

On Kelly-Type Mechanisms for Polymatroids

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Abstract—We consider market-based mechanisms for allocating transmission rates to a set of users whose rates are constrained to lie in a polymatroid capacity region. We study two simple mechanisms, both of which are generalizations of Kelly’s well known allocation mechanism to this setting. We characterize the equilibrium properties of these mechanisms for “price anticipating” users.

I. INTRODUCTION

Recently, there has been much interest in economic-based models for allocating resources in communication networks. In such models, a network is viewed as a seller of resources (e.g. the capacity of each link); the users of the network are consumers of these resources. An allocation mechanism is then used to establish a market in which users can purchase the available resources. Much of this work has been motivated by Kelly’s model for rate allocation in a wireline-network [1]. This model provides a relatively simple mechanism for allocating end-to-end rates to each user in a network; where the allocated end-to-end rates must be a feasible multi-commodity flow vector in the network. In Kelly’s mechanism, each user submits a bid for rate and receives a rate allocation given by their bid divided by a “congestion price.” The main result in [1], is that when users do not account for their effect on the congestion price, there exists a set of bids and congestion prices which are consistent with each other and under which the resulting allocation is efficient (i.e. it maximizes the total utility of the users). Johari and Tsitsiklis studied the effect of *price anticipating* users on the Kelly mechanism, i.e. users who anticipate their effect on prices [3]. Such users can be viewed as playing a game in which their pay-off is their received utility minus their bids. In [3] it is shown that with price anticipating users, the aggregate utility achieved at a Nash Equilibrium (NE) of this game may no longer be efficient. However, for a single-link, the efficiency loss is bounded to be at most 25% of the maximum utility.

In [1], [3], the rates allocated on each link in the network were simply constrained so that their sum was no greater than the link’s capacity. In other words, each link can be viewed as having a capacity region that is an N -dimensional simplex, where N is the number of users. This is a reasonable model for wireline networks. A motivation for this paper is to consider such resource allocation schemes for wireless networks. In such networks, depending on the physical layer

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technology used, the resulting capacity regions may no longer be well modeled as a simplex. Specifically, we are interested in the problem of allocating rates to a set of N users, when the rates are constrained to lie in a capacity region \mathcal{C} , which is a closed, bounded subset of \mathbb{R}_+^N . Here, we focus on the case where this region is a *polymatroid*. This is motivated by the Cover-Wyner capacity region for a Gaussian multiple access channel [4]. However, we note that our analysis only relies on the polymatroid structure of this region and so applies to any resource allocation problem with a polymatroid constraint set.¹

Given a polymatroid capacity region, we study generalizations of the Kelly mechanism and the resulting efficiency losses for price anticipating users. Compared to a simplex, polymatroid capacity regions have two new interesting characteristics. First, there is some rate that each user can utilize without effecting the rates allocated to any other. We refer to this as a user’s “free capacity.” Our results suggest that good resource allocation mechanisms should take into account the existence of this free capacity. Second, a polymatroid is characterized by a number of constraints that is exponential in the number of users. Hence, having each user bid for each constraint separately is not a scalable solution. Therefore, we focus on mechanisms in which each user submits a single one-dimensional bid. Specifically, we present two mechanisms, both of which are equivalent to Kelly’s mechanism when the capacity region is a simplex. In this first mechanism, a NE may not exist for a network of 3 or more users; for the second mechanism, a NE will always exist, but the efficiency loss can approach 100%.

In other related work, in [3], Johari and Tsitsiklis extend their analysis to a network with multiple links. For the network case they no-longer consider Kelly’s mechanism, but instead a mechanism in which the network implements a separate Kelly-type mechanism for each link in the network. For this mechanism the 25% efficiency loss bound still holds. This could be applied to the polymatroid case, but would require an exponential number of bids. In [5], Hajek and Yang also consider a multiple-link model, with the difference that each user submits a single bid, and the network implements a single Kelly-type mechanism to allocate the users’ end-to-end rates. In this case, with more than one link, the efficiency loss may be 100%, and, moreover, a NE can fail to even exist. The mechanism they consider is essentially the same as our first mechanism; our results provide another example in which this mechanism may not have a NE. In

¹Such a setting is referred to as a “submodular resource allocation market” in [8].

[6], [7], a class of network resource allocation mechanisms are presented in which each user submits a one-dimensional bid, but then is charged a price that is similar to a the payment in a Vickery-Clark-Groves (VCG) auction. Under the resulting mechanism there exists Nash equilibria with zero efficiency loss. Such a mechanism can also be applied in the polymatroid case studied here, but place significantly more computational burden on the agents.

II. MODEL AND BACKGROUND

We consider allocating transmission rates to a set of N users whose rates $\mathbf{x} = \{x_i\}_{i=1}^N$ are constrained to lie in a capacity region $\mathcal{C} \subset \mathbb{R}_+^N$, which is closed and bounded. Each user receives a (quasi-linear) utility equal to $u_i(x_i)$ when allocated rate x_i , where $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is concave, and strictly increasing. For this problem, an efficient allocation is defined to be one that solves:

$$\max_{\mathbf{x} \in \mathcal{C}} \sum_{i \in N} u_i(x_i).$$

To begin, consider the case where \mathcal{C} is a single communication link with capacity C , i.e.

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}_+^N : x(N) \leq C\}, \quad (1)$$

where we use the notation $x(N) := \sum_{i \in N} x_i$. We review the mechanism in [1], [3] for allocating rates in this region. In this mechanism, each user $i \in N$ submits a one dimensional bid b_i ; the network then allocates that user a capacity x_i given by $x_i = \frac{b_i}{\pi_C}$. Here, π_C is the congestion price given by²

$$\pi_C = \frac{b(N)}{C}. \quad (2)$$

As noted above, in [1] each user bids assuming the congestion price is fixed, while in [3] the users take into account their effect on the price in (2). In the later case, the N users are playing a game in which the strategy of each user is their choice of bid, b_i and their pay-off is given by³

$$U_i(b_i; \mathbf{b}_{-i}) = u_i(x_i(b_i; \mathbf{b}_{-i})) - b_i.$$

Johari and Tsitsiklis show that under full information, this game has a unique NE which is within $3/4$ of the efficient allocation.

Here, we consider the case where instead of a single link, \mathcal{C} is given by the following polymatroid capacity region,

$$\mathcal{C}(f) := \{\mathbf{x} \in \mathbb{R}_+^N : x(S) \leq f(S), \forall S \subseteq N\},$$

where $f : 2^N \rightarrow \mathbb{R}_+$ is a given polymatroid function, i.e. it is submodular, monotone and $f(\emptyset) = 0$. Furthermore, we assume that f is *strictly submodular*, meaning that

$$f(S) + f(T) > f(S \cup T) + f(S \cap T),$$

for all $S, T \subseteq N$ that are intersecting.⁴ This is motivated by the Cover-Wyner capacity region for a Gaussian multiple

²In the case where all users bid zero, it is assumed that each receives $x_i = 0$.

³Here \mathbf{b}_{-i} indicates the vector of bids for all users except i .

⁴Two sets S and T are *intersecting* if $S \setminus T$, $T \setminus S$ and $S \cap T$ are all non-empty.

access channel, in which $f(S) = \log(1 + \sum_{j \in S} h_j P_j)$, h_j is user j 's channel gain, and P_j is the user's average transmission power.⁵ Given this region, we will consider several "Kelly-type" mechanisms for allocating the transmission rates. In each case the proposed mechanism is equivalent to the Kelly's, when \mathcal{C} is given by (1).

III. MECHANISM I

The first mechanism we consider is the straightforward application of Kelly's algorithm to the polymatroid case. Namely, each agent i submits a bid b_i , which is also that user's payment. Given b_1, \dots, b_N ; the resulting allocation \mathbf{x} is given by solving the following optimization problem:

$$\max_{\mathbf{x} \in \mathcal{C}(f)} \sum_{i \in N} b_i \log(x_i). \quad (P1)$$

Though this problem has an exponential number of constraints, it can be solved in polynomial time. In particular, a polynomial time, combinatorial algorithm for this problem is given in [8]. We state a version of this algorithm in Algorithm 1.⁶

In [8], it is shown that each time step 2) of the algorithm is executed, there is a unique minimal set. After at most N iterations, the algorithm converges to the optimal solution to (P1). The variables p_k^* found by the algorithm correspond to the sum of the Lagrange multipliers for the active constraints in (P1).

Algorithm 1 Algorithm for solving Problem P1

1) Initialize: $S_0 = N$, $k = 1$

2) Let

$$p_k^* = \min_{S \subset S_{k-1}} \frac{b(S_{k-1} \setminus S)}{f(S_{k-1}) - f(S)}$$

and let S_k be the minimal set achieving this maximum.

3) For $i \in S_{k-1} \setminus S_k$, set $x_i = b_i/p_k^*$.

4) If $S_k = \emptyset$ Stop; Else $k = k + 1$, Goto 2.

For the polymatroid $\mathcal{C}(f)$, the *dominate face* refers to those values of $\mathbf{x} \in \mathcal{C}(f)$ such that $x(N) = f(N)$. We say a point is in the *interior* of the dominate face if no other constraints are binding. It is straightforward to see the following:

Lemma 1: Any solution to Problem P1, must lie on the dominate face of the polymatroid, $\mathcal{C}(f)$.

We define the *corresponding single-link model* of a capacity region $\mathcal{C}(f)$ to be a model in which the user's rates are constrained as in (1) with $C = f(N)$. If $\mathcal{C}(f)$ is replaced with its equivalent single link model, then the above mechanism becomes equivalent to that studied in [3]. We refer to the unique NE in that setting as the *JT equilibrium*.

Lemma 2: If a NE rate allocation for Mechanism I lies in the interior of the dominate face of $\mathcal{C}(f)$, then that rate allocation must also be the JT equilibrium of the corresponding single-link model.

⁵Here, we have normalized the noise power and bandwidth to be 1.

⁶This is somewhat different from how the algorithm is presented in [8]; but results in the same algorithm.

This follows by noting the the first order variations in each user's best response around a point in the interior of the dominate face are the same in both settings. Note that the converse may not be true, i.e. if the JT equilibrium is in the interior of the dominate face of $\mathcal{C}(f)$, it may not be a NE.⁷

A. Free capacity

With the above algorithm, one issue is how to deal with the case when a user bids zero.⁸ A natural approach in this setting would be to set that user's allocation to zero. However, this can lead to a user's pay-off being discontinuous at zero. To see this note that for any solution to (P1) when all bids are non-zero, from Lemma 1, each user i will receive at least $x_{i,\min} = f(N) - f(N \setminus i)$. As the next example shows, this discontinuity can lead to no NE existing for the above mechanism.

Consider a system with $N = 2$ agents, where $f(1) = f(2) = 1$ and $f(1, 2) = C$ ($C \in [1, 2]$). Given a set of bids, let μ_{12} be the price (Lagrange multiplier) for the sum rate constraint, and let μ_i be the price for the individual constraint of user i .

Claim 1: Given bids b_1 and b_2 , the resulting rates x_1, x_2 and prices given by the above mechanism satisfy the following:

- 1.) If $C - 1 \leq b_1/b_2 \leq 1/(C - 1)$, then $\mu_{12} = C/(b_1 + b_2)$, $\mu_1 = \mu_2 = 0$, and $x_i = (Cb_i)/(b_i + b_{-i})$.
- 2.) If $b_1/b_2 < C - 1$, then $\mu_{12} = b_1/(C - 1)$, $\mu_2 = b_2 - \mu_{12}$, $\mu_1 = 0$, $x_1 = C - 1$, $x_2 = 1$.
- 3.) If $b_1/b_2 > 1/(C - 1)$, then $\mu_{12} = b_2/(C - 1)$, $\mu_1 = b_1 - \mu_{12}$, $\mu_2 = 0$, $x_2 = C - 1$, $x_1 = 1$.

To see this, one can check that in each case complementary slackness and feasibility are satisfied.

Using this characterization, we show that a NE may not exist. First from lemmas 1 and 2, any NE must be on the dominate face of the polymatroid. If the NE is on the interior of the dominate face it must be the JT equilibrium, in which case the bids will satisfy case 1 in the preceding claim. Assume the JT equilibrium does not lie in the polymatroid. From Lemma 2, any NE must then result in a rate allocation at a corner point of $\mathcal{C}(f)$. Without loss of generality, consider the corner point where $x_1 = C - 1$ and $x_2 = 1$. From case 2 in the claim, we see that no matter what user 2 is bidding, if user 1 decreases his bid, his pay-off will increase. Hence, this can not be a NE. It follows that a NE does not exist in this case.

The above example can be fixed by allocating each user $x_{i,\min}$ for "free" and having them bid only on the excess capacity. Specifically, for each $S \subseteq N$, let

$$g(S) = f(S) - \sum_{i \in S} x_{i,\min}.$$

⁷This is due to the fact discussed below that under Mechanism I, a user's pay-off may not be quasiconcave.

⁸We note that similar issues arise in the multiple link model studied in [3], [5] when all the bids for a given link are zero.

We can then view the users as bidding for rate x_i in the polyhedron⁹

$$\mathcal{C}(g) = \{x_i \in \mathbb{R}_+^N : x(S) \leq g(S), \forall S \subset N\}.$$

Furthermore, it can be seen that $g(S)$ is also (strictly) submodular and increasing, so that $\mathcal{C}(g)$ is again a polymatroid. However, in this polymatroid, clearly

$$g(N) - g(N \setminus i) = 0,$$

i.e., there is no longer any "free capacity" available to some user.

When using $\mathcal{C}(g)$, we note that Algorithm 1, Lemma 1 and Lemma 2 still apply. Also, note that in the special case when $N = 2$, $\mathcal{C}(g)$ will simply be a simplex. It follows that for $N = 2$ users, a unique NE exists and the efficiency loss is at most 25%.

B. No pure strategy NE

Next, we give an example to show that for $N > 2$ users there may not exist a pure strategy NE for this mechanism, even when each user receives $x_{i,\min}$ for free.

Consider a $N = 3$ agent network, where agent 1 has utility $u_1(x_1) = 10x_1$, and agents 2 and 3 have utilities $u_i(x_i) = x_i$. Assume we allocate each user $x_{i,\min}$ for free and then they bid for the remaining rate in $\mathcal{C}(g)$. Finally, assume that the JT equilibrium for the the corresponding single-link problem does not lie in $\mathcal{C}(g)$. It follows from Lemma 2 that if a NE exists it must not lie in the interior of the dominate face but on the boundary.

Consider a boundary rate allocation that satisfies $x_1 = g(1)$ and $x_2 + x_3 = g(N) - g(1)$. We will show that such rate allocation can not be a NE. First consider agent 1, to receive the desired allocation this agent must not be assigned a rate in the first iteration of Algorithm 1, i.e. $1 \in S_1$. For this to be true, it must be that

$$\frac{b_2 + b_3}{g(N) - g(1)} \leq \frac{b_1 + b_2 + b_3}{g(N)}.$$

However, clearly, as long as this condition holds, agent 1 has incentive to lower his bid. Thus for this to be a NE, agent 1's bid must be such that the above expression becomes an equality, i.e.

$$\frac{b_2 + b_3}{g(N) - g(1)} = \frac{b_1 + b_2 + b_3}{g(N)}. \quad (3)$$

Next consider agent 2's bid (the same argument applies to agent 3). Given that (3) holds, then if this agent reduces her bid, it must be that only agents 2 and 3 will be assigned rates in the first iteration of algorithm 1. At a NE, this must result in a negative change in the agent's pay-off. Taking the derivative of the pay-off in this direction, we have

$$\frac{g(N) - g(1)}{b_2 + b_3} \left(1 - \frac{b_2}{b_2 + b_3}\right) - 1 \geq 0. \quad (4)$$

⁹Note that now a users utility will be $u_i(x_{i,\min} + x_i)$.

Likewise, if this agent increases his bid slightly, then all three agents will be assigned rates in the first iteration. Thus we have that

$$\frac{g(N)}{b_1 + b_2 + b_3} \left(1 - \frac{b_2}{b_1 + b_2 + b_3}\right) - 1 \leq 0. \quad (5)$$

Combining (3), (4), and (5), it follows that for such a NE to exist it must be that

$$\left(1 - \frac{b_2}{b_2 + b_3}\right) \geq \left(1 - \frac{b_2}{b_1 + b_2 + b_3}\right)$$

which is not true for any $b_1 > 0$.

A similar argument can be applied to rule out the other possible boundary allocations. Thus a pure strategy NE does not exist.

For this mechanism, it can be shown that each user's best response is not quasiconcave. In particular, the non-quasiconcavity arises at those points at which the sets chosen in Algorithm 1 change. We also note that the payoffs are continuous (as long as we assign users their free capacity) and so a mixed-strategy NE will exist.

IV. MECHANISM II

We next consider a different allocation mechanism defined as follows: Each agent i submits a bid b_i and receives an allocation x_i given by $x_i = b_i t$ for all i where

$$t = \min_{S \subseteq N} \frac{g(S)}{b(S)}. \quad (6)$$

Denote this critical value of t by t^* . Note with this mechanism, we are not longer guaranteed that an allocation is on the dominate face of $\mathcal{C}(g)$.

Now t^* is the solution to the following LP:

$$\max\{t : tb(S) \leq g(S) \forall S \subseteq N\}. \quad (\text{P2})$$

This can be solved in polynomial time via the ellipsoid algorithm. Given a choice of t we can find a violated inequality by solving $\min_{S \subseteq N} g(S) - tb(S)$ which is minimizing a submodular function.

In Algorithm 2, we give an alternative algorithm for solving P2 that does not rely on the ellipsoid algorithm. First, we make a few preliminary observations. Let $h_t(S) := g(S) - tb(S)$. For a given t , let $\mathcal{V}_t := \{S : h_t(S) \leq 0\}$, so that

$$t^* = \min\{t : \mathcal{V}_t \neq \emptyset\}.$$

Lemma 3: For all S , if $S \notin \mathcal{V}_s$, then for all $t \leq s$, $S \notin \mathcal{V}_t$. This follows directly from noting that for all S , $h_t(S)$ is a non-increasing, linear function of t .

Lemma 4: For all $S \subset T$, if $h_s(S) \leq h_s(T)$, then $h_t(S) \leq h_t(T)$ for all $t \leq s$.

This follows directly from the fact that $g(S) < g(T)$.

Lemma 5: For any t , if S and T are two sets in \mathcal{V}_t , and $S \cup T \notin \mathcal{V}_t$, then $h_t(S \cap T) < \min\{h_t(S), h_t(T)\}$.

Proof: As noted above, $h_t(S)$ is submodular. Therefore,

$$h_t(S) + h_t(T) - h_t(S \cup T) \geq h_t(S \cap T).$$

By assumption, $h_t(S) \leq 0$, $h_t(T) \leq 0$ and $h_t(S \cup T) > 0$. Thus, it must be that

$$h_t(S \cap T) < h_t(S) + h_t(T) \leq \min\{h_t(S), h_t(T)\}. \quad \blacksquare$$

Algorithm 2 Algorithm for solving Problem P2

- 1) Initialize: $S_0 = N$, $k = 1$
 - 2) Let $t_k = \frac{g(S_{k-1})}{b(S_{k-1})}$.
 - 3) Find $d_k = \min_{S \subseteq N} h_{t_k}(S)$. Let S_k be the maximal set on which the minimum is obtained.
 - 4) If $d_k = 0$ stop, solution is t_k ; Else $k = k + 1$, Goto 2.
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As noted above, step 3.) of Algorithm 2 is minimizing a submodular function and so can be done in polynomial time.

Claim 2: This algorithm will iterate at most N times and will find the optimal solution to the above LP.

Proof: First note that from Lemma 3, $d_1 \leq 0$ and if $d_1 = 0$ then t_1 is the optimal solution.

Next note that since $h_t(S)$ is non-increasing in t , then $t_1 \geq t_2 \dots$, i.e. at each step we generate a smaller value of t . There are only a finite number of candidate t 's (one corresponding to each set $S \subseteq N$). At each choice of t_i , $S_{i-1} \in \mathcal{V}_{t_i}$. From this it follows that the algorithm converges to the optimal solution in finite time.

It remains to show that the algorithm only requires N iterations. To show this, we will show that the sets S_1, S_2, \dots are in fact laminar.¹⁰ Suppose this is not true, i.e. at the i th iteration of Algorithm 2 the set S_{i+1} is not contained in S_i . Note that by assumption

$$h_{t_i}(S_i) < h_{t_i}(S_i \cup S_{i+1}),$$

otherwise S_i would not be chosen at the i th step. From Lemma 4, it follows that

$$0 = h_{t_{i+1}}(S_i) < h_{t_{i+1}}(S_i \cup S_{i+1}).$$

By definition

$$h_{t_{i+1}}(S_{i+1}) \leq h_{t_{i+1}}(S_i) = 0.$$

It follows that for $t = t_{i+1}$, S_i and S_{i+1} satisfy Lemma 5, and so

$$h_{t_{i+1}}(S_{i+1} \cap S_i) < h_{t_{i+1}}(S_{i+1}),$$

which contradicts the choice of S_{i+1} . \blacksquare

V. CONCAVITY OF PAYOFFS

We define a set $S \subseteq N$ that achieves the minimum in (6) to be a *minimizer*.

Lemma 6: Strict submodularity of g implies that the set of minimizers will be laminar and there will be a unique minimal minimizer.

¹⁰A family of sets is said to be *laminar* if no two sets in the family are intersecting, i.e. the sets are nested.

Proof: First we show that there cannot be two disjoint minimizers, S^1 and S^2 say. Suppose not. Then

$$\begin{aligned} g(S^1 \cup S^2) &\geq x(S^1) + x(S^2) \\ &= t^*b(S^1) + t^*b(S^2) \\ &= g(S^1) + g(S^2) \\ &> g(S^1 \cup S^2), \end{aligned}$$

a contradiction.

Next we show that there cannot be two different intersecting minimizers, S and T . Suppose not. From strict submodularity if S and T are intersecting, then

$$\begin{aligned} g(S) - t^*b(S) + g(T) - t^*b(T) \\ > g(S \cup T) - t^*b(S \cup T) + g(S \cap T) - t^*b(S \cap T). \end{aligned}$$

Hence,

$$\begin{aligned} \max\{g(S) - t^*b(S), g(T) - t^*b(T)\} \\ > \min\{g(S \cup T) - t^*b(S \cup T), g(S \cap T) - t^*b(S \cap T)\}, \end{aligned}$$

contradicting the choice of either S or T . \blacksquare

Recall, the payoff to agent i is $U_i(b_i; \mathbf{b}_{-i}) = u_i(b_i t^*) - b_i$.

Lemma 7: For all i , $u_i(b_i t^*) - b_i$ is concave in b_i .

Proof: Since u_i is concave and increasing it is enough to show that $b_i t^*$ is concave in b_i .

Let S be the minimal minimizer associated with t^* . Suppose first that $i \notin S$. Then agent i 's allocation is $\frac{b_i g(S)}{b(S)}$. The slope of this expression is $\frac{g(S)}{b(S)}$, and this remains the slope as b_i decreases. This is because for all $T \ni i$, $\frac{g(T)}{b(T)}$ increases as b_i decreases which means that S remains the minimal minimizer.

Next, suppose b_i increases. Then S need not remain the minimal set. Increase b_i to the first point where there is another set, $T \ni i$ such that

$$\frac{g(S)}{b(S)} = \frac{g(T)}{b(T)} = t^*.$$

Agent i 's allocation will now be $\frac{b_i g(T)}{b(T)}$. The slope of this object is

$$\frac{g(T)}{b(T)} \left[1 - \frac{b_i}{b(T)} \right] < \frac{g(T)}{b(T)} = \frac{g(S)}{b(S)}.$$

So the slope is decreasing as b_i increases.

Now suppose we started with $i \in S$. If we increase b_i , S remains a minimal minimizer. The slope of agent i 's allocation will be

$$\frac{g(S)}{b(S)} \left[1 - \frac{b_i}{b(S)} \right].$$

If we decrease b_i , the set S will no longer be a minimizer. Some other set T containing i would become the minimal minimizer. In fact, T will be the maximal minimizer (before the change in b_i). To see why, let T be any other minimizer. By Lemma 6, $S \subset T$.

As b_i decreases, the quantity $\frac{g(S)}{b(S)}$ increases at the rate $-\frac{g(S)}{b(S)^2}$. The quantity $\frac{g(T)}{b(T)}$ increases at the rate $-\frac{g(T)}{b(T)^2}$. Since $S \subset T$:

$$-\frac{g(T)}{b(T)^2} < -\frac{g(S)}{b(S)^2}.$$

Notice that the term on the left is minimized when T is chosen to be a maximal minimizer.

Since T becomes the minimal minimizer, the slope of agent i 's allocation becomes $\frac{g(T)}{b(T)} [1 - \frac{b_i}{b(T)}]$. Since $S \subset T$ and $\frac{b(S)}{g(S)} = \frac{g(T)}{b(T)}$, we have

$$\frac{g(S)}{b(S)} \left[1 - \frac{b_i}{b(S)} \right] > \frac{g(T)}{b(T)} \left[1 - \frac{b_i}{b(T)} \right].$$

So, the slope of the allocation again decreases as b_i increases. \blacksquare

Proposition 1: For Mechanism II, a pure strategy NE always exists.

This follows immediately from the preceding lemma and the classical existence theorem of Glicksberg.

Remark: In this section, we never used the fact in $\mathcal{C}(g)$ there is no free capacity. Indeed, this result also applies if we applied use the original region $\mathcal{C}(f)$. However, it can be shown that assigning users their free capacity can reduce the number of NE that exist.

VI. IMPLICATIONS OF NE WHEN THERE IS A UNIQUE MINIMIZER

Consider a given pure strategy NE under which there is a unique minimizer, S^1 , say. Consider an agent k and suppose first that $k \in S^0 = N \setminus S^1$ and $b_k \neq 0$. Then $x_k = b_k t^*$. Further t^* does not depend on b_k . Agent k 's pay-off is

$$u_k(b_k t^*) - b_k.$$

Since $k \in S^0$, i.e., k is not in any minimizer, we can wiggle b_k a little bit without changing the set of minimizers or the value of t^* . Hence the slope of agent k 's utility will be $t^* u'_k(b_k t^*) - 1$. At equilibrium, the slope must be zero. Hence at equilibrium

$$t^* u'_k(b_k t^*) - 1 = 0 \quad \forall k \in \{j \in S^0 : b_j \neq 0\}.$$

If $b_k = 0$ the most we can say is that

$$t^* u'_k(0) - 1 \leq 0 \quad \forall k \in \{j \in S^0 : b_j = 0\}.$$

Now consider an agent k that is in the unique minimizer, S^1 . Then $x_k = b_k t^*$ but t^* may depend on b_k . Agent k 's utility is $u_k(b_k t^*) - b_k$. If agent k decreases b_k this can only increase the value of $\frac{g(T)}{b(T)}$ for any T that contains k and potentially changes the value of t^* . If the increase is small enough, the set S^1 will continue to be the unique minimizer and the slope is

$$\begin{aligned} &\left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \left(1 - \frac{b}{b + b(S^1 \setminus k)} \right) \\ &\times u' \left(b \left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \right) - 1. \end{aligned}$$

Now we examine what happens when agent k increases her bid. This reduces the value of $\frac{g(T)}{b(T)}$ for any T that contains k . This changes the value of t^* . It also changes the set of minimizers. In fact S^1 remains as the only minimizer. Here is why. Any set T that contains k but is not in S^1 is not a minimizer. So before the change $\frac{g(S^1)}{b(S^1)} < \frac{g(T)}{b(T)}$ and this inequality holds for a small change in b_k . If $T \subset S^1$ then $\frac{g(T)}{b(T)}$ decreases at the rate $\frac{g(T)}{b(T)^2}$ while $\frac{g(S^1)}{b(S^1)}$ decreases at the rate $\frac{g(S^1)}{b(S^1)^2}$. Since $T \subset S^1$ it follows that $\frac{g(T)}{b(T)^2} < \frac{g(S^1)}{b(S^1)^2}$.

Since S^1 remains a minimizer, the slope of agent k 's utility will be

$$\left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \left(1 - \frac{b}{b + b(S^1 \setminus k)} \right) \times u' \left(b \left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \right) - 1.$$

At equilibrium, agent $k \in S^1$ must be playing a best response. Hence

$$\left[\frac{g(S^1)}{b_k + b(S^1 \setminus k)} \right] \left(1 - \frac{b_k}{b_k + b(S^1 \setminus k)} \right) \times u' \left(b_k \left[\frac{g(S^1)}{b_k + b(S^1 \setminus k)} \right] \right) - 1 = 0.$$

We note that this is the same as the necessary and sufficient conditions needed for a NE in [3] when all users have a total capacity of $g(S^1)$. In other words, when there is a unique minimizer S^1 , the NE rates of the users in S^1 are the same as what their rates would be if they were simply competing for a single link with capacity $g(S^1)$. The efficient allocation for these users in $\mathcal{C}(g)$ can be no greater than their efficient allocation for a single link with capacity $g(S^1)$. Thus, it follows from [3] that the allocation of these users will be within $3/4$ of their efficient allocation. Note this does not say anything about the total allocation, because we have not accounted for the users in S^0 . Finally, we note that as in [3], it can be shown that $|S^1| \geq 2$, i.e. at least two users must be in this set.

VII. IMPLICATIONS OF NE WITH NO UNIQUE MINIMIZER

Let $S^1 \subset S^2 \dots \subset S^m$ be the set of minimizers and let S^0 be the set of agents not in any minimizer (i.e. $S^0 = N \setminus S^m$). Consider first an agent $k \in S^0$ such that $b_k \neq 0$. Then $x_k = b_k t^*$. Further t^* does not depend on b_k . Agent k 's pay-off is

$$u_k(b_k t^*) - b_k.$$

Since $k \in S^0$, i.e., k is not in any minimizer, we can wiggle b_k a little bit without changing the set of minimizers or the value of t^* . Hence the slope of agent k 's utility will be $t^* u'_k(b_k t^*) - 1$. At equilibrium, the slope must be zero. Hence at equilibrium

$$t^* u'_k(b_k t^*) - 1 = 0, \quad \forall k \in \{j \in S^0 : b_j \neq 0\}.$$

If $b_k = 0$ the most we can say is that

$$t^* u'_k(0) - 1 \leq 0, \quad \forall k \in \{j \in S^0 : b_j = 0\}.$$

Now consider an agent k that is in the minimal minimizer, S^1 . Note that $b_k \neq 0$. Then $x_k = b_k t^*$ but t^* may depend on b_k . Agent k 's pay-off is $u_k(b_k t^*) - b_k$. If agent k increases b_k to b this can only decrease the value of $\frac{g(T)}{b(T)}$ for any T that contains k . The set S^1 will continue to be the minimal minimizer and the slope is

$$\left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \left(1 - \frac{b}{b + b(S^1 \setminus k)} \right) u' \left(b \left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \right) - 1.$$

If agent k decreases her bid to $b < b_k$, then S^1 is no longer the minimal minimizer. However, the set S^m , the maximal minimizer, becomes the new minimal minimizer and the slope of agent k 's payoff becomes:

$$\left[\frac{g(S^m)}{b + b(S^m \setminus k)} \right] \left(1 - \frac{b}{b + b(S^m \setminus k)} \right) u' \left(b \left[\frac{g(S^m)}{b + b(S^m \setminus k)} \right] \right) - 1.$$

Now pick an agent $k \in S^m \setminus S^1$. Note that $b_k \neq 0$. Suppose first that agent k increases her bid to $b > b_k$. Then the smallest minimizer, S^k that contains k becomes the new minimal minimizer. The slope of agent k 's payoff will be

$$\left[\frac{g(S^k)}{b + b(S^k \setminus k)} \right] \left(1 - \frac{b}{b + b(S^k \setminus k)} \right) u' \left(b \left[\frac{g(S^k)}{b + b(S^k \setminus k)} \right] \right) - 1.$$

If agent k decreases her bid to $b < b_k$. Then agent k is no longer part of any minimizer and the slope of her payoff is

$$t^* u'_k(b_k t^*) - 1.$$

Hence, at equilibrium we have the following conditions on these slopes:

- 1) If $k \in S^0$ and $b_k \neq 0$ then

$$t^* u'_k(0) - 1 \leq 0.$$

- 2) If $k \in S^0$ and $b_k = 0$ then

$$t^* u'_k(b_k t^*) - 1 = 0.$$

- 3) If $k \in S^1$ and $b < b_k$ then

$$\left[\frac{g(S^m)}{b + b(S^m \setminus k)} \right] \left(1 - \frac{b}{b + b(S^m \setminus k)} \right) \times u' \left(b \left[\frac{g(S^m)}{b + b(S^m \setminus k)} \right] \right) - 1 \geq 0.$$

- 4) If $k \in S^1$ and $b > b_k$ then

$$\left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \left(1 - \frac{b}{b + b(S^1 \setminus k)} \right) \times u' \left(b \left[\frac{g(S^1)}{b + b(S^1 \setminus k)} \right] \right) - 1 \leq 0.$$

- 5) If $k \in S^m \setminus S^1$, $b < b_k$ then

$$t^* u'_k(b_k t^*) - 1 \geq 0.$$

- 6) If $k \in S^m \setminus S^1$, $b > b_k$ and S^k is the smallest minimizer to contain k then

$$\left[\frac{g(S^k)}{b + b(S^k \setminus k)} \right] \left(1 - \frac{b}{b + b(S^k \setminus k)} \right) \times u' \left(b \left[\frac{g(S^k)}{b + b(S^k \setminus k)} \right] \right) - 1 \leq 0$$

We can restate these conditions as follows:

- 1) If $k \in S^0$ and $b_k \neq 0$ then $u'_k(b_k t^*) = 1/t^*$.
- 2) If $k \in S^0$ and $b_k = 0$ then $u'_k(0) \leq 1/t^*$.
- 3) If $k \in S^1$ from the left $u'_k(b_k t^*) \geq (1 - \frac{b_k}{b(S^m)})^{-1}(1/t^*)$ and from the right $u'_k(b_k t^*) \leq (1 - \frac{b_k}{b(S^1)})^{-1}(1/t^*)$.
- 4) If $k \in S^m \setminus S^1$, then from the left $u'_k(b_k t^*) \geq 1/t^*$ and from the right $u'_k(b_k t^*) \leq (1 - \frac{b_k}{b(S^k)})^{-1}(1/t^*)$.

Note that from this it follows that at a NE, the set of users in S^0 have marginal utilities which are all strictly less than the marginal utilities of the users in S^m . Also note that in this case, we trivially have that $|S^m| \geq 2$.

VIII. PRICED OUT USERS

For either the case where there is a unique minimizer or multiple minimizers, let \tilde{S}^0 denote the subset of S^0 corresponding to the users who bid $b_j = 0$.¹¹ These users will receive no rate allocation except their free capacity. From the preceding first order conditions, such users bids zero because the price per unit rate $1/t^*$ is too high; hence, we refer to them as being “priced out”. To see how this can arise consider the following example. First consider a network with two users. For such users, there will be a unique NE with a corresponding value t^* . We can then add additional users whose marginal utility at zero satisfies $u'_k(0) \leq 1/t^*$. If the original users do not change their bids and the new users all bid zero, it can be seen that this will still be a NE. We can add as many of these users as we want, and so we can drive the total efficiency loss to be arbitrarily close to 100%. We note that the resulting rate allocation is not on the dominate face of $\mathcal{C}(g)$, i.e. there is some available capacity which is not being used. This is similar to the problem which lead us to allocate each users their free capacity. The issue here is that the unused capacity is due to a group of users bidding zero, instead of just a single user. One could consider allocating this capacity among the users in the group for free, but then this can result in a NE no longer existing.

IX. 3 PLAYER CASE

In this section, we consider the efficiency loss in a network with $N = 3$ users. To simplify our arguments, we assume that each agent k has a linear utility $u_k(x_k) = a_k x_k$, with $a_1 \geq a_2 \geq a_3$.¹² Note we are still assuming that each agent is given their free capacity, so that $g(N) = g(N \setminus k)$ for all k .

For this example, the efficient outcome is given by simply maximizing a linear objective over the polymatroid. Hence this is given by

$$\begin{aligned} & a_1(g(1)) + a_2[g(1, 2) - g(1)] + a_3[g(1, 2, 3) - g(1, 2)] \\ &= a_1g(1) + a_2[g(N) - g(1)], \end{aligned}$$

¹¹Note that if users have linear utilities $u_k(x_k) = a_k x_k$ then there exists a NE in which all users in S^0 bid zero; this follows since any user in S^0 bidding $b_j > 0$ will be indifferent to changing her bid.

¹²As in [3], it can be shown that linear utilities are sufficient to characterize the worst case efficiency loss.

where we have used that $g(N) = g(1, 2)$. Note that to be precise this is the excess utility over that which each agent receives from his free capacity. Including this additional term will not effect the following argument.

Next we compare the efficient outcome to an equilibrium outcome. First, consider the case where at an equilibrium S^0 is empty. In this case, there are two possibilities for an equilibrium: (a) N is the only minimizer; and (b) there are two minimizers, $S_1 = \{k\}$, and $S_2 = N$ for some $k \in N$. We can rule out the possibility of $\{k, j\}$ being a minimizer since $g(\{k, j\}) = g(N)$; hence, the only way this could occur is if the third user bids zero, in which case it will be in S^0 . In case (a) there is a single minimizer, and therefore as we have noted in Section VI, the efficiency loss will be at most 1/4 of the efficient outcome.

We next turn to case (b). In case (b), it can be shown that $k = 1$ is the only possibility. This follows from the first order conditions In Section VII. Namely, to establish a contradiction suppose the minimizers are $S_1 = N$ and $S_2 = \{j\}$ for some $j \neq 1$. Then for agent 1, the first order conditions require

$$t^* a_1 \geq 1,$$

and for agent 2,

$$t^* \left(1 - \frac{b_2}{b(N)}\right) a_2 \geq 1.$$

Combining these we have

$$\left(1 - \frac{b_2}{b(N)}\right) \geq \frac{a_1}{a_2}.$$

Since $a_1 \geq a_2$, the right-hand side is clearly no less than 1, while the left-hand side is less than 1, which is a contradiction.

Next, we consider the efficiency of an equilibrium in case (b). The equilibrium outcome is given by

$$a_1 t^* b_1 + a_2 t^* b_2 + a_3 t^* b_3.$$

Using the definition of t^* , we have $a_1 t^* b_1 = a_1 g(1)$. so the first user receives the same allocation as in the efficient outcome. The following lemma shows that the remaining two users’ utility is within 3/4 of the efficient outcome.

Lemma 8: $a_2 t^* b_2 + a_3 t^* b_3 \geq (3/4)a_2[g(N) - g(1)]$.

Proof: First note that from the first order conditions for agents 2 and 3 we have:

$$(1 - \frac{b_2}{b(N)}) a_2 \leq 1/t^*,$$

and

$$1/t^* \leq a_3.$$

Combining these it follows that

$$(1 - \frac{b_2}{b(N)}) \leq \frac{a_3}{a_2}.$$

Next note that, $\frac{b_2}{b(N)} = \frac{x_2}{g(N)}$, so that

$$t^* b_2 = x_2 \geq (1 - \frac{a_3}{a_2})g(N).$$

Also, for the assumed set of minimizers, it must be that $t^*b_2 + t^*b_3 = g(N) - g(1)$. Using these and that $a_3 \leq a_2$, we have

$$\begin{aligned} & a_2t^*b_2 + a_3t^*b_3 \\ & \geq a_2\left(1 - \frac{a_3}{a_2}\right)g(N) + a_3(g(N) - g(1)) - \left(1 - \frac{a_3}{a_2}\right)g(N) \\ & = (a_2 - a_3)g(N) + \frac{a_3^2}{a_2}g(N) - a_3g(1). \end{aligned}$$

Hence, to prove the desired result it is sufficient to show that

$$(a_2 - a_3)g(N) + \frac{a_3^2}{a_2}g(N) - a_3g(1) \geq (3/4)[a_2(g(N) - g(1))] \quad (7)$$

$$\Leftrightarrow \left(\frac{a_2}{4} + \frac{a_3^2}{a_2} - a_3\right)g(N) \geq (a_3 - \frac{3}{4}a_2)g(1). \quad (8)$$

We consider two cases:

Case 1: $a_3 \geq 3/4a_2$. In this case, since $f(N) > f(1)$, a sufficient condition for (8) to be true is that

$$\frac{a_2}{4} + \frac{a_3^2}{a_2} - a_3 \geq a_3 - \frac{3}{4}a_2.$$

Rearranging terms this is equivalent to $(a_2 - a_3)^2 \geq 0$, which is always true.

Case 2: $a_3 < 3/4a_2$. In this case the right-hand side of (8) is negative. Hence a sufficient condition for the inequality to hold it that the left-hand side is non-negative, i.e.

$$\frac{a_2}{4} + \frac{a_3^2}{a_2} - a_3 \geq 0,$$

which is equivalent to $(\frac{1}{2}a_2 - a_3)^2 \geq 0$. ■

Combining the preceding observations, we have that for $N = 3$ users, when S^0 is non-empty the efficiency loss is at most 1/4 the efficient outcome. Next consider the case where S^0 is non-empty. Since S^m must contain at least two user it follows that the only possibility is $S^0 = \{3\}$ and $S^m = \{1, 2\}$. In this case, it must be that $x_3 = 0$ which is the same as in the efficient outcome. Using a similar argument as in the case when S^0 is empty, it can be shown the efficiency loss of the remaining two agents is at most 1/4. Therefore, we have that in a three agent network the efficiency loss is at most 1/4 of the efficient outcome.

X. $N > 3$ PLAYER CASE

Next consider a network with $N > 3$ users. In this case, the efficiency loss can approach 100%; however, we can bound the loss for those users in S^m . Namely we have:

Proposition 2: For a network with N users, in any equilibrium, the total efficiency loss of the users in S^m is no more than 1/4 the efficient outcome. For $N > 3$, the efficiency loss of the uses in S^0 can approach 100%.

The proof of this uses a similar argument as in the $N = 3$ user case above. We note the that key difference between $N = 3$ and $N > 3$ is that for $N > 3$ there may be more than one user in S^0 . While for $N = 3$ there is at most one user. When there is just one user, this does not effect the overall efficiency, since that user will receive zero rate in the efficient outcome.

XI. SUMMARY

To conclude, we summarize the main results of this paper compared to the single link model in [3].

- 1) In [3] there always exists a unique NE. In the polymatroid case, under Mechanism I a NE may not exist for $N > 2$ users. Under mechanism II a NE always exists, but it may not be unique.
- 2) In [3] the unique equilibrium is at least 3/4 of the efficient outcome. In the polymatroid case, under mechanism II, among all the bidders in S^m , their allocation will be within 3/4 of the efficient outcome. However, the overall efficiency loss can approach 100%.

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