

# Information Cascades with Noise

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**Abstract**—Online networks enable agents to better observe the behavior of others and in doing so potentially learn from their actions. A key feature of models for such social learning is that *information cascades* can result, in which agents ignore their private information and blindly follow the actions of other agents. This paper considers the impact of noise in the form of observation errors in such a model. Similar to a model without noise, we show that with noise, both correct and incorrect cascades happen, with the same level of fragility. However, with noise, it is harder to overturn a cascade from one direction to the other. Using Markov chain analysis, we derive the net welfare (payoff) of each agent as a function of his private signal quality and the error rate. We show, somewhat surprisingly, that in certain cases, increasing the observation error rate can lead to higher welfare for all but a finite number of agents. In such cases, we compare and contrast adding additional noise with simply withholding observations from the first few agents. Our analysis assumes that all erroneous observations are available on a common database; however, we also discuss relaxing this assumption. We conclude by discussing the impact of different tie-breaking rules on the probability of wrong cascade and the impact of bounded rationality on the agents’ welfare.

**Index Terms**—Perfect Bayesian Equilibrium, Bayesian Learning, Information Cascades, Herding.

## I. INTRODUCTION

CONSIDER a recommendation system where agents sequentially decide whether to buy an item for which they have some prior knowledge of its quality/utility. The agents’ decisions are reported to a common database that is available to all later agents (e.g., via a website), who can potentially benefit from the information obtained by observing their predecessors’ choices. The study of such a model was initiated in the seminal papers [9], [10], and [11], which cast this in an observational Bayesian learning framework. In these models, each individual (agent) has some prior knowledge or signal about some payoff-relevant state of the world generated according to a commonly known probability distribution. Each agent makes a one-time decision in an exogenous order and his action is observed exactly by all subsequent agents. A common database provides one means for facilitating making this information available. Given these observations and their own signals, agents are assumed to be Bayesian rational, i.e., they sequentially choose the action that reflects their posterior beliefs about the state of the world. To put this in game-theoretic jargon, the setting is equivalent to a dynamic game with asymmetric information. The equilibrium analyzed

in these problems is a Perfect Bayesian Equilibrium (PBE), which uses the common information based belief to determine the strategies<sup>1</sup>.

We highlight two possible outcomes for such a model. First, there is *learning*. This is defined as when the information in the private signals is aggregated efficiently so that agents eventually know the underlying true value of the item and make socially optimal decisions. Second, there is *cascading*. An *information cascade* occurs when it is optimal for the agents to ignore their own signals when taking actions. Though individually optimal, this may lead to the agents making a choice that is socially sub-optimal, which we refer to as a “wrong cascade.” In addition to the possibility of a wrong cascade, an information cascade results in a loss of information about the private signals held by all the agents following the onset of cascade.

In models where a homogeneous population of agents with discrete bounded private signals is assumed (e.g. [9]-[11]), a main result is that a cascade happens in a finite number of agents with probability one. This leads to a positive probability of cascading toward the wrong choice. Once a cascade occurs, all subsequent agents follow suit, given the underlying homogeneity. The ultimate consequence is that all private signals from the onset of cascading are lost; thus learning stops.

In this paper, we consider a similar Bayesian learning model as in [9]-[11] except we introduce *observation errors*. More precisely, we assume observation errors occur when each agent’s action is recorded for all subsequent agents to see, with the recording subject to error, with the statistics of the error process known to all agents. The motivation of having observation errors in our model comes from the following reasons. First, real-world agents can make mistakes. As an example, this could model a setting where agents are asked to report their decisions on a website and agents occasionally misreport (e.g., by simply pressing the wrong button). In communication networks settings, observation errors can also occur as a form of broadcast failure in feedforward networks ([29]). Indeed, even highly sophisticated decision makers such as investors ([28]) and consumers ([27]) can make mistakes in choosing (binary) actions (investing in stocks or purchasing goods). Second, observation errors could come from the mistakes incurred by learning algorithms based on web-log data. For example, consider scenarios when a website uses the customers’ web-log records to infer the customers’ purchasing decisions and then posts such decisions on a database; the errors can occur from both the data and the algorithms used for such inference (e.g., a user could buy an item from a different retailer or could buy it and then return it elsewhere). Finally, this observation error could also result from either strategic

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<sup>1</sup>See [25] where a formal proof is presented.

agents ([32]), or a social planner ([31]), who can manipulate the recording for their own benefit.<sup>2</sup> Even in scenarios when there is no clear incentive for benefit, agents can choose not to express their true opinions, e.g., IMDB review manipulations ([37]). We study the effects of these observation errors on the probabilities of cascades and the welfare obtained for any given agent, and also asymptotically as the number of agents increases without bound.

### A. Contributions

Our work has several contributions. First, we develop a simple Markov chain model that we use to analyze the probabilities of cascades and the expected pay-off of each agent as a function of the error-rates and signal quality. We then study the effect of changing the noise level on both the asymptotic behavior and the behavior for an arbitrary agent by using stochastic ordering and coupling methods for Markov chains. Our results demonstrate a counter-intuitive phenomenon: for certain parameter settings, the asymptotic average payoff can be increased by increasing the noise level. The extent of this phenomenon and the amount of noise to be added depend on the agents' signal quality and the total amount of noise already present.

Second, we extend our conclusions by considering which agents benefit from adding noise in the low noise regime. Interestingly, we show that if one agent benefits, the following agent may not. However, we show several properties of the sequence of benefiting agents, and further show that if one agent benefits at most a finite number of subsequent agents will not benefit.

Third, we compare and contrast the trade-offs of two approaches for improving the agents' welfares: 1) Adding observation noise (whenever it is beneficial to do so); and 2) Forcing the first few agents to be "guinea pigs," i.e., to follow their private signals as in [19]. We see that the second method leads to higher asymptotic pay-offs, but at the cost of lower pay-offs for an initial group of users. We also suggest a hybrid method that for certain model parameters can significantly reduce the number of required guinea pigs, while still achieving better asymptotic welfare than either approach separately.

Fourth, we also discuss the role of a common database in our analysis. We argue that such a database is crucial in enabling the agents being able to infer the exact time in the history where a cascade happens. In particular, we compare the common database setting to one where each agent's observation is an *i.i.d.* erroneous version of the sequence of the true actions taken before. In such case, we show that a cascade may not persist once it starts. Moreover, this is shown to greatly increase the complexity of updating beliefs for each agent, making the Bayesian assumption more questionable.

Finally, our work also highlights the advantage of Markov Chain methods in obtaining a refined and detailed analysis of the problem, in particular, obtaining per agent payoffs and determining sensitivity to parameter changes. This should be

contrasted with the most of the literature, which uses martingale methods to discuss asymptotic performance measures.

The remainder of the paper is organized as follows. We start by discussing related work in Section I-B. In Section II, we state our model. In Section III, we analyze the Bayesian updates for this model and give our Markov chain formulation. We then turn to studying the impact of changing the noise level in Section IV. In Section V, our analysis of when adding noise is better is presented, for both cases of asymptotic and individual welfares. We compare adding noise to the "guinea pigs" model in Section VI, and then consider the case of no common database in Section VII. In Section VIII, we present some discussions of other changes to the model including different tie-breaking rules, the effect of agents' bounded rationality, and the general prior for the true state of the world. We conclude in Section IX.

### B. Related work

Our paper studies a variation of the model in [9]-[11], namely the inclusion of noise in the observations of each agent. There is a rich literature that has studied many other variations of this model, which we briefly review. In [9]-[11] the signals agents receive are binary valued and all agents are homogeneous in their preferences for the given item. The work of Smith and Sorensen [17] relaxed both of these assumptions and showed in particular that allowing for a richer set of signals could result in a model in which learning always occurs (namely, signals that result in an unbounded likelihood ratio). Here, we continue to assume binary signals and homogeneous agents. Another modification studied in [17] was to allow for a fraction of the agents to be "irrational." As discussed in Section VIII, we can adapt our model to also study this case (though this changes how the welfare is calculated). We also note that [17] did not consider how changing the corresponding noise level effects the welfare, which is a main focus of our work.

Another generalization of these models is to consider different information structures. For example, [25] considers the case where each agent only observes the actions of a subset of the previous agents and studies when the resulting network structure leads to asymptotic learning. In [36], this network structure is made endogenous by allowing the agents to select their observations at a cost. Here we stay with the original network structure as in [9]-[11] in which agents observe the actions of all prior agents, which is consistent with our assumed common database.

Another possible change to the information structure is by allowing for "guinea pigs" as we consider in Section VI. Such an approach was studied in [19], which also studied the optimal number of guinea pigs to maximize the social welfare in a finite population. This type of approach could also be viewed through the lens of the multi-armed bandit literature, where the guinea pigs would correspond to agents used for "exploring" as suggested by [7], [13]. From this literature it follows that with  $n$  agents, the optimal number of guinea pigs needed is  $O(\log n)$ , which increases without bound ([13]). Here, our focus is on comparing the use of such guinea pigs to adding additional observation noise.

<sup>2</sup>In the case of strategic agents manipulating reviews, our notion of welfare would need to be modified to account for the preferences of these agents.

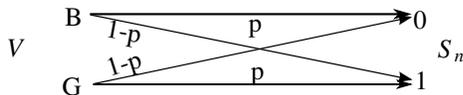
Yet another change to the information structure is to allow the agents to have a richer action space. For example, [12], studies a model with a continuous action space, which allows the agents to fine tune their actions to the posterior. This can ensure that learning occurs but is a fairly strong assumption that does not seem well matched to the type of platform we are considering.

We also note that these types of models are closely related to early work on sequential detection, e.g, [2] and [3]. That literature can be viewed as a scenario in which a planner can design a feedback rule for each agent in an attempt to asymptotically learn the true state of the world. In [2], it was shown that if the planner was limited to binary feedback and each agent had to follow the same rule, then there is a positive probability that learning will not occur. However, if two-bit feedback is allowed or if the encoding rule can vary across agents, then learning can be guaranteed, e.g., [2]. Similar conclusions also hold for more general networks (e.g., [29]), or for a random decision rule (e.g., [35]). In these works, it is not clear whether it is in the agents' best interest to strictly adhere to such set of rules, i.e., the outcomes might not constitute an equilibrium. Here, we instead consider a strategic setting where each agent acts in his own benefit, and the agents are in a PBE. In particular, we limit ourselves to binary feedback in the form of the agent's actions and assume that each agent deterministically chooses his action to optimize his own pay-off (i.e., there is no planner).

## II. MODEL

We consider a model similar to [9] in which there is a countable population of agents, indexed  $n = 1, 2, \dots$  with the index reflecting both the time and the order in which agents act. Each agent  $n$  chooses an action  $A_n$  of either buying ( $Y$ ) or not buying ( $N$ ) a new item. The true value ( $V$ ) of the item can be either good ( $G$ ) or bad ( $B$ ); for simplicity, both possibilities are assumed to be equally likely.<sup>3</sup>

The agents are Bayes-rational utility maximizers<sup>4</sup> whose payoff structure is based on the agent's action and the true value  $V$ . If an agent chooses  $N$ , his payoff is 0. On the other hand, if he chooses  $Y$ , he faces a cost of  $C = 1/2$  and gains one of two amounts depending on the true value of the item: his gain is 0 if  $V = B$  and 1 if  $V = G$ . The total pay-off of an agent choosing  $Y$  is then the gain minus the cost. Thus, the *ex ante* expected payoff of each agent is 0.



**Figure 1:** The BSC over which agents receive signals.

To reflect the agents' prior knowledge about  $V$ , each agent  $n$  receives a private signal  $S_n \in \{1 \text{ (high)}, 0 \text{ (low)}\}$  through a binary symmetric channel (BSC) with crossover probability

$1 - p$ , where  $0.5 < p < 1$ . (See Fig. 1.) In other words, we have

$$\begin{aligned} \mathbb{P}(S_n = 1|V = G) &= \mathbb{P}(S_n = 0|V = B) = p, \text{ and} \\ \mathbb{P}(S_n = 0|V = G) &= \mathbb{P}(S_n = 1|V = B) = 1 - p. \end{aligned} \quad (1)$$

Thus, the private signals are informative, but not revealing. We modify the information structure in [9] by assuming that later agents' observations are noisy versions of their predecessors' actions. Specifically, we assume that each agent reports his action to a public database which is available to all successors. The errors in this process are modeled by passing every action  $A_i$  through another BSC with crossover probability  $\epsilon \in (0, 1/2)$ . This means with probability  $1 - \epsilon$ , the reported action of agent  $i$ ,  $O_i$ , satisfies  $O_i = A_i$ , and with probability  $\epsilon$ ,  $O_i = \bar{A}_i$ , where  $\bar{A}_i$  is the opposite action of  $A_i$ . This assumption reduces the dependence of every agent's decision on the predecessors' choices and drives him toward using his own signal. With this in place, we assume that each agent  $n$  takes a one-time action  $A_n$  based on his private signal  $S_n$  and the (noisy) observations  $O_1, \dots, O_{n-1}$  of all previous agents' actions  $A_1, \dots, A_{n-1}$ .

## III. BAYESIAN UPDATES, CASCADES AND ERROR THRESHOLDS

### A. Bayesian updates

The optimal action for the first agent is always to follow his private signal since no observation history is available. Starting from the second agent, every agent  $n$  considers his private signal  $S_n$  and the observations  $O_1, \dots, O_{n-1}$ . Let the information set of agent  $n$  be  $\{S_n, \mathcal{H}_{n-1}\}$  where  $\mathcal{H}_{n-1} = \{O_1, \dots, O_{n-1}\}$  is the observation history. Based on  $\{S_n, \mathcal{H}_{n-1}\}$ , agent  $n$  will update his posterior probability (using Bayes formula).

**Definition 1.** *The posterior probability of agent  $n$ ,  $\gamma_n$ , is defined as  $\gamma_n(S_n, \mathcal{H}_{n-1}) = \mathbb{P}[V = G|S_n, \mathcal{H}_{n-1}]$ .*

Assuming common rationality<sup>5</sup>, the decision rule for agent  $n$  is as follows:<sup>6</sup>

$$A_n = \begin{cases} Y, & \text{if } \gamma_n > 1/2, \\ N, & \text{if } \gamma_n < 1/2, \\ \text{follows } S_n, & \text{if } \gamma_n = 1/2. \end{cases} \quad (2)$$

Following [17], we can also state this decision rule in terms of a private and public likelihood ratio defined next.

**Definition 2.** *The private likelihood ratio of agent  $n$ ,  $\beta_n$ , is defined as  $\beta_n(S_n) = \mathbb{P}[S_n|V = B]/\mathbb{P}[S_n|V = G]$ .*

Note that the private likelihood ratio is a function of the private signal, and we have  $\beta_n(0) = p/(1 - p)$ ,  $\beta_n(1) = (1 - p)/p$ . For a given  $p \in (0.5, 1)$ , these private ratios are strictly bounded in  $(1, \infty)$  and  $(0, 1)$ , respectively.

**Definition 3.** *The public likelihood ratio after agent  $n$  decides,  $\ell_n$ , is defined as  $\ell_n(\mathcal{H}_n) = \mathbb{P}[\mathcal{H}_n|V = B]/\mathbb{P}[\mathcal{H}_n|V = G]$ . This is available to all subsequent agents  $n + 1, n + 2, \dots$*

<sup>5</sup>As in the PBE in dynamic games.

<sup>6</sup>When equality holds, our decision rule differs from [9], where it is assumed that indifferent agents randomly choose one action. Our assumption not only simplifies the analysis but also proves to be the best tie-breaking rule (see Section VIII-A).

<sup>3</sup>A discussion of how to generalize this is provided in Section VIII-C.

<sup>4</sup>This assumption leads to a PBE that uses the common information based belief to determine strategies as proved in [25].

A simple application of Bayes formula gives  $\gamma_n = \frac{1}{1 + \beta_n \ell_{n-1}}$ . Therefore, using the above formulae for  $\beta_n$  and  $\ell_{n-1}$ , the rule in (2) can be applied to the resulting  $\gamma_n$ , which amounts to comparing  $\beta_n \ell_{n-1}$  to 1, e.g., with  $Y$  being the choice whenever  $\beta_n \ell_{n-1} < 1$ . In this decision-making process, cascading is defined as follows:

**Definition 4.** *An information cascade is said to happen when an agent chooses some fixed action regardless of his private signal.*

It follows that agent  $n$  cascades to the action  $Y(N)$  if  $\gamma_n > 1/2$  ( $< 1/2$ ) for all realizations of  $S_n \in \{0, 1\}$ . In other words, agent  $n$  cascades to  $Y$  if  $\beta_n(0)\ell_{n-1} < 1$  (i.e.,  $\ell_{n-1} < (1-p)/p$ ), and cascades to  $N$  if  $\beta_n(1)\ell_{n-1} > 1$  (or  $\ell_{n-1} > p/(1-p)$ ). It was shown in [9] that when agents are rational and observations are perfect (i.e.,  $O_n = A_n, \forall n = 1, 2, \dots$ ), an information cascade occurs in finite time with probability one. In our paper, we will show that this still holds. Another related phenomenon, *herding*, happens when agents choose to copy the actions of others irrespective of the realizations of their signals. In a model with a homogeneous population and discrete, bounded private signals (e.g. [9]-[11] and our paper), the two concepts cascading and herding are equivalent. Thus, from now on, we will use the two interchangeably.

### B. Cascading properties

In this section, we outline some basic properties of cascading with observation errors. These naturally extend properties for the noiseless case shown in [9], [19], so we omit detailed derivations. For completeness we provide proofs in [38].

**Property 1.** *Until cascading occurs, agent  $n$ 's Bayesian updates depend only on his private signal,  $S_n$ , and the difference in the numbers of  $Y$ 's and  $N$ 's in the observation history  $\mathcal{H}_{n-1}$ .*

In other words, the difference of  $Y$ 's and  $N$ 's in  $\mathcal{H}_{n-1}$  is a sufficient statistic for the observation history of agent  $n$ . This follows from the symmetry of the signal quality and the errors, which enables opposite observations to be ‘‘canceled.’’<sup>7</sup>

Note that once a cascade happens, any new observations do not provide new information about the true value  $V$ , thus the public likelihood ratio stays the same. On the other hand, if an agent  $n$  does not cascade, all agents  $1, 2, \dots, n$  act independently based on their own private signals. Therefore, until a cascade happens, the public likelihood ratio  $\ell_n$  depends only on the difference in the numbers of  $Y$ 's and  $N$ 's in the observation history  $\mathcal{H}_n$ . Specifically, let  $h_n$  be the number of  $Y$ 's minus the number of  $N$ 's in  $\mathcal{H}_n$  (prior to a cascade), then  $\ell_n = \left(\frac{1-a}{a}\right)^{h_n}$ , where for all  $i = 1, 2, \dots, n$ ,

$$a = \mathbb{P}[O_i = Y|V = G] = \mathbb{P}[O_i = N|V = B], \quad (3)$$

gives the probability that a given observation agrees with the optimal action for an agent that is not cascading. It can be shown that  $a = f(\epsilon, p) \in (0.5, p)$ , where:

$$f(x, y) \triangleq x(1-y) + (1-x)y. \quad (4)$$

<sup>7</sup>This also extends to when  $V$  is not equally likely  $G$  or  $B$  as shown in Section VIII-C.

**Property 2.** *Once a cascade happens, it lasts forever.*

Once started, a cascade makes agents stop using their private signals and so they provide no information to the successors. The successors are left in the same situation for the optimal action choice as the first one who started the herd. We stress the importance of having the common database available to all agents. With such information, every agent updates  $h_n$  and knows the exact time when a cascade happens. All other subsequent observations from the onset of a cascade will be ignored. Thus, there is no need to further update  $h_n$ . Next, we provide a condition for determining the onset of cascades given the signal quality,  $p$ , and observation error,  $\epsilon$ .

### C. Error thresholds

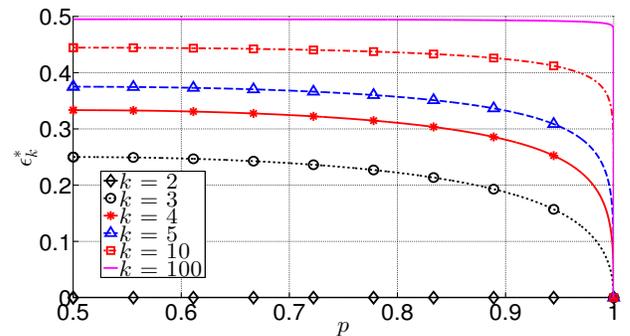
Without noise ( $\epsilon = 0$ ), it was shown in [9] that cascading starts with the first agent  $n$  to have  $|h_{n-1}| = 2$ , i.e., the first agent to observe one action two times more than the other. However, as  $\epsilon$  increases, each observation provides less information, which in turn can increase the value of  $|h_n|$  needed to start a cascade. This is characterized in the following lemma.

**Lemma 1.** *Let  $\alpha = (1-p)/p$ . For any  $k \geq 2$ , define the increasing sequence of thresholds  $\{\epsilon_k^*\}_{k=2}^\infty$ , where the  $k^{\text{th}}$  threshold  $\epsilon_k^*$  is given as:*

$$\epsilon_k^* = (1 - \alpha^{\frac{k-2}{k-1}}) / (1 - \alpha^{\frac{k-2}{k-1}} + \alpha^{\frac{1}{k-1}} - \alpha). \quad (5)$$

Define  $\mathcal{I}_k \triangleq [\epsilon_k^*, \epsilon_{k+1}^*)$  as the  $k^{\text{th}}$   $\epsilon$ -interval. Then for  $\epsilon \in \mathcal{I}_k$ , any agent  $n$  starts a cascade as soon as  $|h_{n-1}| = k$ .

The proof follows from direct calculation of  $\gamma_n$ ; a detailed version is given in [38]. Note also that the parameter  $\alpha$  here is the same as  $\beta_n(1)$ . Fig. 2 shows the thresholds  $\epsilon_k^*$  for different values of  $k$  and  $p$ . For  $k = 2$ , this lemma yields  $\epsilon_2^* = 0$ , which is the case of noiseless observations as in [9].



**Figure 2:** Examples of the thresholds  $\epsilon_k$  for the indicated values of  $k$  versus  $p$ .

A direct consequence of this lemma is that we can deduce  $k$  from a given  $\epsilon$  as:

$$k = \left\lceil \log_{(1-a)/a} \alpha \right\rceil + 1. \quad (6)$$

Here, recall that  $a$  is a function of  $\epsilon$ . Thus, given the signal quality  $p$ , the observation error  $\epsilon \in \mathcal{I}_k$  and the observation history  $\{O_1, O_2, \dots, O_{n-1}\}$ , an agent  $n$  can track the difference  $h_{n-1}$  and declare a  $N$  cascade as soon as  $h_{n-1} = -k$ , and a  $Y$

cascade as soon as  $h_{n-1} = k$ . In the limit of  $\epsilon \rightarrow 0.5$ , we have  $k \rightarrow \infty$ , and thus agents only use their own signals and herding never starts. In this case, no information is passed through each observation. Therefore, no learning occurs either.

Also note that  $\epsilon_k^*$  is a decreasing function of  $p$ . Therefore, the more accurate the private signal, the sooner cascades may start for a fixed  $\epsilon$ . In particular, Fig. 2 shows that if the private signal quality is high, with an intermediate level of noise a cascade may occur very early.

In the models without error ([9]-[11]), the authors also discussed how *fragile* a cascade can be. In fact, for the noiseless case, an information cascade occurs merely on the agreement of the running count of an action exceeding the other by two. Assuming that any of the subsequent agents is endowed with an additional private signal that disagrees with the cascading actions, the cascade stops. This provides some intuition to why many asset markets are volatile, or why changes in fashions happen frequently and unpredictably. With noisy observations, our results show that information cascades have the same level of fragility as in the noiseless case. In fact, for  $\epsilon \in \mathcal{I}_k$ , it would require only one additional opposite private signal to reduce the difference to  $|h_{n-1}| < k$ , so that the cascade stops. However, with noise,  $k$  can exceed 2 and it is harder to reverse a cascade to the opposite direction: one would need a longer sequence of agents with private signals indicating the other action choice. Therefore, given a realistic setting where the information is noisy, it may take a long time to reverse certain trends.

#### D. Markovian analysis of cascades

By symmetry, first consider the case  $V = G$ .<sup>8</sup> By Property 1, a non-cascading agent  $n$ 's observation history can be summarized by  $h_n$ . Thus, viewing each agent as a time-epoch, the agent's observation history can be represented as a state of a discrete-time Markov chain (DTMC). Each state  $i$  represents values of  $h_{n-1}$  that agent  $n$  may see before making his decision. Note that the first agent starts at state 0 since no observation history is available.

For the rest of the paper, assume  $\epsilon \in \mathcal{I}_k$  so that an agent  $n$  will start a cascade if and only if  $|h_{n-1}| = k$ . Thus, to model cascades, we simply make the states  $\pm k$  absorbing, and so we are left with a finite state DTMC. The two events  $N$  cascade and  $Y$  cascade translate into hitting the left ( $-k$ ) and the right ( $k$ ) walls (or absorbing states), respectively. The probability of moving one step to the right is the probability that one more  $Y$  is added to the observation history, i.e.,  $a = \mathbb{P}[O_n = Y|V = G] = f(\epsilon, p) > 0.5$ . Likewise, the probability of moving one-step to the left is  $1 - a$ . Hence, this DTMC is a simple random walk as shown in Fig. 3.

<sup>8</sup>If  $V$  is not equally likely  $G$  or  $B$ , the dynamics of the Markov chains for this analysis are unchanged since these are conditioned on the true  $V$ . Moreover, the (unconditional) probabilities of cascades and expected welfare can be easily adjusted according to the prior knowledge of  $V$ . See details in Section VIII-C.

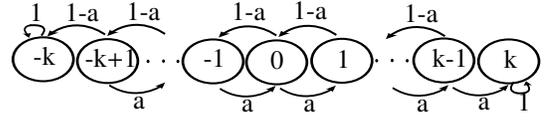


Figure 3: Transition diagram of the random walk when  $V = G$ .

The state transition matrix of this DTMC is given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1-a & 0 & a & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1-a & 0 & a \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

Since  $a > 0.5$ ,  $Q$  is a row stochastic matrix with a drift to the right. We will use methods in [1] to calculate the probability of being at either absorbing state at any given time. Since the agent indices correspond to the time index, this yields the probability of cascade for each agent. Assume that the process starts at state  $i$ . Let  $u_{i,n}^*, v_{i,n}^*$  be the probabilities of being at the left wall,  $-k$ , and the right wall,  $k$ , at the  $n^{\text{th}}$  step, respectively. Let  $u_{i,n}, v_{i,n}$  be the probabilities of hitting the left wall and the right wall for the first time at the  $n^{\text{th}}$  step, respectively. Obviously,  $u_{0,n}^* = v_{0,n}^* = 0$  for  $1 \leq n \leq k-1$ . By [33],  $u_{0,n} = v_{0,n} = 0$ , for  $n - k$  odd. Thus, for  $n \geq k$ :

$$u_{0,n}^* = \sum_{m=k}^n u_{-k,n-m}^* u_{0,m} = \sum_{m=k}^n u_{0,m}, \quad (8)$$

$$v_{0,n}^* = \sum_{m=k}^n v_{k,n-m}^* v_{0,m} = \sum_{m=k}^n v_{0,m}, \quad (9)$$

where the explicit expressions for the terms on the right-hand side in (8) and (9) are given in the next lemma. The proof can be found in Lemma 2 of [33].

**Lemma 2.** For  $n - k$  even:

$$u_{0,n} = \frac{2^n}{k} a^{\frac{n-k}{2}} (1-a)^{\frac{n+k}{2}} A_{k,n}, \quad v_{0,n} = \frac{2^n}{k} a^{\frac{n+k}{2}} (1-a)^{\frac{n-k}{2}} A_{k,n}, \quad (10)$$

where  $A_{k,n} = \sum_{j=1, \text{ odd}}^{j < k} \cos^{n-1} [j\pi/(2k)] \sin [j\pi/(2k)] (-1)^{\frac{j-1}{2}}$ .

By symmetry, if  $V = B$ , the wrong and correct cascade probabilities for agent  $n$  are  $v_{0,n}^*$  and  $u_{0,n}^*$ , respectively. The following lemma states some useful relations between the probabilities of cascade.

**Lemma 3.** For  $\epsilon \in \mathcal{I}_k$ , and any arbitrary agent  $n \geq k$ :

- 1) If  $k$  is even, then  $u_{0,n}^* = u_{0,n+1}^*, v_{0,n}^* = v_{0,n+1}^*$  for  $n$  even;
- 2) If  $k$  is odd, then  $u_{0,n}^* = u_{0,n+1}^*, v_{0,n}^* = v_{0,n+1}^*$  for  $n$  odd;
- 3) The probabilities of correct and wrong cascade are related by:

$$v_{0,n}^*/u_{0,n}^* = [a/(1-a)]^k. \quad (11)$$

The proof of part 1) and 2) follows directly from (8) and (9), whereas the proof for part 3) follows by using the formulae from Lemma 2 in (8) and (9).

#### IV. EFFECT OF VARYING OBSERVATION ERROR RATES

We next turn to studying the effect of the error rates on the cascade probabilities and agent welfare.

### A. Cascade probabilities

Lemma 2 shows that the probabilities of wrong and correct cascade depend on  $k$  and  $a$ . From Lemma 1 and since  $a = f(\epsilon, p)$ , it follows that for a fixed  $p$ , these probabilities are determined by the error probability  $\epsilon$ .

The next theorem characterizes the effect of varying the error  $\epsilon$  on the probabilities of wrong cascade,  $u_{0,n}^*$ , and correct cascade,  $v_{0,n}^*$ , for an arbitrary agent  $n$ .

**Theorem 1.** For  $\epsilon \in \mathcal{I}_k$ :

- 1) The wrong cascade probability increases with  $\epsilon$ .
- 2) The correct cascade probability decreases with  $\epsilon$ .

**Proof.** Consider  $\epsilon' < \epsilon''$ , both in  $\mathcal{I}_k$ , and let  $\{Z'_n\}_{n \geq 0}$  and  $\{Z''_n\}_{n \geq 0}$  be the two corresponding DTMCs on the same state space  $\mathcal{S} = \{-k, -k+1, \dots, 0, \dots, k-1, k\}$ . The following concept of *stochastic ordering* ([6]) compares these two DTMCs:

**Definition 5.** Let  $X$  and  $Y$  be two discrete random variables taking values on the same set  $\mathcal{S}$  and let  $\mathbf{x}$  and  $\mathbf{y}$  be their corresponding probability distribution vectors.  $X(\mathbf{x})$  is said to be larger than  $Y(\mathbf{y})$  in stochastic ordering, denoted by  $X \geq_{st} Y(\mathbf{x} \geq_{st} \mathbf{y})$ , if

$$\sum_{i \geq j} x_i \geq \sum_{i \geq j} y_i, \text{ for all } j \in \mathcal{S}. \quad (12)$$

**Definition 6.** The DTMC  $\{Z'_n\}$  is said to be larger than the DTMC  $\{Z''_n\}$  in stochastic ordering, denoted by  $\{Z'_n\} \geq_{st} \{Z''_n\}$ , if

$$Z'_n \geq_{st} Z''_n, \text{ for all } n \geq 0. \quad (13)$$

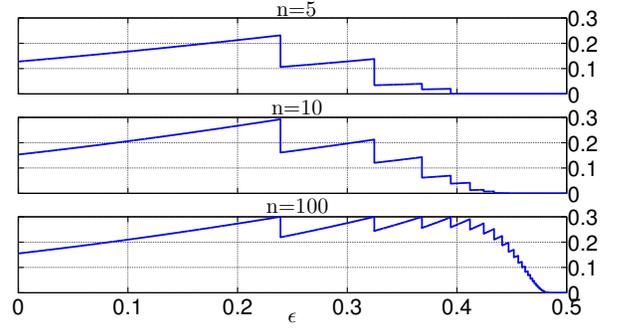
**Definition 7.** A transition matrix  $Q$  is said to be stochastically increasing if for all  $i, i-1 \in \mathcal{S}$ :

$$Q_i \geq_{st} Q_{i-1} \quad (14)$$

where  $Q_i$  denotes the  $i^{\text{th}}$  row of  $Q$  (which is a distribution vector for some random variable).

The proof continues by noting that the corresponding right transition probabilities of the two DTMC's satisfy  $a' > a'' > 0.5$ . Thus,  $Q'_i \geq_{st} Q''_i$  for all  $i \in \mathcal{S}$ . Moreover, the transition matrices for each DTMC are stochastically increasing, and both  $\{Z'_n\}$  and  $\{Z''_n\}$  start from the same state 0. Therefore, by Theorem 4.2.5a and equation (4.2.16) in [6], we have  $\{Z'_n\} \geq_{st} \{Z''_n\}$ . By Defn. 6, it follows that at any arbitrary time  $n$ ,  $Z'_n \geq_{st} Z''_n$ . Let  $z'$  and  $z''$  be the corresponding probability distribution vectors at time  $n$ .

Setting  $j = k$  in Defn. 5, we have  $v_{0,n}^{*'} = z'_k \geq z''_k = v_{0,n}^{*''}$ . Now, letting  $j = -k+1$  yields  $\sum_{i \geq -k+1} z'_i \geq \sum_{i \geq -k+1} z''_i$ , so that  $u_{0,n}^{*'} \leq u_{0,n}^{*''}$ . In fact, we can use a coupling argument to prove that equality does not hold for  $n > k$ , i.e.,  $v_{0,n}^{*'}$  is strictly greater than  $v_{0,n}^{*''}$  for  $n > k$ . These details are shown in Appendix E in [38]. Similarly,  $u_{0,n}^{*''} > u_{0,n}^{*'}$ . Since Lemma 2 shows that  $u_{0,n}^*$  and  $v_{0,n}^*$  are continuous functions of  $\epsilon$  (in the specified interval), this completes the proof.  $\square$



**Figure 4:** Probability of wrong cascade for agent  $n$ , with  $p = 0.70$ .

As a demonstration of Theorem 1, Fig. 4 shows the probability of wrong cascade when  $p = 0.70$ , for agents  $n = 5, 10$  and 100. Notice that for any agent  $n$ , Lemma 1 shows that there exists  $\epsilon$  close enough to 0.5 that yields  $k \geq n$ . The probability of wrong cascade will thus go to zero for large enough  $\epsilon$ , as shown in the figure. Note also that for each agent  $n$ , as  $\epsilon$  increases, the probability of wrong cascade discontinuously decreases at a countable number of points (and continuously increases in between). These points correspond exactly to the values  $\epsilon_k^*$  for different choices of integer  $k \geq 2$ .

### B. Agent Welfare

Let  $\pi_n$  be the payoff or welfare of agent  $n$ . From Section II, we have  $\pi_n = 0$  if  $A_n = N$ , while if  $A_n = Y$ ,  $\pi_n$  is either  $1/2$  or  $-1/2$  corresponding to  $V = G$  or  $V = B$ , respectively. All agents  $1 \leq n \leq k$  use their own signals; thus by averaging over  $V \in \{G, B\}$  they all have the same welfare given by:

$$\begin{aligned} E[\pi_n] &= \{\mathbb{P}[A_n = Y|V = G] - \mathbb{P}[A_n = Y|V = B]\} / 4 \\ &= (2p - 1)/4 \triangleq F. \end{aligned} \quad (15)$$

For agents  $n \geq k+1$ :

$$\begin{aligned} E[\pi_n] &= \{\mathbb{P}[A_n = Y|V = G] - \mathbb{P}[A_n = Y|V = B]\} / 4 \\ &= F + \left[ (1-p)v_{0,n-1}^* - p u_{0,n-1}^* \right] / 2. \end{aligned} \quad (16)$$

For a fixed observation error  $\epsilon$ , (16) explicitly relates the welfare of an agent  $n$ , who faces the possibility of cascade, to the probability of wrong and correct cascades created by the immediate previous agent  $n-1$ . Using results from Theorem 1, the following theorem summarizes some properties of the agents' welfares.

**Theorem 2.** With the same signal quality  $p$  and  $\epsilon \in \mathcal{I}_k$ :

- 1) The expected welfare for each agent is at least equal to the expected welfare of his predecessors. Thus,  $E[\pi_n] \geq F$  and is non-decreasing in  $n$ .
- 2)  $\lim_{n \rightarrow \infty} E[\pi_n]$  exists and equals:

$$\Pi(\epsilon) = F + (1/2) \left\{ 1 / \left[ 1 + ((1-a)/a)^k \right] - p \right\}. \quad (17)$$

- 3) For every agent  $n$ ,  $E[\pi_n]$  decreases continuously as  $\epsilon$  increases over  $\mathcal{I}_k$  so that:

$$\lim_{\epsilon \downarrow \epsilon_k^*} E[\pi_n] > E[\pi_n] > \lim_{\epsilon \uparrow \epsilon_{k+1}^*} E[\pi_n] = F.$$

**Proof** 1) When a cascade happens, every user takes the same action and so achieves the same expected welfare. Thus, we are left to show  $E[\pi_n] \geq F$  for all  $n \geq k+1$ . Using (16) and the form of  $v_{0,i-1}^*$ ,  $u_{0,i-1}^*$ , we only need to show:

$$(1-p)a^{\frac{i+k}{2}}(1-a)^{\frac{i-k}{2}} - pa^{\frac{i-k}{2}}(1-a)^{\frac{i+k}{2}} \geq 0. \quad (18)$$

This can be seen by noting that since  $\epsilon_k^* \leq \epsilon < \epsilon_{k+1}^*$ , we have:

$$0 < ((1-a)/a)^k < (1-p)/p \leq ((1-a)/a)^{k-1} < 1. \quad (19)$$

2) For  $t \in \mathbb{R}$ , let  $V_0(t)$  and  $U_0(t)$  be the probability generating functions for the first hitting time of state  $k$  and  $-k$ , respectively. Using these, the limiting welfare can be written as:

$$\Pi(\epsilon) - F = (1/2)[(1-p)V_0(1) - pU_0(1)]. \quad (20)$$

Expressions for these generating functions are given in equations (47) and (48) in Appendix D of the archived version in [38]; evaluating these at  $t = 1$  yields (17).

3) For a fixed  $p$ , (15) shows that  $F$  decreases in  $\epsilon$ . The proof follows by using (16) and Theorem 1.  $\square$

Theorem 2 suggests a few interesting observations. First, even though a model with no observation errors yields maximum welfare for any agent, the welfare is not monotonically decreasing in  $\epsilon$ . In fact, as the given  $\epsilon$  approaches each  $\epsilon_k^*$  from below, it is better to increase  $\epsilon$  to exactly  $\epsilon_k^*$  so that the welfare for each agent moves to the next local maximum. This comes from the fact that the probability of wrong cascade drops discontinuously as  $\epsilon$  crosses over  $\epsilon_k^*$  due to the structure of the underlying DTMC changing. Second, the first observation raises the question of when and how much observation error should be added, especially when some pre-set error is already present in this model. We will consider this question in the next section.

## V. WHEN IS MORE NOISE BETTER?

### A. Asymptotic welfare

Assume that there is a fixed observation error  $\epsilon$  in the model. Suppose a social planner is allowed to randomly change the history of observations with probability  $\epsilon_s$ . This introduces an effect that is equivalent to increasing the observation error to

$$\epsilon_{total} = \epsilon(1 - \epsilon_s) + (1 - \epsilon)\epsilon_s \triangleq f(\epsilon, \epsilon_s). \quad (21)$$

Fig. 5 shows an example of the asymptotic welfare as a function of  $\epsilon_s$ , when the pre-set observation error is  $\epsilon = 0.15$  and  $p = 0.70$ . It is clear from this figure that the asymptotic welfare is maximized when  $\epsilon_s \approx 0.12$ , i.e., more noise is beneficial. By part 1) of Theorem 2, this also maximizes the (Cesàro) average social welfare of the entire population.

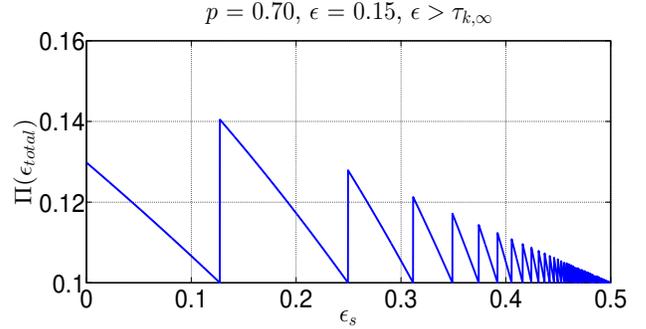


Figure 5: Asymptotic welfare as a function of additional noise.

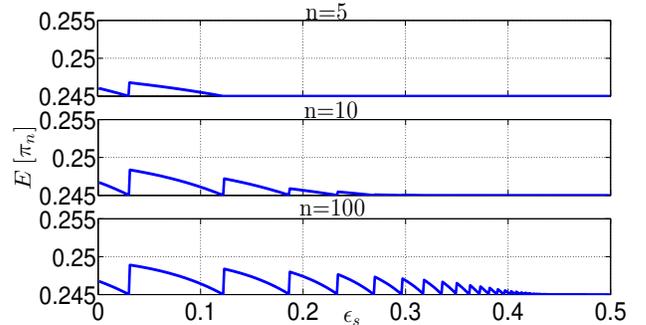


Figure 6: Expected welfare for agent  $n$  with high signal quality.

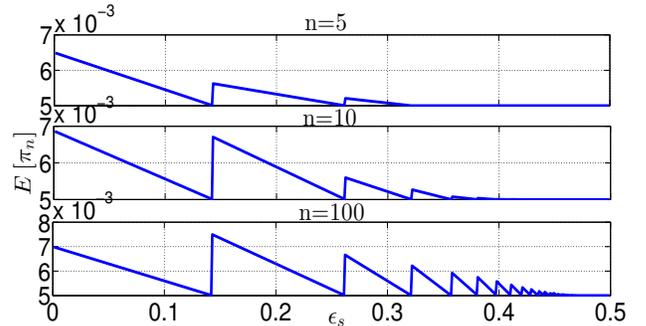


Figure 7: Expected welfare for agent  $n$  with low signal quality.

Whether more noise is beneficial depends on the existing noise  $\epsilon$  and on the signal quality  $p$ . To better understand this, for each  $k$ , define  $\tau_{k,n} \in \mathcal{I}_k$  such that adding some  $\epsilon_s > 0$  will improve the welfare of agent  $n$  if and only if  $\epsilon > \tau_{k,n}$ . Likewise, let  $\tau_{k,\infty}$  denote this threshold for the asymptotic welfare. For the case in Fig. 5, since there exists a  $\epsilon_s > 0$  that improves the welfare, it must be that  $\epsilon > \tau_{k,\infty}$ . Note also that the optimal choice of  $\epsilon_s$  corresponds to a discontinuity of the welfare, which from our previous analysis corresponds to making the effective noise,  $\epsilon_{total}$ , equal to  $\epsilon_{k+1}^*$ . We formalize this phenomenon in the following theorem.

**Theorem 3.** Assume that  $\epsilon \in \mathcal{I}_k$ :

- 1) The asymptotic social welfare is maximized at either  $\epsilon_s = 0$  or  $\epsilon_s = \frac{\epsilon_{k+1}^* - \epsilon}{1 - 2\epsilon} > 0$ .
- 2) The latter case happens when  $\epsilon_{k+1}^* > \epsilon > \tau_{k,\infty} > \epsilon_k^*$ , where

$$\tau_{k,\infty} = \left[ 1 / \left( 1 + \alpha^{(k+1)/k^2} \right) - p \right] / (1 - 2p). \quad (22)$$

**Proof.** 1) The proof of 1) follows by part 3 of Theorem 2.  
2) To prove 2), we need to find  $\epsilon_s$  such that  $\Pi(\epsilon) < \Pi(\epsilon_{total})$ . Using (17), we obtain the lower bound  $\tau_{k,\infty}$ .  $\square$

Note that when it is optimal for the social planner to add  $\epsilon_s > 0$ , the welfare of every individual agent is not necessarily improved. This effect is shown numerically in Fig. 6 and 7 with the same pre-set value  $\epsilon = 0.15$ , for high and low signal quality, respectively. These figures each shows the expected welfare for three agents corresponding to  $n = 5, 10$  and  $100$  as a function of  $\epsilon_s$ . For high signal quality ( $p = 0.99$ ), increasing  $\epsilon_s$  is beneficial for all three agents, i.e.,  $\epsilon > \max(\tau_{k,5}, \tau_{k,10}, \tau_{k,100})$ . On the other hand, for low signal quality ( $p = 0.51$ ), increasing  $\epsilon_s$  will increase the welfare of agent  $n = 100$  and decrease the welfare of agents  $n = 5, 10$ , i.e.,  $\epsilon > \tau_{k,100}$ , but  $\epsilon < \min(\tau_{k,5}, \tau_{k,10})$ .

### B. Individual welfares in the low-noise region

In Fig. 6 and 7, we observe that not all agents' welfares are improved by adding noise. In this section, we take a deeper look and consider the low-noise region, defined as when the pre-set observation error  $\epsilon \in \mathcal{I}_2$ . In this region, it takes only two (consecutive) agents to start a cascade as in the noise-free model. We derive a few important results for the sequence  $\{\tau_{2,n}\}_{n=1}^{\infty}$ , where  $n$  is the agent index. For notational simplicity, we suppress  $k = 2$  and abbreviate  $\tau_{2,n}$  as  $\tau_n$ . Using part 3) of Lemma 3, (16) is rewritten as:

$$E[\pi_n] = F + (1/2) \left[ (1-p)(a/(1-a))^k - p \right] u_{0,n-1}^*. \quad (23)$$

Now using part 1) and 2) of Lemma 3, if  $n > k$  then  $E[\pi_n] = E[\pi_{n-1}]$  is true for  $k$  and  $n$ , both odd or both even. Thus, we can simplify the welfare in (23) for the low noise region, as stated in the next lemma.

**Lemma 4.** Let  $\tilde{a} = a(1-a)$ .

1) For  $\epsilon \in \mathcal{I}_2$ ,  $m \geq 1$ , both  $E[\pi_{2m+2}]$ ,  $E[\pi_{2m+1}]$  are equal to:

$$F + \left[ (1-p)a^2 - p(1-a)^2 \right] \left[ 1 - (2\tilde{a})^m / (1-2\tilde{a}) \right] / 2. \quad (24)$$

2) For  $\epsilon \in \mathcal{I}_3$ ,  $m \geq 2$ , both  $E[\pi_{2m+1}]$ ,  $E[\pi_{2m}]$  are equal to:

$$F + \left[ (1-p)a^3 - p(1-a)^3 \right] \left[ 1 - (3\tilde{a})^{m-1} / (1-3\tilde{a}) \right] / 2. \quad (25)$$

**Proof** See details in the archived version in [38].  $\square$

We remind the reader that by Theorem 2, the expected welfare  $E[\pi_n]$  is non-decreasing in  $n$ . Lemma 4 suggests that the welfare strictly increases between pairs of two consecutive agents, depending on whether  $k$  is odd or even.

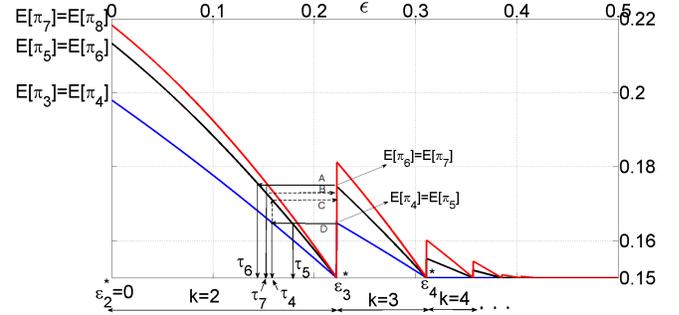
Note that for  $k = 2$ , agents  $n \leq 3$  can never benefit from adding  $\epsilon_s$ . Using Lemma 4, the following theorem characterizes the sequence  $\{\tau_n\}_{n=4}^{\infty}$ . In particular, Theorem 4 says that the sequence  $\{\tau_n\}_{n=4}^{\infty}$  can be divided into even and odd sub-sequences, where both sub-sequences are decreasing and have the same limit  $\tau_{\infty}$  as in (22) with  $k = 2$ . Thus, if adding noise improves the limit welfare, then it will benefit all but the first few finite number of agents.

**Theorem 4.** For  $m \geq 2$ :

1) Both sub-sequences  $\{\tau_{2m}\}$  and  $\{\tau_{2m+1}\}$  decrease and have the same limit  $\tau_{\infty}$ , and

2)  $\tau_{2m+1} > \tau_{2m}$ .

**Proof.** Here, we outline the sketch of the proof for  $\tau_n$ ,  $n \in \{4, 5, 6, 7\}$  with a graphical illustration as in Fig. 8, which plots the welfares of these agents as a function of  $\epsilon$ . To show that part 1) is true, we essentially show  $\tau_4 > \tau_6$  and  $\tau_5 > \tau_7$ . These correspond to showing point A is higher than point C, and point B is higher than point D, respectively. To prove part 2), we essentially show that point C is higher than point D. The complete proof can be found in the archived version in [38].



**Figure 8:** An illustration of ordering for  $\tau_n$ ,  $n \in \{4, 5, 6, 7\}$ .

$\square$

Moreover, Theorem 4 also suggests an interesting fact: with the same signal quality  $p$ ,  $\epsilon \in \mathcal{I}_2$ , and  $m \geq 2$ , if adding noise benefits agent  $2m$  it will also benefit agents  $2m+2, 2m+4, \dots$ ; and if adding noise benefits agent  $2m+1$  then it will also benefit agents  $2m$  and  $2m+3, 2m+5, \dots$ . A consequence of this result is that while adding noise never benefits the first three agents, all other agents' welfares, including agent 4's, are better off as long as agent 5's welfare is.

## VI. ADDING NOISE VERSUS USING GUINEA PIGS

In this section, our goal is to study the trade-offs of two schemes (mentioned below) for welfare improvements. Assume that  $\epsilon \in \mathcal{I}_k$  and  $\epsilon > \tau_{k,\infty}$ , thus increasing the observation error to  $\epsilon_{k+1}^*$  is beneficial for the average asymptotic social welfare. We refer to this as Scheme 1. Extrapolating from Theorem 4, it stands to reason that in Scheme 1 there exists a smallest agent index  $n_1$  who benefits, and there also exists an agent index  $n_2$  beyond which all subsequent agents  $n \geq n_2$  benefit. On the other hand, for agents  $n \in (n_1, n_2)$ , some benefit and some do not. The second scheme we consider is motivated by [19] where the author restricts the first  $M+1$  agents (called "guinea pigs") to use only their private signals. The purpose of this scheme is to skew the initial state distribution toward a correct cascade, owing to the informativeness of the signal quality. Thus, the asymptotic social welfare is improved at the cost of the guinea pigs' payoffs. We summarize these two schemes as follows:

- 1) *Scheme 1:* Increase  $\epsilon$  to the next threshold,  $\epsilon_{k+1}^*$  (keeping  $M = 1$  fixed).
- 2) *Scheme 2:* Sacrifice the first  $M+1$  agents as guinea pigs (keeping  $\epsilon$  fixed).

In Sections VI-A and VI-B below, we characterize the expected welfare for each of these schemes by using (23) and modifying the corresponding probability of wrong cascade.

#### A. Scheme 1: Adding observation errors

As we increase  $\epsilon$  to  $\epsilon_{k+1}^*$ , note that  $\left(\frac{a}{1-a}\right)^k = 1/\alpha$  and  $\frac{v_{0,n}^*}{u_{0,n}^*} = \left(\frac{a}{1-a}\right)^{k+1}$ , for all  $n > 1$ . Thus, (23) becomes:

$$E[\pi_n]_{\epsilon=\epsilon_{k+1}^*} = F + p \left[ \alpha^{-1/k} - 1 \right] u_{0,n-1}^*/2. \quad (26)$$

#### B. Scheme 2: Sacrificing guinea pigs

Using notation from [19], assume that there are  $M+1$  guinea pigs who have to use only private signals.<sup>9</sup> For these agents, we have  $E[\pi_n] = F$ ,  $\forall n = 1, 2, \dots, M+1$ . The effect of having  $M+1$  guinea pigs is equivalent to changing the distribution for the initial state of the Markov chain and then starting this chain with agent  $M+2$ . For each  $i \in \{-M-1, -M, \dots, M, M+1\}$ , denote  $\Delta_i$  as the probability that the  $M+1$  guinea pigs result in  $h_{M+1} = i$  for the subsequent agents  $n \geq M+2$ . For agents  $n \geq M+2$ , let  $\tilde{u}_{i,n}^*$  be the probability of wrong cascade of agent  $n$ , conditioned on  $h_{M+1} = i$ . Their corresponding expected welfare can be written as:

$$F + (1/2) \sum_{|i| \leq M+1} \Delta_i \left[ (1-p) \frac{a}{(1-a)^k} - p \right] \tilde{u}_{i,n-1}^*, \quad (27)$$

where the initial distribution for  $h_{M+1}$  with  $M+1-i$  even is:

$$\Delta_i = \mathbb{P}[h_{M+1} = i] = \binom{M+1}{(M+1-i)/2} a^{\frac{M+1+i}{2}} (1-a)^{\frac{M+1-i}{2}}, \quad (28)$$

and  $\Delta_i = 0$  for  $M+1-i$  odd.

For  $-k < i < k$ , again using techniques from [1], the probability of a wrong cascade first occurring at time  $n \geq M+2$  starting from state  $i$  is given by:

$$\tilde{u}_{i,n} = \begin{cases} 0, & \tilde{n} - (k+i) \text{ odd}, \\ \frac{2^{\tilde{n}}}{k} \left(\frac{1-a}{a}\right)^{\frac{k+i}{2}} (a(1-a))^{\frac{\tilde{n}}{2}} \tilde{B}_{k,\tilde{n}}, & \tilde{n} - (k+i) \text{ even}, \end{cases} \quad (29)$$

where  $\tilde{B}_{k,\tilde{n}} = \sum_{y=1}^{y < k} \cos^{\tilde{n}-1} \left( \frac{y\pi}{2k} \right) \sin \left( \frac{y\pi}{2k} \right) \sin \left( \frac{y\pi(k+i)}{2k} \right)$  and  $\tilde{n} = n - M - 1$ . As a result, the probability of wrong cascade for agent  $n$ ,  $\tilde{u}_{i,n}^*$ , is zero except for the following two cases:  $\tilde{u}_{i,n}^* = 1$  if  $i \leq -k$ , and  $\tilde{u}_{i,n}^* = \sum_{m=k+i}^{\tilde{n}} u_{i,m}$  if  $\tilde{n} \geq k+i$ .

Thus, revisiting (27), the summation in the welfare of every agent  $n$  can be further divided into three components corresponding to three different ranges of  $i$ :  $i < -k$ ,  $-k \leq i \leq k$ , and  $i > k$ . Whether all these three ranges exist depends on the values of  $M+1$  and  $k$ . It is obvious that for a fixed  $k$ , letting  $M+1$  increases increases the number of terms for which  $i > k$ . In other words, for high enough  $M$ , the initial state after the last guinea pig is more likely equal to the correct herding state, thus improving average social welfare. However, this happens at the cost of sacrificing the first  $M+1$  agents. In the next section, we highlight this trade-off and offer an alternative solution that combines the two schemes. We refer to this as:

3) *Scheme 3*: sacrificing the first  $M+1$  agents as guinea pigs, and at the same time increasing  $\epsilon$  to  $\epsilon_{k+1}^*$ .

<sup>9</sup>In [19], the first agent always follows his own signal and the author denotes  $M$  as the number of additional agents forced to follow their own signals. In our case, more than the first agent may always follow their signals, depending on the noise. To be consistent with [19], here we also denote  $M+1$  as the total number of agents that follow their own signals.

### C. Comparisons and trade-offs

1) *Individual welfare*: Figures 9-10 show numerical results for the three schemes mentioned above. Each figure shows  $E[\pi_n]$  versus  $n$  for these schemes; the choice of  $p$  and  $\epsilon$  varies in different figures. In Figs. 9 the signal quality is high, and the initial noise is high (so that  $k = 8$ ). Figs. 10 shows analogous curves with a low signal quality, where the initial noise is low (so that  $k = 2$ ). Other scenarios when  $p$  is high while  $\epsilon$  is low, and vice versa, are shown in [38]. In each figure, the curves labeled  $M = m$  for some integer  $m$  correspond to Scheme 2 with  $m+1$  guinea pigs (where  $M = 1$  is the baseline case since the first two agents always follow their own signals). Scheme 1 is shown in the curve labeled "adding noise" and Scheme 3 is shown in the curve labeled by " $M = k$  and adding noise," where again  $k+1$  is the number of guinea pigs. Note that trade-offs exist for various values of  $p$  and  $\epsilon$ . For high  $M$ , Scheme 2 eventually gives better individual welfare as compared to Scheme 1. However, as the observation error  $\epsilon$  increases, more agents need to be sacrificed. This trade-off is alleviated by using Scheme 3, which yields better welfares while sparing almost half of the would-be guinea pigs.

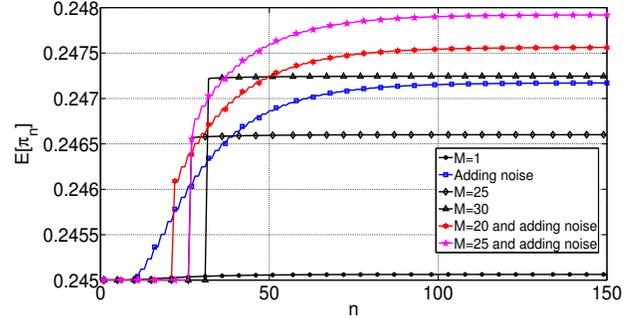


Figure 9: Agent  $n$  welfare for  $p = 0.99$ ,  $k = 8$ ,  $\epsilon = 0.357 \in (\tau_{k,10}, \epsilon_3^*)$ .

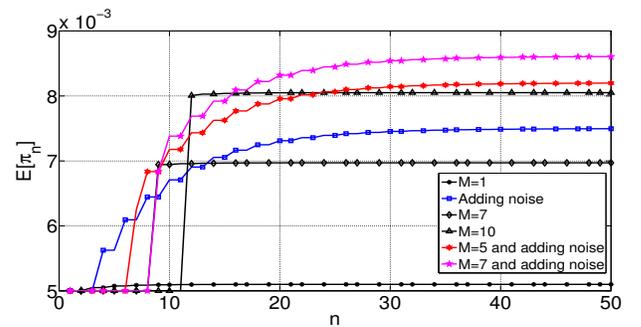


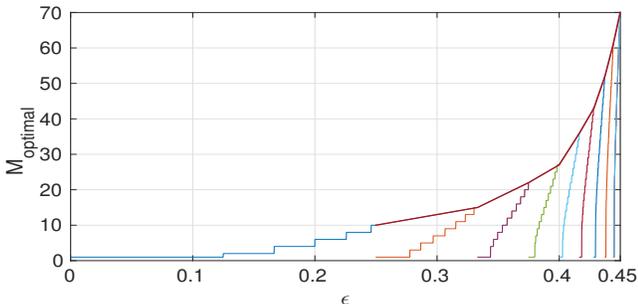
Figure 10: Agent  $n$  welfare for  $p = 0.51$ ,  $k = 2$ ,  $\epsilon = 0.245 \in (\tau_{k,5}, \epsilon_3^*)$ .

2) *Asymptotic welfare*: By the law of large numbers, as  $M+1$  increases, the probability that agent  $M+2$  starts a correct cascade increases to 1. Thus, for large enough  $M$ , Scheme 2 achieves better asymptotic welfare as compared to Scheme 1. In this section, we aim to give a lower bound on the minimum number of guinea pigs,  $M+1$ , that makes Scheme 2 asymptotically better than Scheme 1.

**Proposition 1.** For  $\epsilon \in \mathcal{I}_k$  and  $\epsilon > \tau_{k,\infty}$  Scheme 2 achieves higher asymptotic welfare than Scheme 1 only if  $M+1 > k$ .

**Proof.** We refer to the case where the social planner does nothing as Scheme 0 (i.e. neither uses guinea pigs nor increases  $\epsilon$ ). Assume that  $M + 1 \leq k$ . For any  $\epsilon \in \mathcal{I}_k$ , by Lemma 1 all agents  $n = 1, \dots, k$  must use private signals even when agent  $n$  is not a guinea pig. By this argument, for a given sequence of private signals, Scheme 2 and Scheme 0 have identical samples path, thus yielding identical asymptotic welfare. Moreover, since  $\epsilon > \tau_{k,\infty}$ , Scheme 1 is better than Scheme 0. Thus, the number of guinea pigs,  $M + 1$  needs to be strictly higher than  $k$  to achieve better asymptotic welfare.  $\square$

As a consequence of Proposition 1, as  $\epsilon \rightarrow 0.5$ , i.e.,  $a \rightarrow 0.5$ , we have  $k \rightarrow \infty$ . Thus Scheme 2 cannot give better asymptotic welfare for a finite number of guinea pigs. For  $k = 2, 3, \dots, 10$ , Fig. 11 illustrates how the required minimum  $M$  (obtained numerically) increases as  $\epsilon$  approaches 0.5 for low signal quality. In addition to plotting this for each given  $k$ , we also show a curve that connects the maximum optimal  $M$  for each  $k$ , which is clearly increasing in  $\epsilon$ . This shows that for high noise levels, Scheme 2 is only viable if one is willing to sacrifice a large number of initial users. A similar figure for high signal quality is shown in [38].



**Figure 11:**  $p = 0.51$ , from left to right:  $k = 2, 3, \dots, 10$

## VII. CASCADES WITH NO COMMON DATABASE

So far, our model has assumed that there is observation noise by letting the agents report their actions on a common database which all other agents can see. Even though the reporting process is in error with probability  $\epsilon$ , the observation errors are common knowledge. In other words, what one agent observes reflects exactly what any of the previous agents observed. In some settings, this may be a wrong assumption, e.g., if agents may obtain information using different methods or through different channels. Additionally, such a common database may not be valid in other social networks settings different from online recommendation systems. In this section, we step away from this assumption and look at in some sense the opposite case. In particular, we consider the scenario when each agent individually observes an erroneous realization of the previous agents' true actions, where all errors are assumed to be *i.i.d.* Again, the distribution of the errors is still common knowledge. In such a scenario, not only are the agents blind to other agents' private signals, but they also do not know what their predecessors have observed.

### A. Will cascades persist?

We remind the reader that the key to previous analysis of cascades with a common database is that every subsequent

agent knows the exact answers to two questions: 1) Has a cascade started yet? and 2) If the answer to the previous question is Yes, then when did that cascade start? The reasons are two-fold: 1) The distributions of the prior knowledge about  $V$ , the private signal, and the observation errors are given; and 2) A common observation database is available to all agents. Without the second assumption, no agent can infer if and when herding has started.

Again, since the pre-cascading actions are independent, Property 1 still holds as in the common database scenario. However, the following property shows that, without a common observation database, herding is not recurrent.

**Property 2'** *Without the common database, a cascade needs not be permanent.*

**Proof** Assume that the statement is not true. We will provide a counter-example to this using just four agents. Denote  $O_j^i$  as the observations of agent  $i$  about the action  $A_j$ . Again, assume that  $O_j^i$  is a noisy version of  $A_j$ , so that  $O_j^i$  differs from  $A_j$  in an *i.i.d.* manner with error probability  $\epsilon$  for all agents  $i$  and their predecessors  $1 \leq j \leq i - 1$ . Now the information set of each agent  $i$  includes his private signal  $S_i$  and his observations  $O_1^i, \dots, O_{i-1}^i$  of the true actions  $A_1, \dots, A_{i-1}$ .

The decision rule for agents 1 and 2 is to follow their private information, the same as before. Since the first two agents both use their own signals, agent 3 takes account of the independence of  $O_1^3, O_2^3$ . Thus, agent 3's decision rule remains the same: herds only if  $O_1^3, O_2^3$  are identical and  $\epsilon < \epsilon_3^*$ . Now assuming that  $\epsilon < \epsilon_3^*$ , the Bayesian update of agent 4 is:

$$\gamma_4 = \frac{\mathbb{P}[O_1^4, O_2^4, O_3^4 | V = G] \mathbb{P}[S_4 | V = G]}{\text{numerator} + \mathbb{P}[O_1^4, O_2^4, O_3^4 | V = B] \mathbb{P}[S_4 | V = B]} \quad (30)$$

Now, conditioning on  $V \in \{G, B\}$ , we can write:

$$\begin{aligned} \mathbb{P}[O_1^4, O_2^4, O_3^4] &= \mathbb{P}[O_3^4 | O_1^4, O_2^4] \mathbb{P}[O_1^4, O_2^4] \\ &= (1 - \epsilon) \mathbb{P}[A_3 = Y, O_1^4, O_2^4] + \epsilon \mathbb{P}[A_3 = N, O_1^4, O_2^4], \end{aligned} \quad (31)$$

where (31) holds if  $O_3^4 = Y$ . For  $O_3^4 = N$ ,  $\epsilon$  and  $1 - \epsilon$  are interchanged. The calculation of  $\gamma_4$  is possible if we can evaluate  $\mathbb{P}[A_3, O_1^4, O_2^4]$  for both  $A_3 = Y, N$  and  $V = G, B$ . This can be done by conditioning on the 4 possible realizations of agent 3's observations  $\{O_1^3, O_2^3\} = \{YY, YN, NY, NN\}$ .

As an example, assume that the information set of agent 4 is  $\{S_4, O_1^4, O_2^4, O_3^4\} = \{0, Y, Y, N\}$ . The above procedure gives:

$$\gamma_4 = \frac{[\epsilon \alpha_G + (1 - \epsilon) \beta_G + 2(1 - a) \delta_G] (1 - p)}{\text{numerator} + [\epsilon \alpha_B + (1 - \epsilon) \beta_B + 2a \delta_B] p}, \quad (32)$$

where subscripts correspond to true value  $V \in \{G, B\}$ , and:  $\alpha_G = [a - \epsilon(1 - \epsilon)]^2$ ,  $\alpha_B = [(1 - a) - \epsilon(1 - \epsilon)]^2$ ,  $\beta_G = \beta_B = [\epsilon(1 - \epsilon)]^2$ ,  $\delta_G = \epsilon(1 - \epsilon)[a - \epsilon(1 - \epsilon)]$ ,  $\delta_B = \epsilon(1 - \epsilon)[(1 - a) - \epsilon(1 - \epsilon)]$ . Based on the decision rule in (2), one would need to compare  $\gamma_4$  and  $1/2$ . Using the assumption that  $0 \leq \epsilon < \epsilon_3^*$ , which means  $\frac{a}{1-a} \leq \frac{p}{1-p} < \left(\frac{a}{1-a}\right)^2$ , we deduce  $\gamma_4 < 1/2$ , thus agent 4 follows his own signal and chooses  $N$ . For other realizations of agent 4's observation history, similar reasoning applies and agent 4 herds only when all of his three observations are identical. Now given that agent 3 cascades to the first two agents' action, agent 4 would choose the

same action only if there were no observation errors, i.e., with probability  $(1-\epsilon)^3$ , which is bounded away from 1. Thus in this scenario the cascade does not persist with probability one.  $\square$

### B. Complexity analysis

We now discuss the complexity of updating the posterior probability for an arbitrary agent  $n$  without a common database. Again, assume that  $\epsilon \in \mathcal{I}_k$ . We have the following:

1) For agents  $n = 1, \dots, k$ ,  $|h_{n-1}| < k$  thus cascades cannot happen. The decision rule is simple: always follow one's signal. Thus complexity is  $O(1)$ .

2) However, any agent  $n > k$  faces the possibility that some prior agent was in a cascade. While agent  $n$ 's decision rule remains the same as in (2), the complexity of his Bayesian update of the posterior probability,  $\gamma_n$ , increases exponentially as  $n$  grows. In fact, conditioned on his own observations sequence, every agent  $n$  needs to account for  $2^{n-2}$  possible observations sequences of agent  $n-1$ . For each one of those  $2^{n-2}$  possibilities, agent  $n$  needs to know the decision rule of agent  $n-1$ , which depends on another  $2^{n-3}$  possible realizations of agent  $n-2$ 's history. Since the pre-herding agent  $n=k$ 's update is of order  $O(k)$  (including first calculating the sufficient statistics  $h_{n-1}$  and then comparing that to  $\pm k$ ), the complexity of the update for agent  $n > k$  is:

$$\left(\prod_{m=k}^{n-1} 2^m\right) O(k) = O(k2^{n^2-k^2}), \quad (33)$$

which blows up exponentially as  $(n-k)$  increases.

For comparison, we also evaluate the complexity when a common database is available. For agents  $n = 1, \dots, k$ , the complexity is also  $O(1)$  by simply checking if  $n \leq k$ . For any agent  $n > k$ , the decision process is done by simply computing the sufficient history statistic  $h_{n-1}$  and comparing that with  $\pm k$  at each step. Note that  $|h_{n-1}|$  never exceeds  $k$  throughout the updating process, and so in the worst case agent  $n$  endures a complexity of  $O(n)$ . This illustrates a benefit of having a common online platform for aggregating information. On one hand, such a platform lets agents get more information than they probably could by just having their own observations. On the other hand, it also provides a computational advantage such that if there is noise present, having this common platform makes it easier for agents to learn. The complexity incurred without a common database for Bayes-rational agents suggests that in such case, it is worth considering non-Bayesian approaches in the same spirit as [26]. We leave such approaches for future work.

## VIII. OTHER GENERALIZATIONS

In this section, we discuss other generalizations of our model. We start next by examining different tie-breaking rules.

### A. Comparison of different tie-breaking rules

When an agent's posterior probability satisfies  $\gamma_n = 1/2$ , its expected pay-off under either action is the same. In [9], the authors assumed that when such a tie occurs, agents randomize their choices. Another option for breaking a tie is to follow the previous action whenever a tie happens. In our model,

we instead assume that an agent chooses to follow his own private signal when he is indifferent. Our assumption not only simplifies the analysis, but as we state in the next proposition, also is the better tie-breaking rule as compared to the other two: by following their own signals, agents who face a tie have lower wrong cascade probabilities and hence, higher welfare.

**Proposition 2.** *For a given integer  $k \geq 2$ :*

- 1) *When  $\epsilon \in (\epsilon_k^*, \epsilon_{k+1}^*)$ , all agents are never in a tie. Thus all three tie-breaking rules perform equivalently.*
- 2) *When  $\epsilon = \epsilon_k^*$ , agents who face a tie achieve the lowest (resp. highest) wrong cascade probability if they follow their own signals (resp. follow the previous action).*

**Proof.** We start by reminding the reader that a tie happens when the posterior probability satisfies  $\gamma_n = 1/2$ , i.e., there exists an integer  $k \geq 2$  such that  $\left(\frac{a}{1-a}\right)^k = \frac{p}{1-p}$ . To prove the first part, note that when  $\epsilon \in (\epsilon_k^*, \epsilon_{k+1}^*)$ , the above equality cannot happen. Thus all tie-breaking rules give the same outcome.

To prove the second part, assume that  $\epsilon = \epsilon_k^*$  for some  $k \geq 2$ . If an agent  $n$  encounters a history with  $|h_{n-1}| = k-1$  that is opposite to his private signal, he has to resort to the tie-breaking rule. If he follows the previous action, a cascade happens where the absorbing states of the underlying Markov chain are  $\pm(k-1)$ . On the other hand, if he follows his private signal, a cascade does not happen yet, as the corresponding absorbing states are  $\pm k$ . Using the equations (8)-(10), the latter cases give a lower wrong cascade probability and higher correct cascade probability. As a consequence by (16), when agent  $n$  follows his own signal, he has a higher welfare. Moreover, this also yields a higher asymptotic welfare as seen from (17). This completes the proof.  $\square$

The intuition of Proposition 2 is as follows: if the tie-breaking rule is to follow private signals, the underlying Markov chain has the two absorbing states expanded by one unit on both sides as compared to when following the previous action. This induces a longer expected time until a cascade event (on either side). However, in both scenarios the chains have the same drift, which is always in favor of the correct cascade. This effect decreases (increases) the wrong (correct) cascade probabilities. This can also be seen in Fig. 4: each point of discontinuity happens exactly when  $k$  is increased by 1 unit where the probability of wrong cascades drops. If agents follow the previous actions, this yields the upper end of the discontinuity; while if agents follow their private signals, this yields the lower end of the discontinuity. Moreover, any randomized tie-breaking rule results in a linear combination of these two end points, which also yields a higher wrong cascade probability than when agents follow their own signals.

### B. Cascades with bounded rationality

Next, we consider the possibility that agents might make mistakes in choosing their optimal action. This captures a more realistic assumption since it is well documented that real-world agents are not always rational. In fact, the reasons underlying human irrationality have been well argued in [20] where the authors presented the numerous cognitive deficiencies that could account for human systematic deviation from the

perceived normative behaviors. Further, from the viewpoint of Prospect Theory ([5]), the action error can also be argued to be a consequence of inconsistency in preferences when agents tend to act differently under the possibilities of gains (facing a correct herding) and losses (being in a wrong herd). Finally, such a deviation from full rationality has also been used for equilibrium selection, in particular, for trembling-hand perfection equilibrium ([4]).

Indeed, there is a long history of using noise to model imperfect rationality (see e.g. [8], [14]) resulting in agents having so-called “noisy best responses.” Here, we follow the approach in those works, and examine a simple form of bounded rationality. In particular, assume that all agents derive the exact posterior probability of the true value of the item, but then choose the sub-optimal action with probability  $\epsilon_a \in (0, 0.5)$ ; we call this an “action error.” Assume that this probability is known to all other agents, and is the same across all agents. Again, as before, assume that the observation database is still common knowledge, with *i.i.d.* observation errors with probability  $\epsilon \in (0, 0.5)$ .

The analysis is similar as in the setting with no action errors, so we omit the details. The reason being we can combine the two types of errors,  $\epsilon$  and  $\epsilon_a$ , to an equivalent error type,  $f(\epsilon, \epsilon_a)$ , and use that in updating the posterior probability as before. As a result, the probabilities of wrong and correct cascades,  $u_{0,n}^*$  and  $v_{0,n}^*$ , respectively, are as those in a model with only observation errors of rate  $f(\epsilon, \epsilon_a)$ . However, the agent welfares are now reduced due to the probability of choosing the non-optimal action,  $\epsilon_a$ . In particular, for agents  $1 \leq n \leq k$ , they all have the same welfare given by:

$$\begin{aligned} E[\pi_n] &= \{\mathbb{P}[A_n = Y|V = G] - \mathbb{P}[A_n = Y|V = B]\} / 4 \\ &= (1 - 2\epsilon_a)(2p - 1)/4 = (2a_1 - 1)/4 \triangleq F_1. \end{aligned} \quad (34)$$

where  $a_1 = f(\epsilon_a, p)$ . For agents  $n \geq k + 1$ :

$$\begin{aligned} E[\pi_n] &= \{\mathbb{P}[A_n = Y|V = G] - \mathbb{P}[A_n = Y|V = B]\} / 4 \\ &= F_1 + (1 - 2\epsilon_a) \left[ ((1 - p)/2)v_{0,n-1}^* - (p/2)u_{0,n-1}^* \right]. \end{aligned} \quad (35)$$

By comparing (34)-(35) against (15)-(16), it is clear that the action error  $\epsilon_a$  only reduces the welfare of every agent by a factor of  $1 - 2\epsilon_a$  as compared to a model with only observation errors with probability  $f(\epsilon, \epsilon_a)$ . Thus, the expected welfare of agents in a quasi-rational model inherits many properties from the fully rational setting with noisy observations as in Theorem 2 (e.g., the overall welfare is not monotonic in the error rate).

### C. General prior for the true value, $V$

In this section, we consider the scenario where the binary true value  $V$  of the item might not be equally likely, but the *ex-ante* pay-off is still restricted to be zero.<sup>10</sup> In particular, assume that  $\mathbb{P}[V = G] = \lambda \in (0, 1)$ . Let the rewards (resp. losses) of buying when  $V = G$  (resp.  $V = B$ ) be  $x$  (resp.  $-y$ ) where  $x, y \geq 0$  and  $x + y > 0$ . In all previous sections, we have assumed  $x = 1, y = 0$  and  $\lambda = 1/2$ . In this general scenario,

setting the *ex-ante* payoff to 0 leads to the new cost of the item being  $C = \lambda x - (1 - \lambda)y$ . Putting these all together, the general payoff is given by:

$$\pi_n = \begin{cases} 0, & \text{if } A_n = N, \\ x - C = (1 - \lambda)(x + y) & \text{if } A_n = Y, V = G, \text{ and} \\ -y - C = -\lambda(x + y) & \text{if } A_n = Y, V = B. \end{cases} \quad (36)$$

The *ex-post* expected payoff of agent  $n$  when taking an action  $A_n \in \{Y, N\}$  is calculated as:

$$\begin{cases} E[\pi_n|A_n = N] = 0, \text{ and} \\ E[\pi_n|A_n = Y] = (x + y)(\gamma_n - \lambda). \end{cases} \quad (37)$$

By (37), the new decision rule is based on comparing the posterior probability,  $\gamma_n$ , to  $\lambda$ . However, as  $\lambda$  changes, the posterior probability is a function of  $\lambda$  and is given by:

$$\gamma_n = \mathbb{P}[V = G|S_n, \mathcal{H}_n] = \frac{1}{1 + \beta_n \ell_{n-1} \frac{1-\lambda}{\lambda}} \quad (38)$$

Thus, the new (user) optimal decision rule compares  $\beta_n \ell_{n-1}$  and 1. Moreover, since  $\beta_n, \ell_{n-1}$  are conditioned on  $V$  and, thus, do not change as  $\lambda$  changes, the new decision rule with respect to  $\beta_n \ell_{n-1}$  is the same as before. Conditioned on  $V$ , the transition probabilities of the underlying Markov chain are identical to the case  $\lambda = 1/2$ . As a result, conditioned on  $V$ , the probabilities of cascades for an agent  $n$  are also given by:

$$\begin{cases} \mathbb{P}[\text{wrong}|V = G] = u_{0,n}^* = 1 - v_{0,n}^*, \text{ and} \\ \mathbb{P}[\text{wrong}|V = B] = v_{0,n}^* = 1 - u_{0,n}^*. \end{cases} \quad (39)$$

However, the (unconditional) probabilities of wrong and correct cascades are given as  $\lambda u_{0,n}^* + (1 - \lambda)v_{0,n}^*$  and  $\lambda v_{0,n}^* + (1 - \lambda)u_{0,n}^*$ , respectively. Given that  $\lambda, x, y$  are fixed, the welfare for agents are scaled by a constant factor as compared to when  $V$  is equally likely  $G$  or  $B$ . We show this next. All agents  $1 \leq n \leq k$ , who use their own signals, have the same payoff:

$$\begin{aligned} E[\pi_n] &= (1 - \lambda)E[\pi_n|V = B] + \lambda E[\pi_n|V = G] \\ &= (1 - \lambda)(-\lambda)(x + y)P[A_n = Y|V = B] \\ &\quad + \lambda(1 - \lambda)(x + y)P[A_n = Y|V = G] \\ &= \lambda(1 - \lambda)(x + y)(2p - 1) \triangleq F_2. \end{aligned} \quad (40)$$

For agents  $n \geq k + 1$ , who face the possibility of cascades:

$$\begin{aligned} E[\pi_n] &= \lambda(1 - \lambda)(x + y) \{\mathbb{P}[A_n = Y|V = G] - \mathbb{P}[A_n = Y|V = B]\} \\ &= \lambda(1 - \lambda)(x + y) \left[ v_{0,n-1}^* + p(1 - v_{0,n-1}^* - u_{0,n-1}^*) \right] \\ &\quad - \lambda(1 - \lambda)(x + y) \left[ u_{0,n-1}^* + (1 - p)(1 - u_{0,n-1}^* - v_{0,n-1}^*) \right] \\ &= F_2 + 2\lambda(1 - \lambda)(x + y) \left[ v_{0,n-1}^* - u_{0,n-1}^* \right]. \end{aligned} \quad (41)$$

Thus, comparing (40) and (41) to (15) and (16), the welfare of all agents are scaled by a factor of  $4\lambda(1 - \lambda)(x + y)$ . If  $x + y > 0$  is fixed while  $\lambda$  can change in  $(0, 1)$ , the welfare is maximized if the true value  $V$  is equally likely  $G$  or  $B$ , i.e.  $\lambda = 1/2$ .

<sup>10</sup>The case of zero *ex-ante* payoff is the interesting case as at the outset agents have no preference for either action, and thus the additional information from the signals and past observations could be helpful.

## IX. CONCLUSIONS AND FUTURE WORK

This paper studied the effect of observation error in a simple Bayesian information cascade where agents can be either rational or quasi-rational. We showed that despite the presence of observation errors, cascades happen in finite time with probability 1. However, observation errors can delay the onset of a cascade; we also determined the error thresholds that increase this delay. In addition, cascades have the same level of fragility with or without observation errors, and in both cases they can be broken by an additional private signal. Using a Markov chain based analysis we determined the probabilities of cascades for an arbitrary agent and used these to calculate the agents' welfare based on the given signal quality and the error. Our main result shows that for certain ranges of parameters, adding a controlled amount of observation error can lead to higher welfare for all but a finite number of agents. In such scenarios, we compared the trade-offs of different methods for improving the agents' welfares. These results may be helpful for a platform operator who has the option to weigh the trade-offs and choose the best method that would benefit his platform. Our results are strongly based on the existence a common database, which stores all the erroneous observations of past actions. Moreover, we argued that such a database provides an important computational benefit to agents in performing Bayesian updates. In future work we plan on generalizing to agents with heterogeneous private signals and observation errors, and allowing each agent to only observe subsets of past agents' actions prior to taking actions.

## REFERENCES

- [1] W. Feller, *An introduction to Probability Theory and Its Applications*, vol. I, 3rd ed., New York Wiley, 1968.
- [2] T. Cover, *Hypothesis Testing with finite statistics*, Ann. Math. Stat., vol. 40, No. 3, pp. 828-835, 1969.
- [3] M. Hellman, *Learning with finite memory*, Ph.D. Dissertation, Stanford University, 1969.
- [4] R. Selten, *A reexamination of the perfectness concept for equilibrium points in extensive games*, IJGT, vol. 4, pp. 25-55, 1975.
- [5] D. Kahneman, A. Tversky, *Prospect Theory: An Analysis of Decision under Risk*, Econometrica, vol. 47, issue 2, pp. 263-292, 1979.
- [6] D. Stoyan, *Comparison Methods for Queues and Other Stochastic Models*, John Wiley and Sons, 1983.
- [7] T.L. Lai, H. Robbins, *Asymptotically efficient adaptive allocation rules*, Adv. Appl. Math., vol. 6, pp. 4-22, 1985.
- [8] R. Rosenthal, *A Bounded Rationality Approach to the Study of Noncooperative Games*, IJGT, vol. 18, pp. 273-92, 1989.
- [9] S. Bikhchandani, D. Hirshleifer, and I. Welch. *A Theory of Fads, Fashion, Custom and Cultural Change as Informational Cascades*, J. Polit. Econ., vol. 100, No. 5, pp. 992-1026, 1992.
- [10] A. Banerjee, *A Simple Model of Herd Behavior*, Q. J. Econ., vol. 107, pp. 797-817, 1992.
- [11] I. Welch, *Sequential Sales, Learning, and Cascades*, J. Finance, vol. 47, issue 2, pp. 695-732, 1992.
- [12] I. Lee, *On the Convergence of Informational Cascades*, J. Econ. Theory, vol. 61, pp. 396-411, 1993.
- [13] R. Agrawal, *Sample mean based index policies with  $O(\log n)$  regret for the multi-armed bandit problem*, Adv. Appl. Math., vol. 27, pp. 1054-1078, 1995.
- [14] R. McKelvey, T. Palfrey, *Quantal Response Equilibria for Normal Form Games*, GEB, vol. 10, pp. 6-38, 1995.
- [15] S. Bikhchandani, D. Hirshleifer, and I. Welch. *Learning from the Behavior of Others: Conformity, Fads, and Informational Cascades*, J. Econ. Perspect., vol. 12, No. 3, pp. 151-170, 1998.
- [16] V. Bala, S. Goyal, *Learning from Neighbours*, Rev. Econ. Stud., Wiley Blackwell, vol. 65, issue 3, pp. 595-621, 1998.
- [17] L. Smith, P. Sorensen, *Pathological Outcomes of Observational Learning*, Econometrica, vol. 68, pp. 371-398, 2000.
- [18] P. Sorensen, *Rational Social Learning*, Ph.D. Dissertation, Massachusetts Institute of Technology, 1996.
- [19] D. Sgroi, *Optimizing Information in the Herd: Guinea Pigs, Profits, and Welfare*, GEB, vol. 39, pp. 137-166, 2002.
- [20] E. Shafir, L. Robyn, *Rationality*, Ann. Rev. Psychol., vol. 53, pp 491-517, 2002.
- [21] F. Dalmagro, J. Jimenez, R. Jimenez and H. Lugo, *Co-evolution of bounded rational agents in adaptive social networks*, CEF, No. 354, 2005.
- [22] J. Surowiecki, *The Wisdom of Crowds*, New York: Anchor Books, 2005.
- [23] D. Ariely, *Predictably Irrational: The Hidden Forces That Shape Our Decisions*, Revised and Expanded Edition, Harper Perennial, 2010.
- [24] S. Veeraraghavan, L. Debo, *Herding in queues with waiting costs: Rationality and regret*, M&SOM, vol. 13, issue 3, pp. 329-346, 2011.
- [25] D. Acemoglu, M. Dahleh, I. Lobel, and A. Ozdaglar, *Bayesian learning in social networks*, Rev. Econ. Stud., vol. 78, pp. 1201-1236, 2011.
- [26] A. Jadbabaie, P. Molavi, A. Sandroni, and A. Tahbaz-Salehi, *Non-Bayesian social learning*, GEB, vol. 76, pp. 210-225, 2012.
- [27] D. McFadden, *The new science of pleasure: consumer choice behavior and the measurement of well-being*, NBER Work. Pap. No. 18687, 2013.
- [28] R. Greenwood, A. Shleifer, *Expectations of Returns and Expected Returns*, NBER Work. Pap. No. 18686, 2013.
- [29] Z. Zhang, E. Chong, A. Pezeszki, W. Moran, *Hypothesis Testing in Feedforward Networks With Broadcast Failures*, JSTSP, vol. 7, issue 5, pp. 797-810, 2013.
- [30] K. Drakopoulos, A. Ozdaglar, J. N. Tsitsiklis, *On Learning With Finite Memory*, IEEE Trans. Inf. Theory, vol. 59, issue 10, pp. 6859-6872, 2013.
- [31] A. Gionis, E. Terzi, P. Tsaparas, *Opinion maximization in social networks*, SIAM SDM, pp. 387-395, 2013.
- [32] M. Förster, A. Mauleon, V. Vannetelbosch, *Trust and manipulation in social networks*, Network Science, vol. 4, issue 2, pp. 216-243, 2014.
- [33] T. Le, V. Subramanian, R. Berry, *The Value of Noise for Informational Cascades*, IEEE ISIT Proceeding, pp. 1101-1105, 2014.
- [34] T. Le, V. Subramanian, R. Berry, *The Impact of Observation and Action Errors on Informational Cascades*, IEEE CDC Proceeding, pp. 1917-1922, 2014.
- [35] Y. Wang, P. Djuric, *Social learning with Bayesian agents and random decision making*, IEEE Trans Sig. Process, vol. 63, no. 12, 2015.
- [36] Y. Song, *Social Learning with Endogenous Network Formation*, submitted to J. Econ. Theory, 2015.
- [37] S. Etesami, S. Bolouki, A. Nedić, T. Başar, *Conformity versus Manipulation in Reputation Systems*, IEEE CDC proceeding, pp. 4451-4456, 2016.
- [38] T. Le, V. Subramanian, R. Berry, *Information Cascades with Noise*, longer archived version of this paper, available at: [https://archive.org/details/info\\_cascades\\_noise\\_journal](https://archive.org/details/info_cascades_noise_journal)