiver is constrained to receive and decode each user separately, treating the other user as noise. This models a situation in which CDMA is used with single user detectors and single user decoding. Let $P(g^1, g^2, u^1, u^2)$ be the minimum total power needed for the mutual information rate between user i and the receiver to be $u^i W/N$ in this situation. In other words,

$$\begin{aligned}
 P(g^1, g^2, u^1, u^2) &= \min_{P^1, P^2} P^1 + P^2 \\
 \text{subject to:} \quad u^1 W/N &\leq W \log \left(1 + \frac{|g^1|^2 P^1}{N_0 W + |g^2|^2 P^2} \right) \\
 u^2 W/N &\leq W \log \left(1 + \frac{|g^2|^2 P^2}{N_0 W + |g^1|^2 P^1} \right)
 \end{aligned} \tag{7.11}$$

For a given sum power \bar{P} and joint channel state (g^1, g^2) , the set of rates (u^1, u^2) for which there exists a solution to (7.11) with $P(g^1, g^2, u^1, u^2) \leq \bar{P}$, will always have the structure shown in Fig. 7-3. From this it follows that (7.11) does not define a good power function.

Next consider the situation in (7.11) but assume that the transmitters can use a randomized policy as discussed above. Now let $P(g^1, g^2, u^1, u^2)$ be the minimum total power needed for the expected mutual information rate between user i and the receiver to be $u^i W/N$ using a randomized policy, when the channel state is (g^1, g^2) . From Fig. 7-3, it is apparent that when a randomized policy is allowed, this problem reduces to (7.10). In other words, with single user detectors, only one user should optimally transmit during each block.⁷

As in the single user case, P can also be defined to correspond to a given variable rate modulation scheme or a given bound on the power needed for a fixed probability of error. For example, suppose the channel is to be dynamically allocated between the two users as in (7.10). However, now assume than when a given user is transmitting, a

⁷This means that using a randomized allocation with FDMA or TDMA is always “better” than using CDMA, with single user detectors/decoding. Of course, one reason for using a CDMA systems is to avoid having to do the scheduling required for dynamic allocation. Also, we are only considering flat fading channels, and thus don’t see the multi-path diversity advantages of a CDMA system.

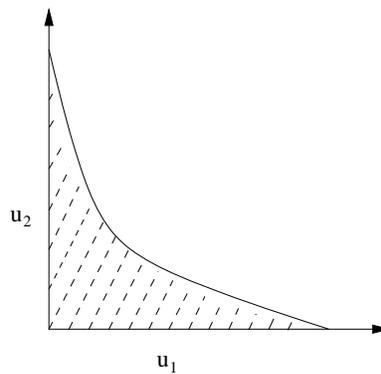


Figure 7-3: The shaded region in this figure indicates the “capacity region” corresponding to (7.11) with a fixed total power.

variable rate modulation scheme is used which has a good single user power function. In this case, defining the multi-user power function as in (7.10) results in a multi-user good power function.

Finally, one could consider a model with a fixed number of codewords, as in Sect. 4.3, where each codeword takes a variable length of time to transmit. There are several difficulties in extending the model from Sect. 4.3 to this situation, *i.e.*, a model where each transmitter stops transmitting a given codeword when enough “exponent” has been received. We will not consider such models in the following, but wish to point out some of these difficulties. First, the amount of exponent required depends on the decoding method. If joint decoding is used, then there are three error exponents, corresponding to the three possible error types [Gal85] and one of the exponents depends on both users’ transmitted signals. This makes any natural extension of the single user model difficult. If stripping is used instead of joint decoding, then there are only two error exponents – one corresponding to each user. However, in this case the amount of exponent received during a block depends upon the stripping order. From the above discussion, to minimize the sum power, it would be desirable to vary the stripping order from block to block, depending on the channel gains - but this can not be done until a user is decoded. One possible way around this is to split each user into two “virtual users” – one corresponding to each stripping order. In this

case some of the ideas in [Yeh00] may be useful. Finally with stripping, the second user's codeword can not be decoded until after the first user's codeword is decoded. Thus the delay experienced by the second user depends not only on its "transmission rate" but on the other user's action. This is further complicated by the fact, that the codewords from each user will, in general, not be block synchronized.

Now we define the analogue to $\mathcal{P}_a(\bar{A})$ (cf. (5.20)) for the multiple-access case. Let ψ^1 and ψ^2 be functions from $\mathcal{G}^1 \times \mathcal{G}^2$ to \mathbb{R}^+ . These functions are to be interpreted as rate allocations which only depend on the joint channel state. For any good power function P let

$$\begin{aligned} \mathcal{P}_a(\bar{A}^1, \bar{A}^2) &= \min_{\psi^1, \psi^2} \mathbb{E}_{\mathbf{G}} P(\mathbf{G}, \psi^1(\mathbf{G}), \psi^2(\mathbf{G})) \\ \text{subject to: } \mathbb{E}_{\mathbf{G}} \psi^1(\mathbf{G}) &= \bar{A}^1 \\ \mathbb{E}_{\mathbf{G}} \psi^2(\mathbf{G}) &= \bar{A}^2. \end{aligned} \tag{7.12}$$

The solution to this optimization problem is the minimum convex combination of power needed for each user i to transmit at average rate \bar{A}^i , without any delay constraints. Some characteristics of the function $\mathcal{P}_a(\bar{A}^1, \bar{A}^2)$ are summarized next.

Proposition 7.1.2 $\mathcal{P}_a(\bar{A}^1, \bar{A}^2)$ is a strictly convex function of \bar{A}^1, \bar{A}^2 and if $\bar{A}^1 < \tilde{A}^1$ and $\bar{A}^2 < \tilde{A}^2$ then $\mathcal{P}_a(\bar{A}^1, \bar{A}^2) < \mathcal{P}_a(\tilde{A}^1, \tilde{A}^2)$.

These follow from the corresponding properties of $P(g^1, g^2, u^1, u^2)$ by similar arguments to those in Ch. 6 (cf. Thm. 6.1.1).

7.1.4 Buffer cost & Markov decision formulation

Let \mathcal{S}^i denote the buffer state space of user i . We assume that at each time n , each user i chooses U_n^i based on a stationary policy $\mu^i : \mathcal{S}^1 \times \mathcal{S}^2 \times \mathcal{G}^1 \times \mathcal{G}^2 \mapsto \mathbb{R}^+$. Let $\boldsymbol{\mu} = (\mu^1, \mu^2)$ denote the joint policy of the two users. Under such a policy, the

expected time average power⁸ is given by

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E}(P(G_n^1, G_n^2, \mu^1(S_n^1, S_n^2, G_n^1, G_n^2), \mu^2(S_n^1, S_n^2, G_n^1, G_n^2))). \quad (7.13)$$

This quantity is denoted by \bar{P}^μ . To simplify the notation we will often write an expression such as (7.13) as

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E}(P(\mathbf{G}_n, \boldsymbol{\mu}(\mathbf{S}_n, \mathbf{G}_n)))$$

where $\mathbf{S}_n = (S_n^1, S_n^2)$. As in the single user case, we are interested in minimizing this quantity but also in minimizing some measure of delay or buffer overflow.

Let $b^i : \mathcal{S}^i \mapsto \mathbb{R}^+$ be the “buffer cost” for user i . As in the single user case, we assume that $b^i(s)$ is a non-decreasing, convex function of s . We could also consider a buffer costs which are a function of s , a and u , as in Ch. 4, but for the sake of simplicity, we restrict our attention to costs which only depend on s . For a given policy $\boldsymbol{\mu}$, the expected time average buffer cost for user i is given by

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \mathbb{E}b^i(S_n^i). \quad (7.14)$$

We denote this quantity by $\bar{b}^{\boldsymbol{\mu}, i}$. Again, the two prototypical examples are

$$b^i(s) = \mathbf{1}_{L^i}(s) = \begin{cases} 1 & \text{if } s = L^i \\ 0 & \text{otherwise} \end{cases} \quad (7.15)$$

and $b^i(s) = s/\bar{A}^i$. In the first case $\bar{b}^{\boldsymbol{\mu}, i}$ corresponds to the fullness probability of user i 's buffer and in the later case $\bar{b}^{\boldsymbol{\mu}, i}$ corresponds to the average buffer delay (assuming $L^i \rightarrow \infty$). Once again, there is a trade-off between minimizing the average power in (7.13) and the average buffer costs for each user. This can be thought of as a multi-

⁸More precisely we should say “the expected time average convex combination of the user’s transmitted powers”, but we will shorten this when it is clear that we are referring to the convex combination of the transmitted power.

objective optimization problem with three objective functions - or by considering each users power separately, a problem with four objectives. We consider a Markov decision problem, where the cost is a weighted combination of these objectives. In other words, we want to find a policy μ which minimizes

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \mathbb{E} \left(\sum_{n=1}^m P(\mathbf{G}_n, \mu(\mathbf{S}_n, \mathbf{G}_n)) + \beta^1 b^1(S_n^1) + \beta^2 b^2(S_n^2) \right) \quad (7.16)$$

where $\beta^1 > 0$ and $\beta^2 > 0$ are weighting factors.⁹ This is the sum of the average power used plus β^1 times the average buffer cost of user 1 plus β^2 times the average buffer cost of user 2.

Much of the analysis in Ch. 5 can be repeated for this problem. For example, if the state space is finite and no overflows occur, then it can be shown that there must exist an optimal policy such that each user's transmission rate will be non-decreasing in its own buffer state, if the other user's buffer state is fixed. Also an optimal power/delay curve for the multi-access problem can be defined as in Sect. 5.3. In this case we define $P^*(B^1, B^2)$ to be the minimum of \bar{P}^μ over all policies such that $\bar{b}^{\mu,i} \leq B^i$ for $i = 1, 2$. When both user's have infinite buffers it can be shown that this is a convex function of (B^1, B^2) and non-increasing, meaning that if $B^i \geq \hat{B}^i$, for $i = 1, 2$, then $P^*(B^1, B^2) \leq P^*(\hat{B}^1, \hat{B}^2)$. Thus every point on this surface can be found by solving (7.16) for some choice of β^1 and β^2 ; these correspond to all the Pareto optimal points of interest.

7.2 Asymptotic Analysis for Multiple-Access Model

In this section, we analyze an asymptotic version of the multiple-access buffer control problem. Specifically, we show that the results from Sect. 6.2 can be generalized to the multiple-access case. Recall, in Sect. 6.2 we considered the single user problem where the buffer cost corresponded to probability of buffer overflow; we examined the limiting case as $L \rightarrow \infty$. Here, we consider the multiple-access model, where again each user's buffer cost corresponds to probability of overflow and we let both user's

⁹as in the single user case these can also be interpreted as Lagrange multipliers.

buffer sizes grow. More precisely, assume that user i has a buffer of size L^i and $b^i(s)$ is given (7.15). We then let $L^1 \rightarrow \infty$ and $L^2 \rightarrow \infty$ while keeping $L^1/L^2 = \alpha$. To simplify notation, let $L^2 = L$ and thus $L^1 = \alpha L$; the above limits will then be denoted by $L \rightarrow \infty$.

Let $\mathcal{S}_L = [0, L]$ indicate a buffer state space of size L . Thus when user i has a buffer of size L^i , $\mathcal{S}^i = \mathcal{S}_{L^i}$. Let $\boldsymbol{\mu}_L$ be a policy for the buffer state space $\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}$. The average cost under this policy is $\bar{P}^{\boldsymbol{\mu}_L} + \beta^1 \bar{b}^{\boldsymbol{\mu}_L, 1} + \beta^2 \bar{b}^{\boldsymbol{\mu}_L, 2}$. In this section we look at the behavior of this quantity as $L \rightarrow \infty$.

Assume that under every $\boldsymbol{\mu}$, the resulting Markov chain $\{(S_n^1, S_n^2, G_n^1, G_n^2)\}$ is ergodic. Thus, there is a unique steady-state distribution on the joint state space; this is denoted by $\pi_{\mathbf{S}, \mathbf{G}}^{\boldsymbol{\mu}}$. Let $\pi_{\mathbf{S}}^{\boldsymbol{\mu}}$ denote the marginal steady-state distribution on the joint buffer state space and let $\pi_{S^i}^{\boldsymbol{\mu}}$ denote the marginal steady-state distribution for user i 's buffer.¹⁰ Then, $\bar{b}^{\boldsymbol{\mu}_L, i} = \pi_{S^i}^{\boldsymbol{\mu}_L}(L^i)$. Let $\bar{P}^{\boldsymbol{\mu}}(\mathbf{s})$ denote the average power used when the joint buffer state is $\mathbf{s} = (s^1, s^2)$ under policy $\boldsymbol{\mu}$, *i.e.*,

$$\bar{P}^{\boldsymbol{\mu}}(\mathbf{s}) = \mathbb{E}(P(\mathbf{G}, \boldsymbol{\mu}(\mathbf{S}, \mathbf{G})) | \mathbf{S} = \mathbf{s}).$$

where $(\mathbf{S}, \mathbf{G}) \sim \pi_{\mathbf{S}, \mathbf{G}}^{\boldsymbol{\mu}}$. Thus

$$\bar{P}^{\boldsymbol{\mu}} = \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \bar{P}^{\boldsymbol{\mu}}(\mathbf{s}) d\pi_{\mathbf{S}}(\mathbf{s}).$$

As in the single user case, if β^1 and β^2 are large enough, $\mathcal{P}_a(\bar{A}^1, \bar{A}^2)$ is a lower bound to the average cost for all L and all policies.¹¹ In other words,

$$\bar{P}^{\boldsymbol{\mu}} + \beta^1 \pi_{S^1}^{\boldsymbol{\mu}_L}(L^1) + \beta^2 \pi_{S^2}^{\boldsymbol{\mu}_L}(L^2) \geq \mathcal{P}_a(\bar{A}^1, \bar{A}^2)$$

As $L \rightarrow \infty$ this bound is achievable. Following Sect. 6.2 we bound the rate at which this quantity can be approached and then demonstrate a sequence of strategies whose rate is near this bound. Before doing this we establish some useful notation

¹⁰It would be more consistent with our previous notation to denote this by $\pi_{S_{L^i}}^{\boldsymbol{\mu}}$, but the above notation is somewhat more compact.

¹¹In the following it will always be assumed that β^1 and β^2 are large enough for this to hold.

and preliminary results.

7.2.1 Preliminaries

The notation and results in this section deal primarily with vectors in \mathbb{R}^2 . These ideas extend to higher dimensional Euclidean space in a direct manner.

The vector $(1, 1)'$ is denoted by \mathbf{e} , the zero vector is denoted by $\mathbf{0}$, and the Euclidean norm on \mathbb{R}^2 is denoted by $\|\cdot\|$. Let $\mathbf{x} = (x^1, x^2)'$ and $\mathbf{y} = (y^1, y^2)'$ be two vectors in \mathbb{R}^2 . We write $\mathbf{x} < \mathbf{y}$ if and only if $x^1 < y^1$ and $x^2 < y^2$. This relation partially orders \mathbb{R}^2 . This ordering is compatible with addition on \mathbb{R}^2 , *i.e.*, if $\mathbf{x} < \mathbf{y}$ then $\mathbf{x} + \mathbf{z} < \mathbf{y} + \mathbf{z}$ for any $\mathbf{z} \in \mathbb{R}^2$. Also note that if $\mathbf{x} > \mathbf{0}$ and $\mathbf{y} > \mathbf{0}$ then $\mathbf{x}'\mathbf{y} > 0$, where $\mathbf{x}'\mathbf{y} = x^1y^1 + x^2y^2$ is the usual inner product in \mathbb{R}^2 . Related notation such as $\mathbf{x} \leq \mathbf{y}$ is taken to have the obvious meaning.

Another order relation on \mathbb{R}^2 will be useful. Define $\mathbf{x} \prec \mathbf{y}$ if and only if $\mathbf{x}'\mathbf{e} < \mathbf{y}'\mathbf{e}$. This relation linearly orders \mathbb{R}^2 and is also compatible with addition. If $\mathbf{x} < \mathbf{y}$ then $\mathbf{x}'\mathbf{e} < \mathbf{y}'\mathbf{e}$, while the converse is not always true.

The notion of a projection of a vector onto a closed, convex subset of \mathbb{R}^2 will also be useful. Specifically, Let C be such a subset and let $\mathbf{x} \in \mathbb{R}^2$ be an arbitrary vector. A vector $\mathbf{y} \in C$ is the *projection* of \mathbf{x} onto C if $\|\mathbf{x} - \mathbf{y}\| = \inf\{\|\mathbf{x} - \mathbf{w}\| : \mathbf{w} \in C\}$. This is a well defined operation as stated in the following theorem:

Theorem 7.2.1 *Let C be a closed, convex subset of \mathbb{R}^2 . For all $\mathbf{x} \in \mathbb{R}^2$, there exists a unique vector $\mathbf{y} \in C$ such that \mathbf{y} is the projection of \mathbf{x} onto C .*

This is one part of the *projection theorem* for convex sets. We refer to [Ber99] for a proof. Note that when C is a subspace, this is the finite dimensional version of the Hilbert space projection theorem.

7.2.2 Bound on rate of convergence

For $\bar{P}^{\mu_L} + \beta^1 \pi_{S^1}^{\mu_L}(L) + \beta^2 \pi_{S^2}^{\mu_L}(\alpha L) \rightarrow \mathcal{P}_a(\bar{A}^1, \bar{A}^2)$, it must be that both $\bar{P}^{\mu_L} \rightarrow \mathcal{P}_a(\bar{A}^1, \bar{A}^2)$ and $\beta^1 \pi_{S^1}^{\mu_L}(L) + \beta^2 \pi_{S^2}^{\mu_L}(\alpha L) \rightarrow 0$. As in the single user case, we bound the overall rate of convergence by considering these terms separately. We restrict ourselves to the following class of admissible buffer control policies.

Definition: A sequence of stationary policies, $\{\boldsymbol{\mu}_L\}$ is *admissible* if it satisfies the following conditions:

1. Under every policy $\boldsymbol{\mu}_L$, $\{(\mathbf{S}_n, \mathbf{G}_n)\}$ is an ergodic Markov chain.
2. There exists an $\epsilon > 0$, a $\delta > 0$ and a $M > 0$ such that for all $L > M$ and for all $(s^1, s^2) \in \mathcal{S}_L \times \mathcal{S}_{\alpha L}$:

$$\Pr((\mathbf{A} - \boldsymbol{\mu}_L(s^1, s^2, \mathbf{G}))' \mathbf{e} > \delta | \mathbf{S}^{\mu_L} = (s^1, s^2)) > \epsilon.$$

where $\mathbf{A} \sim \pi_{\mathbf{A}}$, and \mathbf{S}^{μ_L} and \mathbf{G} are random variables whose joint distribution is $\pi_{\mathbf{S}, \mathbf{G}}^{\mu_L}$, the steady-state distribution of $\{(\mathbf{S}_n, \mathbf{G}_n)\}$ under the policy $\boldsymbol{\mu}_L$.

The second requirement means that for large enough L , there is a positive steady-state probability that the sum of the buffer states increases for any joint buffer state. This generalizes the definition of an admissible sequence of policies considered in Sect. 6.2. If $\Pr(\mathbf{G} = \mathbf{0}) > \epsilon$ and $A_{min} > 0$, the above condition will hold for any L .

Recall, we are assuming that the arrival process $\{\mathbf{A}_n\}$ is memoryless, but, in this section, we do allow the channel process to have memory. Also, we assume that at $\mathbf{x} = (\bar{A}^1, \bar{A}^2)$ both $\nabla \mathcal{P}_a(\mathbf{x})$ and $\nabla^2 \mathcal{P}_a(\mathbf{x})$ exist.

The bound on the rate of convergence is given in the following theorem:

Theorem 7.2.2 *For any admissible sequence of policies $\{\boldsymbol{\mu}_L\}$, the rate of convergence of the average power required and the buffer overflow probability have the following relationship: if $\beta^1 \pi_{S^1}^{\mu_L}(L) + \beta^2 \pi_{S^2}^{\mu_L}(\alpha L) = o((1/L)^2)$, then $\bar{P}^{\mu_L} - \mathcal{P}(\bar{A}^1, \bar{A}^2) = \Omega((1/L)^2)$.*

Before proving Thm. 7.2.2 we first prove the following two lemmas. These lemmas are analogous to the corresponding lemmas in Sect. 6.2 which were used in the proof of the Thm. 6.2.1. For $\mathbf{s} \in \mathcal{S}_{L^1} \times \mathcal{S}_{L^2}$, let $\Delta^{\mu_L}(\mathbf{s}) = (\bar{A}^1, \bar{A}^2) - \mathbb{E}(\boldsymbol{\mu}_L(\mathbf{S}^{\mu_L}, \mathbf{G}) | \mathbf{S}^{\mu_L} = \mathbf{s})$. This is the expected drift in both buffers when the joint buffer state is \mathbf{s} and any overflows are ignored. The first lemma bounds the average of $\Delta^{\mu_L}(\mathbf{s})$ over the joint buffer state space.

Lemma 7.2.3 *For any stationary policy, $\boldsymbol{\mu}$,*

$$\mathbf{0} \leq \left(\int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \Delta^\mu(\mathbf{s}) d\pi_{\mathbf{S}}^\mu(\mathbf{s}) \right) \leq (A_{max}^1 \pi_{S^1}^\mu(L^1), A_{max}^2 \pi_{S^2}^\mu(L^2))$$

PROOF. Let $\mathbf{S}_n = (S_n^1, S_n^2)$ denote the joint buffer process. Likewise, let $\mathbf{U}_n = (U_n^1, U_n^2)$ be the number of bits transmitted. Consider the following decomposition of \mathbf{S}_n . Let $\mathbf{F}_n = \mathbf{A}_n - \mathbf{U}_{n-1}$ be the net change in the buffer occupancy, ignoring overflows, between time $n-1$ and n . For $i = 1, 2$, let $E_n^i = \sum_{m=0}^{n-1} [S_m^i + A_{m+1}^i - U_m^i - L^i]^+$; this is the total number of the i th user's bits lost due to overflows up till time n . Assuming that the buffer is empty at $n = 0$, we have

$$\mathbf{S}_n = \sum_{m=1}^n \mathbf{F}_m - (E_n^1, E_n^2), \quad (7.17)$$

and thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \mathbf{S}_n = \lim_{n \rightarrow \infty} \left(\mathbb{E} \frac{1}{n} \sum_{m=1}^n \mathbf{F}_m - \mathbb{E} \frac{1}{n} (E_n^1, E_n^2) \right). \quad (7.18)$$

By similar arguments to those in the proof of Lemma 6.2.2, it follows that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \mathbf{S}_n = (0, 0), \quad (7.19)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \sum_{m=1}^n \mathbf{F}_m = \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \Delta^\mu(\mathbf{s}) d\pi_{\mathbf{S}}^\mu(\mathbf{s}), \quad (7.20)$$

and,

$$\mathbf{0} \leq \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} (E_n^1, E_n^2) \leq (A_{max}^1 \pi_{S^1}^\mu(L^1), A_{max}^2 \pi_{S^2}^\mu(L^2)). \quad (7.21)$$

After substituting these into (7.18), the desired relationship follows. ■

The next lemma bounds the drift, averaged over an appropriate “tail” region of the buffer. From this lemma it follows that if $\beta^1 \pi_{S^1}^{\boldsymbol{\mu}_L}(L) + \beta^2 \pi_{S^2}^{\boldsymbol{\mu}_L}(\alpha L) = o((1/L)^2)$, then $[\Delta^{\boldsymbol{\mu}_L}(\mathbf{s})]'\mathbf{e}$ averaged over this tail region must eventually become negative (recall we assume that $L^1 = L$ and $L^2 = \alpha L$).

Lemma 7.2.4 *Let $\{\boldsymbol{\mu}_L\}$ be an admissible sequence of policies, and let M , ϵ and δ*

be as in the definition of admissibility. Then for all $L > M$, there exists a $s_L \in (0, L^1 + L^2]$ such that

$$\int_{\{\mathbf{s}: \mathbf{s}'\mathbf{e} > s_L\}} [\Delta^{\mu_L}(\mathbf{s})]' \mathbf{e} d\pi_{\mathbf{S}^L}^{\mu_L}(\mathbf{s}) \leq \frac{-\epsilon\delta^2}{4(L^1 + L^2)} + A_{max}^1 \pi_{S^1}^{\mu_L}(L^1) + A_{max}^2 \pi_{S^2}^{\mu_L}(L^2).$$

PROOF. Assume that $L > M$. Without loss of generality, assume that δ divides $L^1 + L^2 = (1 + \alpha)L$. Consider partitioning the joint buffer state $\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}$ into $m = 2(L^1 + L^2)/\delta$ disjoint regions as follows. The first region is $\mathcal{Q}_1 = \{\mathbf{s} : \mathbf{s}'\mathbf{e} \leq \delta/2\}$ and for $i = 2, 3, \dots, m$, the i th region, \mathcal{Q}_i will be given by $\mathcal{Q}_i = \{\mathbf{s} : (i-1)\delta/2 < \mathbf{s}'\mathbf{e} \leq i\delta/2\}$. Let c be the index of a region which has maximal probability with respect to $\pi_{\mathbf{S}^L}^{\mu_L}$, so that

$$\pi_{\mathbf{S}^L}^{\mu_L}(\mathcal{Q}_c) \geq \frac{1}{m} = \frac{\delta}{2(L^1 + L^2)}. \quad (7.22)$$

Let $s_L = c\delta/2$. Consider the convex set $\{\mathbf{s} : \mathbf{s}'\mathbf{e} \geq s_L\} \cap (\mathcal{S}_{L^1} \times \mathcal{S}_{L^2})$. Define the new process $\{\hat{\mathbf{S}}_n\}$ to be the projection of $\{\mathbf{S}_n\}$ onto this set (cf. Thm. 7.2.1). Figure 7-4 illustrates these steps.

Let \mathbf{F}_n and (E_n^1, E_n^2) be defined as in the previous lemma. Let $\hat{\mathbf{F}}_n$ be defined by:

$$\hat{\mathbf{F}}_n = \begin{cases} \mathbf{F}_n & \text{if } \hat{\mathbf{S}}'_{n-1}\mathbf{e} > s_L \text{ and } \hat{\mathbf{S}}'_n\mathbf{e} > s_L \\ \mathbf{F}_n + \mathbf{S}_{n-1} - \hat{\mathbf{S}}_{n-1} & \text{if } \hat{\mathbf{S}}'_{n-1}\mathbf{e} = s_L \text{ and } \hat{\mathbf{S}}'_n\mathbf{e} > s_L \\ \hat{\mathbf{S}}_n - \hat{\mathbf{S}}_{n-1} & \text{otherwise.} \end{cases} \quad (7.23)$$

Thus $\hat{\mathbf{F}}_n$ is the net change in $\hat{\mathbf{S}}_n$ ignoring any overflows when $\mathbf{S}'_n\mathbf{e} > s_L$. Let $(\hat{E}_n^1, \hat{E}_n^2) = \sum_{m=1}^n \hat{\mathbf{F}}_m - \hat{\mathbf{S}}_n$, so \hat{E}_n^i is the number of bits lost due to overflow by the i th component of the $\hat{\mathbf{S}}_n$ process until time n . Thus we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{\mathbf{S}}_n = \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=1}^n \hat{\mathbf{F}}_m - \lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} (\hat{E}_n^1, \hat{E}_n^2) \quad (7.24)$$

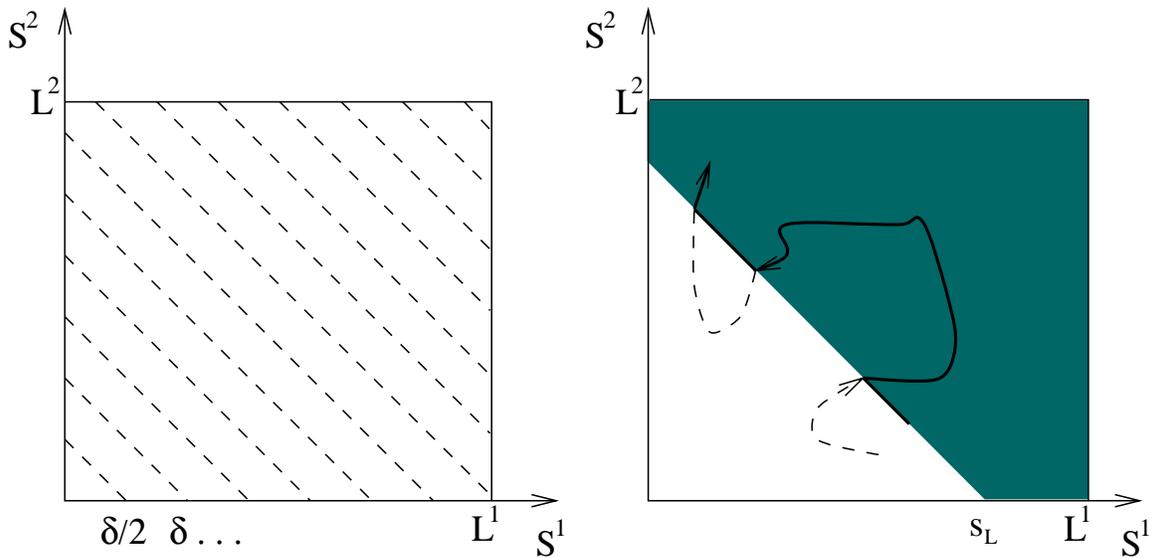


Figure 7-4: This figure illustrates some steps in the proof of Lemma 7.2.4. On the left, the partitioning of $\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}$ into the m regions, $\{\mathcal{Q}_i\}$ is illustrated. On the right a sample trajectory of the \mathbf{S}_n process is shown as a dashed line and a sample trajectory of the $\hat{\mathbf{S}}_n$ process is shown as a solid line. The shaded region, is the convex set $\{\mathbf{s} : \mathbf{s}'\mathbf{e} \geq s_L\} \cap (\mathcal{S}_{L^1} \times \mathcal{S}_{L^2})$.

Once again, using similar arguments to the proof of Lemma 6.2.3, we have:

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{\mathbf{S}}_n = \mathbf{0} \quad (7.25)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \sum_{m=0}^{n-1} \hat{\mathbf{F}}_m = \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \quad (7.26)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{n} \hat{E}_n^i \leq A_{max}^i \pi_{S^i}^{\mu_L}(L^i) \quad \text{for } i = 1, 2. \quad (7.27)$$

Here, $\lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})$ is the steady state expected value of $\hat{\mathbf{F}}_n$ conditioned on $\mathbf{S}_{n-1} = \mathbf{s}$. Using these in (7.24) yields,

$$\int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \leq (A_{max}^1 \pi_{S^1}^{\mu_L}(L^1), A_{max}^2 \pi_{S^2}^{\mu_L}(L^2)). \quad (7.28)$$

Therefore,

$$\int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \leq A_{max}^1 \pi_{S^1}^{\mu_L}(L^1) + A_{max}^2 \pi_{S^2}^{\mu_L}(L^2). \quad (7.29)$$

Next we bound $\lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e}$ in three different cases. These corresponds to \mathbf{s} being in one of three different regions of the buffer state space.

1. First assume $\mathbf{s}' \mathbf{e} > s_L$. In this case, if $\mathbf{F}'_n \mathbf{e} \geq 0$ then $\mathbf{F}'_n \mathbf{e} = \hat{\mathbf{F}}'_n \mathbf{e}$, and if $\mathbf{F}'_n \mathbf{e} < 0$ then $\mathbf{F}'_n \mathbf{e} \leq \hat{\mathbf{F}}'_n \mathbf{e}$. Thus

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} \geq \lim_{l \rightarrow \infty} \mathbb{E}(\mathbf{F}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} = [\Delta^{\mu_L}(\mathbf{s})]' \mathbf{e}. \quad (7.30)$$

2. Next assume $\mathbf{s} \in \mathcal{Q}_c$ (recall $c = 2s_L/\delta$). In this case $\hat{\mathbf{F}}'_l \mathbf{e}$ is a non-negative random variable and $\mathbf{F}'_l \mathbf{e} < \hat{\mathbf{F}}'_l \mathbf{e} + \delta/2$. Thus

$$\mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} = \mathbb{E}(\hat{\mathbf{F}}'_l \mathbf{e} | \mathbf{S}_{l-1} = \mathbf{s}) \quad (7.31)$$

$$\geq \delta/2 \Pr(\hat{\mathbf{F}}'_l \mathbf{e} > \delta/2 | \mathbf{S}_{l-1} = \mathbf{s}) \quad (7.32)$$

$$\geq \delta/2 \Pr(\mathbf{F}'_l \mathbf{e} > \delta | \mathbf{S}_{l-1} = \mathbf{s}). \quad (7.33)$$

The first inequality is due to the Markov inequality. Taking the limit and using that $\boldsymbol{\mu}$ is admissible we have

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} \geq \lim_{l \rightarrow \infty} \delta/2 \Pr(\mathbf{F}'_l \mathbf{e} > \delta | \mathbf{S}_{l-1} = \mathbf{s}) \geq \frac{\delta \epsilon}{2}. \quad (7.34)$$

3. Finally, for all \mathbf{s} such that $\mathbf{s}' \mathbf{e} \leq (c-1)\delta/2$,

$$\lim_{l \rightarrow \infty} \mathbb{E}(\hat{\mathbf{F}}_l | \mathbf{S}_{l-1} = \mathbf{s})' \mathbf{e} \geq 0. \quad (7.35)$$

Substituting each of these bounds into (7.29) yields:

$$\int_{\mathcal{Q}_c} \frac{\delta \epsilon}{2} d\pi_{\mathbf{S}}^{\boldsymbol{\mu}_L}(\mathbf{s}) + \int_{\{\mathbf{s}: \mathbf{s}' \mathbf{e} > s_L\}} [\Delta^{\boldsymbol{\mu}_L}(\mathbf{s})]' \mathbf{e} d\pi_{\mathbf{S}}^{\boldsymbol{\mu}_L}(\mathbf{s}) \leq A_{max}^1 \pi_{S^1}^{\boldsymbol{\mu}_L}(L^1) + A_{max}^2 \pi_{S^2}^{\boldsymbol{\mu}_L}(L^2). \quad (7.36)$$

Finally, using $\pi_{S^L}^{\boldsymbol{\mu}_L}(\mathcal{Q}_c) \geq \frac{\delta}{2(L^1+L^2)}$ the desired relationship follows. ■

We are now ready to prove Theorem 7.2.2. The basic ideas of the proof generalize those of the single user case.

PROOF. Assume that $\{\boldsymbol{\mu}_L\}$ is a sequence of admissible policies, such that $\beta^1 \pi_{S^1}^{\boldsymbol{\mu}_L}(L) + \beta^2 \pi_{S^2}^{\boldsymbol{\mu}_L}(\alpha L) = o(1/L^2)$. Since $\pi_{S^i}^{\boldsymbol{\mu}_L}(L^i)$ is non-negative for $i = 1, 2$, it follows that for any $\alpha^1 > 0$ and $\alpha^2 > 0$, $\alpha^1 \pi_{S^1}^{\boldsymbol{\mu}_L}(L) + \alpha^2 \pi_{S^2}^{\boldsymbol{\mu}_L}(\alpha L)$ is also $o(1/L^2)$. In particular, this is true when $\alpha^1 = A_{max}^1$ and $\alpha^2 = A_{max}^2$.

For each policy $\boldsymbol{\mu}_L$, the average transmission rate when $\mathbf{S}_n = \mathbf{s}$ is $(\bar{A}^1, \bar{A}^2) - \Delta^{\boldsymbol{\mu}_L}(\mathbf{s})$. Therefore,

$$\bar{P}^{\boldsymbol{\mu}_L}(\mathbf{s}) \geq \mathcal{P}_a((\bar{A}^1, \bar{A}^2) - \Delta^{\boldsymbol{\mu}_L}(\mathbf{s})).$$

Thus, the over-all average power is lower bounded by

$$\bar{P}^{\boldsymbol{\mu}_L} \geq \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \mathcal{P}_a((\bar{A}^1, \bar{A}^2) - \Delta^{\boldsymbol{\mu}_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\boldsymbol{\mu}_L}(\mathbf{s}). \quad (7.37)$$

The first order Taylor expansion of $\mathcal{P}_a(\mathbf{x})$ around $\mathbf{x} = (\bar{A}^1, \bar{A}^2)$ is

$$\mathcal{P}_a(\mathbf{x}) = \mathcal{P}_a(\bar{A}^1, \bar{A}^2) + [\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2)]'(\mathbf{x} - (\bar{A}^1, \bar{A}^2)) + Q(\mathbf{x} - (\bar{A}^1, \bar{A}^2)). \quad (7.38)$$

The remainder function $Q(\mathbf{x})$ has the following characteristics: (1) it is convex; (2) for $\mathbf{x} \neq \mathbf{0}$, $Q(\mathbf{x}) > 0$, while $G(\mathbf{0}) = 0$; and (3) $\nabla Q(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. These follow from the strict convexity and monotonicity of \mathcal{P}_a . Also note that $\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2) > 0$; this follows from Prop. 7.1.2.

Combining (7.37) and (7.38) yields

$$\begin{aligned} \bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}^1, \bar{A}^2) &\geq [\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2)]' \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} (-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \\ &\quad + \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} Q(-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}). \end{aligned} \quad (7.39)$$

Lemma 7.2.3 can be used to bound the first term on the right hand side as

$$\begin{aligned} -[\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2)]'(A_{max}^1 \pi_{\mathcal{S}^1}^{\mu_L}(L), A_{max}^2 \pi_{\mathcal{S}^2}^{\mu_L}(\alpha L)) \\ \leq [\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2)]' \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} (-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \leq 0. \end{aligned} \quad (7.40)$$

Here we used the fact that $\mathbf{x} > 0$ and $\mathbf{y} > 0$ imply $\mathbf{x}'\mathbf{y} > 0$ (see Sect. 7.2.1). Thus, it follows that $\left| [\nabla \mathcal{P}_a(\bar{A}^1, \bar{A}^2)]' \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} (-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \right| = o((1/L)^2)$.

Let s_L be as defined in Lemma 7.2.4 and let V_L denote the corresponding convex set $\{\mathbf{s} : \mathbf{s}'\mathbf{e} \geq s_L\} \cap (\mathcal{S}_{L^1} \times \mathcal{S}_{L^2})$. Using the convexity and non-negativity of Q , the second term in (7.39) can be bounded as:

$$\int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} Q(-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \geq \pi_{\mathbf{S}}^{\mu_L}(V_L) \int_{V_L} \frac{1}{\pi_{\mathbf{S}}^{\mu_L}(V_L)} Q(-\Delta^{\mu_L}(\mathbf{s})) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \quad (7.41)$$

$$\geq \pi_{\mathbf{S}}^{\mu_L}(V_L) Q\left(\int_{V_L} \frac{-\Delta^{\mu_L}(\mathbf{s})}{\pi_{\mathbf{S}}^{\mu_L}(V_L)} d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s})\right) \quad (7.42)$$

$$\geq Q\left(\int_{V_L} -\Delta^{\mu_L}(\mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s})\right). \quad (7.43)$$

Let $\mathbf{x}_L = \int_{V_L} -\Delta^{\mu_L}(\mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s})$. Taking the second order Taylor series of $Q(\mathbf{x})$ around $\mathbf{x} = \mathbf{0}$ we have:

$$Q(\mathbf{x}_L) = \frac{1}{2} \mathbf{x}_L' \nabla^2 Q(\mathbf{0}) \mathbf{x}_L + o(\|\mathbf{x}_L\|^2). \quad (7.44)$$

Since Q is strictly convex, $\nabla^2 Q(\mathbf{0})$ is positive definite, and therefore has positive real eigenvalues, λ_1, λ_2 . Assuming (without loss of generality) $\lambda_1 \leq \lambda_2$, we have: $\mathbf{x}'_L \nabla^2 Q(\mathbf{0}) \mathbf{x}_L \geq \lambda_1 \|\mathbf{x}_L\|^2$.

Note that by Lemma 7.2.4, we have for $L > M$

$$\mathbf{x}'_L \mathbf{e} \geq \frac{\epsilon \delta^2}{4(L + \alpha L)} - A_{max}^1 \pi_{S^1}^{\mu_L}(L) - A_{max}^2 \pi_{S^2}^{\mu_L}(\alpha L). \quad (7.45)$$

By assumption $A_{max}^1 \pi_{S^1}^{\mu_L}(L) + A_{max}^2 \pi_{S^2}^{\mu_L}(\alpha L) = o((1/L)^2)$. Thus there exists constants $B > 0$ and $\tilde{M} > M$ such that for $L > \tilde{M}$,

$$\mathbf{x}'_L \mathbf{e} \geq \frac{B}{L} \geq 0. \quad (7.46)$$

It follows that for $L > \tilde{M}$, $\|\mathbf{x}_L\|^2 \geq \frac{1}{2} \left(\frac{B}{L}\right)^2$, and therefore

$$G(\mathbf{x}_L) \geq \lambda_1 \frac{1}{2} \left(\frac{B}{L}\right)^2 + o\left(\left(\frac{1}{L}\right)^2\right) = \Omega\left(\left(\frac{1}{L}\right)^2\right). \quad (7.47)$$

Combining (7.47) with (7.41) thru (7.43) shows that the second term in (7.39) is $\Omega((1/L)^2)$. We showed earlier that the first term in (7.39) is $o((1/L)^2)$. Thus, we have $\bar{P}^{\mu_L} - \mathcal{P}(\bar{R}^1, \bar{R}^2) = \Omega((1/L)^2)$ as desired. ■

7.2.3 A nearly optimal simple policy

In this section we show that, as in the single user case, if the fading is memoryless then there exists a sequence of “simple” policies which achieve convergence rates near the bound in Thm. 7.2.2. To begin, we generalize the notion of a simple policy as defined for one user. A natural way to do this is described in the following. Consider partitioning the joint buffer state space, $\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}$ into the four disjoint sets:

$$\begin{aligned} V^{ll} &= [0, L^1/2) \times [0, L^2/2), & V^{lu} &= [0, L^1/2) \times [L^2/2, L^2], \\ V^{ul} &= [L^1/2, L^1] \times [0, L^2/2), & V^{uu} &= [L^1/2, L^1] \times [L^2/2, L^2]. \end{aligned}$$

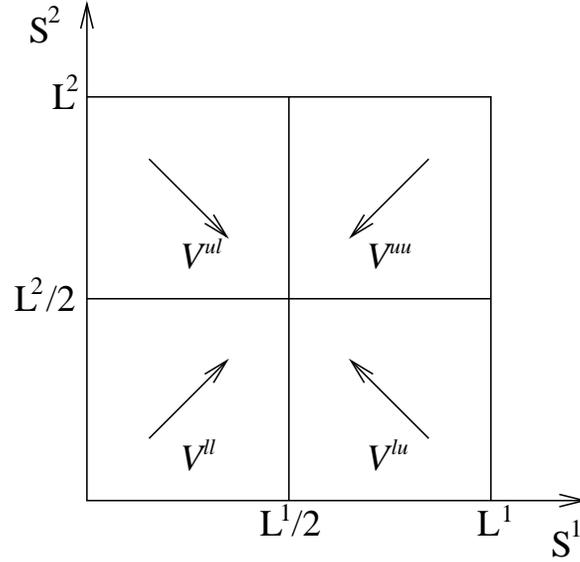


Figure 7-5: This figure illustrates a two user simple policy. The arrows illustrate the drifts, $(\bar{A}^1, \bar{A}^2) - \mathbb{E}(\Psi^{xx}(\mathbf{G}))$, in each region V^{xx} .

Corresponding to each set, V^{xx} , define a rate allocation $\psi^{xx} : \mathcal{G}^1 \times \mathcal{G}^2 \mapsto \mathbb{R}^2$ which depends only on the joint channel state. For a given (v^1, v^2) such that $0 < v^i < \bar{A}^i$ for $i = 1, 2$ define a *simple policy with drift* (v^1, v^2) to be a policy $\boldsymbol{\mu}$ such that

$$\boldsymbol{\mu}(\mathbf{g}, \mathbf{s}) = \psi^{xx}(\mathbf{g}) \text{ if } \mathbf{s} \in V^{xx},$$

and

$$\begin{aligned} \mathbb{E}(\psi^{ll}(\mathbf{G})) &= (\bar{V}^1 - v^1, \bar{V}^2 - v^2), & \mathbb{E}(\psi^{lu}(\mathbf{G})) &= (\bar{V}^1 - v^1, \bar{V}^2 + v^2), \\ \mathbb{E}(\psi^{ul}(\mathbf{G})) &= (\bar{V}^1 + v^1, \bar{V}^2 - v^2), & \mathbb{E}(\psi^{uu}(\mathbf{G})) &= (\bar{V}^1 + v^1, \bar{V}^2 + v^2). \end{aligned}$$

The rate allocation under a simple policy depends only on the joint channel state and on which of the regions V^{xx} the buffer process is in. The expected drift in each buffer state is toward the “center” of the joint buffer. Figure 7-5 illustrates such a policy.

While in one of the sets V^{xx} , the buffer process under a simple policy is again a random walk; only now it is a random walk in the plane. Following the development

for the single user case, we would now like to bound the rate of convergence of the overflow probability for a sequence of such simple policies. This presents the following difficulty. In the single user case, we bounded the overflow probability by considering a renewal process, where renewals occurred every time the buffer occupancy reached L . The reciprocal of the inter renewal time is then $\pi_{S^L}^{\mu_L}(L)$. In the multiple user case, the event of the buffer process reaching the boundary, is no longer a renewal event. Therefore we can't bound the fullness probability in the same way.

We avoid the above difficulty by considering a special class of simple policies. Specifically, define a simple policy to be *separable* if there exist real-valued functions $\psi^{u,1}$, $\psi^{u,2}$, $\psi^{l,1}$, and $\psi^{l,2}$, each defined on $\mathcal{G}^1 \times \mathcal{G}^2$, such that

$$\begin{aligned} \psi^{ll}(\mathbf{g}) &= (\psi^{l,1}(\mathbf{g}), \psi^{l,2}(\mathbf{g})), & \psi^{lu}(\mathbf{g}) &= (\psi^{l,1}(\mathbf{g}), \psi^{u,2}(\mathbf{g})), \\ \psi^{ul}(\mathbf{g}) &= (\psi^{u,1}(\mathbf{g}), \psi^{l,2}(\mathbf{g})), & \psi^{uu}(\mathbf{g}) &= (\psi^{u,1}(\mathbf{g}), \psi^{u,2}(\mathbf{g})). \end{aligned}$$

Note with such a policy, each user's rate allocation is determined by only the joint channel state and the user's own buffer state. Thus each user's individual buffer process behaves as if it were determined by a single user simple policy. However, a user's transmission power will depend not only on its own buffer state and the joint channel state, but also on the other user's buffer state. One might think that restricting our attention to such policies would limit the attainable performance. In the following, we show that a separable sequence of policies can attain performance near the bound in Thm. 7.2.2. Thus, at least asymptotically, the effect of requiring the policy to be separable is minor. The performance attainable by such a sequence is stated in the following proposition. Note this is the same growth rate that was attainable in the single user case.

Proposition 7.2.5 *For any $K \geq 2$, there exists a sequence of separable simple policies, $\{\boldsymbol{\mu}_L\}$, such that $\beta^1 \pi_{S_1^L}^{\mu_L}(L) + \beta^2 \pi_{S_2^L}^{\mu_L}(\alpha L) = o((1/L)^K)$ and $\bar{P}^{\mu_L} - \mathcal{P}(\bar{A}^1, \bar{A}^2) = O(\frac{\ln^2 L}{L^2})$.*

PROOF. First we construct a sequence of policies. Then we show that this sequence has the desired performance.

Let $\Psi^* = (\Psi^{*,1}, \Psi^{*,2}) : \mathcal{G}^1 \times \mathcal{G}^2 \mapsto \mathbb{R}^2$ be a rate allocation with average rate (\bar{A}^1, \bar{A}^2) which achieves $\mathcal{P}_a(\bar{A}^1, \bar{A}^2)$ (cf. (7.1.2)). For a given $(v^1, v^2) \in (0, \bar{A}^1) \times (0, \bar{A}^2)$, let

$$\begin{aligned}\Psi^{u,1} &= \left(\frac{\bar{A}^1 + v^1}{\bar{A}^1} \right) \Psi^{*,1}, & \Psi^{u,2} &= \left(\frac{\bar{A}^2 + v^2}{\bar{A}^2} \right) \Psi^{*,2}, \\ \Psi^{l,1} &= \left(\frac{\bar{A}^1 - v^1}{\bar{A}^1} \right) \Psi^{*,1}, & \Psi^{l,2} &= \left(\frac{\bar{A}^2 - v^2}{\bar{A}^2} \right) \Psi^{*,2}.\end{aligned}$$

This determines a separable, simple policy with drift (v^1, v^2) .

Consider a sequence of such policies $\{\boldsymbol{\mu}_L\}$ indexed by L where for $i = 1, 2$,

$$v_L^i = \frac{K \text{Var}(\Psi^{*,i}(\mathbf{G})) \ln L}{L^i}.$$

Assume that L is large enough so that $v_L^i < \bar{A}^i$. We use a subscript L to denote the component functions described above corresponding to policy $\boldsymbol{\mu}_L$, e.g. $\Psi_L^{u,1}$. As noted above, since each policy $\boldsymbol{\mu}_L$ is separable we can think of each user's buffer process as being governed by a single user, simple policy. Thus we can apply Lemma 6.2.5 for each user. This yields, for $i = 1, 2$

$$\pi_{S^i}^{\boldsymbol{\mu}_L}(L^i) = o\left(\exp\left(-\frac{1}{2} r^{*,i}(v_L^i) L^i\right)\right), \quad (7.48)$$

where $r^{*,i}(v_L^i)$ is the unique positive root of the semi-invariant moment generating function of $A^i - \Psi_L^{u,i}(\mathbf{G})$.

For any $\mathbf{g} \in \mathcal{G}^1 \times \mathcal{G}^2$, note that $\Psi^{u,1} = \left(\frac{\bar{A}^1 + v^1}{\bar{A}^1}\right) \Psi^{*,1}(\mathbf{g})$ is a continuous and differentiable function of v^1 . Using this observation, it can be shown that

$$\left. \frac{dr^{*,1}(v)}{dv} \right|_{v=0} = \frac{2}{\text{Var}(\Psi^{*,1}(\mathbf{G}))}. \quad (7.49)$$

Note this is exactly the same result given in Lemma 6.2.6, and it is proved in the same way. The corresponding result also holds for $r^{*,2}(x)$.

Using (7.49), we have

$$r^{*,1}(v_L^1) = r^{*,1}(0) + \left. \frac{dr^{*,1}(v)}{dv} \right|_{v=0} \cdot v_L^1 + O((v_L^1)^2) \quad (7.50)$$

$$= \frac{2v_L^1}{\text{Var}(\Psi^{*,1}(\mathbf{G}))} + O((v_L^1)^2), \quad (7.51)$$

where we have used that for $v^1 = 0$, $r^{*,1}(v^1) = 0$. Using the given choice of v_L^1 we have

$$r^{*,1}(v_L^1) = \frac{2K \ln L}{L} + O\left(\left(\frac{\ln L}{L}\right)^2\right). \quad (7.52)$$

Substituting this into (7.48) yields

$$\begin{aligned} \pi_{S^1}^{\mu_L}(L) &= o\left(\exp\left(-K \ln L + O\left(\frac{(\ln L)^2}{L}\right)\right)\right) \\ &= o\left(\left(\frac{1}{L}\right)^K\right). \end{aligned} \quad (7.53)$$

Similarly, for the second user, we have

$$r^{*,2}(v_L^2) = \frac{2K \ln L}{\alpha L} + O\left(\left(\frac{\ln L}{\alpha L}\right)^2\right) \quad (7.54)$$

and thus

$$\pi_{S^2}^{\mu_L}(\alpha L) = o\left(\left(\frac{1}{L}\right)^K\right) \quad (7.55)$$

Therefore,

$$\beta^1 \pi_{S^1}^{\mu_L}(L) + \beta^2 \pi_{S^2}^{\mu_L}(\alpha L) = o\left(\left(\frac{1}{L}\right)^K\right), \quad (7.56)$$

and we have the desired rate of convergence of the overflow probability. Next we show that the total power for this sequence of policies also converges at the desired rate.

Under the L th policy, for any joint buffer state $\mathbf{s} \in V^u$ and any joint channel

state, \mathbf{g} , we have

$$\Psi^l(\mathbf{g}) - \Psi^*(\mathbf{g}) = \begin{bmatrix} -v_L^1/\bar{A}^1 & 0 \\ 0 & -v_L^2/\bar{A}^2 \end{bmatrix} \Psi^*(\mathbf{g}).$$

This can be rewritten as

$$\Psi^l(\mathbf{g}) = \Psi^*(\mathbf{g}) + \mathbf{U}\Delta^{\mu_L}(\mathbf{s}), \quad (7.57)$$

where

$$\mathbf{U} = \begin{bmatrix} -\Psi^{*,1}(\mathbf{g})/\bar{A}^1 & 0 \\ 0 & -\Psi^{*,2}(\mathbf{g})/\bar{A}^2 \end{bmatrix}, \quad (7.58)$$

and $\Delta^{\mu_L}(\mathbf{s}) = (v_L^1, v_L^2)'$ is the average drift in state \mathbf{s} . The power used in this state is then $P(\mathbf{g}, \Psi^l(\mathbf{g}))$. Expanding $P(\mathbf{g}, \mathbf{u})$ in a Taylor series in \mathbf{u} about $\mathbf{u} = \Psi^*(\mathbf{g})$ we have

$$P(\mathbf{g}, \Psi^l(\mathbf{g})) = P(\mathbf{g}, \Psi^*(\mathbf{g})) + [\nabla_{\mathbf{u}}P(\mathbf{g}, \Psi^*(\mathbf{g}))]' \mathbf{U}\Delta^{\mu_L}(\mathbf{s}) + o((v_L^1)^2 + (v_L^2)^2). \quad (7.59)$$

Taking the expected value over the fading state, we have for any $\mathbf{s} \in V^l$,

$$\bar{P}^{\mu_L}(\mathbf{s}) = \mathcal{P}_a(\bar{A}^1, \bar{A}^2) + \mathbf{v}'\Delta^{\mu_L}(\mathbf{s}) + o((v_L^1)^2 + (v_L^2)^2), \quad (7.60)$$

where $\mathbf{v}' = \mathbb{E}_{\mathbf{G}}([\nabla_{\mathbf{u}}P(\mathbf{G}, \Psi^*(\mathbf{G}))]' \mathbf{U})$. Exactly the same expression holds for a buffer state in any of the other sets V^{xx} . In these cases of course, $\Delta^{\mu_L}(\mathbf{s})$ will be the appropriate drift for the region under consideration.

Next taking the expected value with respect to the buffer state, we have

$$\bar{P}^{\mu_L} - \mathcal{P}_a(\bar{A}^1, \bar{A}^2) = \mathbf{v}' \left(\int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \Delta^{\mu_L}(\mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \right) + o((v_L^1)^2 + (v_L^2)^2). \quad (7.61)$$

Recall from Lemma 7.2.3,

$$\mathbf{0} \leq \int_{\mathcal{S}_{L^1} \times \mathcal{S}_{L^2}} \Delta^{\mu_L}(\mathbf{s}) d\pi_{\mathbf{S}}^{\mu_L}(\mathbf{s}) \leq (A_{max}^1 \pi_{S^1}^{\mu_L}(L), A_{max}^2 \pi_{S^2}^{\mu_L}(\alpha L)). \quad (7.62)$$

Thus the first term in (7.61) is $o\left(\left(\frac{1}{L}\right)^K\right)$. Using our choice for (v_L^1, v_L^2) , the second term in (7.61) is $O\left(\left(\frac{\ln L}{L}\right)^2\right)$. Therefore we have

$$\bar{P}^{\mu_L} - \mathcal{P}(\bar{A}^1, \bar{A}^2) = O\left(\left(\frac{\ln L}{L}\right)^2\right) \quad (7.63)$$

as desired. ■

To summarize, we have considered a two user multiple-access model, where the buffer cost of each user corresponds to probability of buffer overflow. If each user has a constant arrival rate, this corresponds to the probability a maximum delay constraint is violated. We have shown that the single user results generalize to this situation. Specifically, we bounded the rate at which a weighted combination of the average power and each user's overflow probabilities converge as the buffer sizes increase. We have also shown that using a separable simple policy is asymptotically nearly optimal when the fading is memoryless. Let us reiterate some of the assumptions that were made in this section: First, each user had a memoryless arrival process. However, different users may have different arrival process and the processes of the two users can be correlated. Also the buffer size of each user can be different, as long as they increase proportionally. With constant arrival rates, this corresponds to each user having a different maximum delay constraint. Finally, in proving the bound on the rate of convergence, the sequence of joint channel states $\{\mathbf{G}_n\}$ was allowed to have memory. In proving that a simple policy was nearly optimal we assumed that $\{\mathbf{G}_n\}$ was an i.i.d. sequence, Again, the individual components, G_n^1 and G_n^2 , were not assumed to be either independent or identically distributed.

7.2.4 Generalizations

Now we comment on several generalizations of this model:

1. First consider a model with more than 2 user's communicating over a multiple-access channel. For M users, the power cost will be a map $P(g^1, \dots, g^M, u^1, \dots, u^M)$ which gives the minimum convex combination of power required for each user i to transmit at rate $u^i W/K$ when the joint fading state is (g^1, \dots, g^M) . A good

power function P will be convex and increasing in (u^1, \dots, u^M) . Theorem 7.2.2 generalizes directly to this case and so does Prop. 7.2.5. A separable simple policy will still be defined by dividing each user's buffer in half; this results in partitioning the joint buffer state space into 2^M subsets of \mathbb{R}^M .

2. Next one could consider generalizing the buffer cost. In the previous section, we looked at the case where the buffer cost corresponded to the probability of buffer overflow. We conjecture that the single user results for average delay should also generalize to the multiple-access model. An even more interesting situation would be to allow different user's to have different buffer costs. For example one user could have an average delay constraint and one user a maximum delay constraint. This may correspond to a situation in which one user is transmitting real-time traffic, while another is transmitting data.
3. Finally we could consider the fading process, the arrival process, or both to have memory; in other words we could assume that $\{\mathbf{G}_n\}$ and/or $\{\mathbf{A}_N\}$ is a stationary ergodic Markov chain. As noted above, the bound in Thm. 7.2.2 was proven under the assumption the arrivals are memoryless but the fading process had memory. The memoryless assumption was made for both the fading and arrival processes in proving Prop. 7.2.5. We conjecture that these assumptions are not needed for this proposition to hold. As in the single user case, generalizing this proof involves considering Markov modulated random walks.

7.2.5 Required overhead

We want to make a few comments on the overhead required to implement the multi-user policies discussed above. A general policy for M users requires knowledge of the joint channel state and the joint buffer state. Using a simple policy still requires knowledge of the joint fading state, but only 1 bit of information per user about the joint buffer state. In either case, some amount of communication overhead is required to convey this information to the necessary locations. As the number of user's gets large, this overhead may become prohibitive. In this section we look at these issues in more detail.

The first issue to be addressed is where information is initially known. This depends on the overall architecture. First consider the channel state information. As we have mentioned previously, this information may be estimated in a variety of different ways. Each transmitter could estimate its own channel state, using either a pilot signal broadcast by the receiver or data derived measurements from information received on a reverse link. Of course, for the channel state to be learned in this way, the reverse signal needs to be transmitted in the same coherence time and coherence bandwidth in which the forward communication will take place. Alternatively, the receiver could estimate the transmitter's channel state based on either a training sequence or data derived measurements.¹² Regarding the buffer state information, each transmitter has access to its own buffer state. If each user has a constant arrival rate, then any user's buffer state can be calculated by any location which has access to the joint channel state. If there are random arrival rates or imperfect state information, then the buffer state information must be communicated. In this case, simple policies clearly require much less overhead than an arbitrary policy.

The next issue is where does information need to be in order to implement the control policy. Once again the answer to this may vary with the overall architecture. First note every transmitter needs to know its own control actions, that is the power and rate at which to transmit. In general the receiver also needs to know the control action of each transmitter to reliably decode. Thus one approach is to communicate the necessary channel and buffer state information to each transmitter as well as the receiver, then each location can calculate the current control action. For each transmitter to calculate its control action, it is not generally necessary for the transmitter to know the complete state description. For example, assume that each user has identical and independent fading and identical buffer requirements. Additionally assume that $|\mathcal{G}| = m$ and that a simple policy is used. Then the transmitter only needs to know the number of users in each fading state, with buffer states in each of the two portions. For a large number of users this information could be encoded using a Huffman code, which could further reduce the required overhead. An alternative is to communicate the necessary state information to the receiver; the receiver then

¹²Another important issue is the amount of overhead required for a training sequence and/or a pilot signal. We don't address this here.

calculates the control action of each user and forwards this decision to the user. If there are only a small number of control choices and many different joint states result in the same control action, then this approach could require less overhead.

7.3 Future Directions

We conclude this chapter by mentioning two other directions in which the model in this chapter could be extended.

Other Multi-User Situations: The multi-access channel studied in this chapter is a specific example of a multi-user network. We briefly discuss extending these results to other multi-user networks. Note that the asymptotic analysis in Sect. 7.2 only required that $P(\mathbf{g}, \mathbf{u})$ be a good power function, where \mathbf{g} is a vector representing the channel gains for each user and \mathbf{u} is a vector of transmission rate. For other multi-user situations in which a function P can be defined with these characteristics, the above analysis can be repeated. For example, suppose a single transmitter is broadcasting to two users, where each user is receiving data from different applications and the data for each user is stored in a separate buffer at the transmitter. Let $P(g^1, g^2, u^1, u^2)$ be the minimum power required to send u^1 bits to user 1 and u^2 bits to user 2 when their respective channel gains are g^1 and g^2 . Then if P is a good power function the above analysis will also apply to this situation.

More generally one could consider an arbitrary single hop network. In other words, suppose that there is a set of M transmitters and N receivers. Suppose each transmitter has to send data to each receiver. This data arrives at each transmitter according to a given random process and is stored in a buffer; a separate buffer being used for each receiver. Let $\mathbf{u} \in \mathbb{R}^{NM}$ be a vector representing the transmission rate between each pair of transmitters and receivers, and let $\mathbf{g} \in \mathbb{C}^{NM}$ be the vector representing the channel gains between each pair of transmitters and receivers. Let $P(\mathbf{g}, \mathbf{u})$ be the minimum total power required to send at rate \mathbf{u} when the channel gains are \mathbf{g} . Then if P is a good power function, the above analysis can again be repeated. Of course as MN gets large it becomes increasingly difficult to get all of the needed state information. A variation of this type of model is to assume that each

transmitter has a single shared buffer, instead of a separate buffer for each receiver. In this case the analysis from the previous section would have to be modified.

Incomplete state information: The model in the preceding section assumes that both users have complete state information. In many cases a more realistic assumption would be to assume that each user has only local knowledge; in other words, each user only knows its own buffer occupancy and channel gain. With this assumption, the actual amount of mutual information received from each user in a block will be a random variable depending on the other users' actions. One way to model this situation is to assume that a random number of bits are reliably received. A second way to model this is to assume that each user chooses its transmission power and rate, so that independent of the other users' actions, this rate can be received. In either case, if the average power required is given by a function P with appropriate properties, then the above analysis should be able to be modified to apply here.

CHAPTER 8

Conclusions

8.1 Summary

In this thesis we have considered several simple models of communication over fading channels subject to delay constraints. Our emphasis has been on understanding the possible trade-offs between the average power needed to communicate reliably and some cost related to the delay incurred. The first part of this thesis was primarily concerned with modeling issues. First we discussed models of fading channels; here we focused on a narrow-band block fading model. Various notions of capacity for such channels were discussed. In particular, we considered in detail several definitions of capacity which arise by viewing the channel as a compound channel, such as the delay-limited capacity. Some extensions of these ideas were provided, in particular for the case where the transmitter has causal channel state information. We argued that these compound channel formulations are only of limited use for the problems we consider. Next several buffer models were developed. In each case, we looked at situations where the transmitter can vary the transmission rate and power based on its knowledge of the channel state, the buffer state and the arrival state. In this setting we formulated a Markov decision problem where the per-stage cost is a weighted combination of two terms; one term corresponding to the average transmission power used, and the other term is intended to model a quantity related to delay.

In Chapter 5 we demonstrated some structural characteristics of the optimal policy

for the aforementioned Markov decision problem. We also discussed the behavior of the optimal solution as the relative weighting of the two cost components is varied. Next, in Chapter 6, we considered asymptotic versions of this problem. We analyzed two cases in detail. First we considered the case where the buffer cost corresponds to probability of buffer overflow. We considered such problems as $L \rightarrow \infty$ where L is the buffer size. For a constant arrival rate, this case corresponds to a maximum delay constraint. Next we looked at the case where the buffer cost corresponds to average buffer delay. Here we considered such problems as the average buffer delay became infinitely large. In both of these cases we bounded the rate that the average costs converge to their limiting values. We also demonstrated a sequence of simple buffer control policies with nearly optimal convergence rates. Finally, in chapter 7, we considered problems with more than one user. Here, we focused on a multiple-access situation with two users, where each user had a buffer cost corresponding to probability of buffer overflow. The single user results from Chap. 6 were generalized for this case.

8.2 Further Directions

In the previous chapters, we have mentioned several directions in which this work could be extended. We will not repeat these here, but rather discuss some broader architectural issues related to this work. The problems addressed in this thesis lie at the boundary of physical layer and higher network layer issues. Specifically issues such as the modulation and coding rate are generally addressed at the physical layer, while buffer control would generally be thought of as a higher layer issue. From an architectural point of view there are many advantages to separating network layers. However, as this work makes clear, in a mobile wireless network the boundaries between these layers may not necessarily have the same characteristics as in a fixed wire-line network. One way to think about this is to ask what is a good “black box” abstraction for higher layers to have of the physical layer in such a network. In a wired point-to-point network, this abstraction is typically that the physical layer is a “packet pipe” which can deliver packets at a fixed rate, fixed delay and some small probability of error. In a wireless network, this pipe can potentially have a variable

rate, a variable delay and/or a variable probability of error. Furthermore these can be thought of as parameters that the next layer can adjust along with the transmission power. With such a physical layer, network issues such as routing and flow control may need to be thought about from a new perspective, particularly in a packet-radio network or another all-wireless network architecture.

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