MIMO Precoding with Limited Rate Feedback: Simple Quantizers Work Well

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Abstract— **Transmitter precoding is a crucial technique for harnessing the potential of multiple-input multiple-output (MIMO) fading channels. In many practical wireless systems, a limited amount of feedback from the receiver is available at the transmitter, which can be used to direct the choice of the precoder from a codebook to match the channel state. Assuming noiseless, limited-rate feedback, this work studies the design of simple, efficient quantization and feedback schemes which achieve near-optimal ergodic channel capacity. In the case the precoder takes the form of a beamforming vector for modulating a single symbol stream, it is found that simple scalar quantization of the elements of the vector is nearly optimal over a wide range of feedback rates; it typically costs a fraction of a dB higher SNR to achieve the same capacity as that of far more sophisticated vector quantization schemes. In the case a precoding matrix consisting of multiple beams is used to modulate multiple symbol streams, separate encoding of the beams using scalar quantization also performs well. Roughly speaking, the rate loss due to separate encoding of the beams increases linearly with the number of beams but appears to be constant over a wide range of SNRs. The loss can be reduced substantially by more sophisticated encoding of each beam, e.g., two-state trellis coded quantization. The complexity of such quantization schemes is linear in the number of antennas and the number of feedback bits.**

I. Introduction

It is well-known that with full channel state information (CSI) at the transmitter, precoding in the form of water-filling across the eigenstates of a multiple-input multiple-output (MIMO) channel can improve its capacity. In many wireless systems, the channel state can only be measured by the receiver, which then directs the transmitter's precoder through a limited amount of feedback. With limited-rate feedback, which can be less than one bit per channel coefficient, the design of the precoding codebook is crucial for exploiting the potential of the MIMO channel.

Suppose B bits of noiseless feedback are available at the beginning of each coherent block; the precoding problem can be simply formulated as follows. A codebook consisting of 2^B precoders is designed and made available to both the transmitter and the receiver. The receiver selects the precoder from the codebook which best matches the channel matrix and sends its index (B bits) to the transmitter. Clearly, the problem can also be regarded as quantization of the channel state. Ideally, the codebook should be designed to maximize the ergodic capacity of the MIMO channel averaged over the fading statistics.

The design and analysis of codebooks for MIMO precoding has been studied extensively [1]–[5]. In the special case of a single receive antenna, the precoder reduces to a beamformer. It has been shown that codebook optimization can be interpreted as maximizing the minimum distance between points in a Grassmannian space [2]. The optimal Grassmannian codebook [2] and an asymptotically optimal scheme known as random vector quantization (RVQ) [3] have been proposed. With multiple receive antennas, the precoder may consist of multiple beams for modulating multiple symbol streams to exploit the dimensions of the received signal space. In such cases, the codebook design is discussed in [5], where a metric for distance between two matrices is proposed and the Lloyd algorithm is used to compute the optimal codebook iteratively.

The aforementioned feedback schemes are impractical when the codebook is relatively large (e.g., when $B \geq 8$) because the codebook is not designed with simple structures to facilitate search and storage at the receiver. To find the optimal precoder, the computational complexity is in general exponential in the number of feedback bits. The difficulty is magnified in multicarrier systems where a precoder has to be selected for each sub-channel.

This work addresses the following question: Can much simpler feedback coding schemes be designed to approach the channel capacity achieved by RVQ? The answer is shown to be positive by introducing specific schemes. In particular, we first investigate scalar quantization (SQ), namely, separate quantization of the coefficients of the beams corresponding to the maximum eigenstates of the MIMO channel using phase-shift keying (PSK) alphabets. In the case of beamforming, the performance is found to be surprisingly good. Numerical results suggest a loss of merely a fraction of a dB in signal-to-noise ratio (SNR) compared to the far more complicated RVQ scheme. In the case of multiple beams, scalar quantization still delivers a competitive trade-off in terms of performance and complexity. Roughly speaking, the loss in terms of capacity is multiplied by the number of beams, but otherwise stays essentially constant for all moderate to high SNRs.

We note that the design of structured codebooks is also considered in [6]–[8]. Reference [6] considers MIMO beam-

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forming with no less than one bit of feedback per complex coefficient, where noncoherent sequence detection is used to search over the codebook, which is more complicated than scalar quantization. A similar approach is considered in [7]. Reference [8] considers codebook based on Kerdock codes which consists of multiple mutually unbiased unitary matrices with quaternary entries and the identity matrix. The reduction of search complexity with this codebook comes from the fact that the entries of the codebook are scaled version of $\{1, -1, j, -j\}$, and the multiplication operation reduces to flipping signs and/or swapping real and imaginary components. However, the search complexity is still exponential in the number of feedback bits.

In order to reduce the gap between scalar quantization and RVQ, we also consider encoding of each beam using trellis coded quantization (TCQ), which is analogous to trellis coded modulation [9], [10]. Using the Viterbi algorithm to search the codebook, the TCQ scheme entails only linear complexity in the number of feedback bits per coherent block. TCQ was first proposed for quantization in [11] and has recently been independently proposed for beamformer design in [7]. The trellis considered here consists of only two states and has much lower complexity than the code presented in [7].

This paper also presents a simple large-system analysis of the performance of SQ and TCQ. Specifically, we consider the case where the number of symbol streams (i.e., the number of beams, which is also the rank of the precoding matrix) is fixed while the number of feedback bits and transmit antennas goes to infinity with fixed ratio. It is shown that in the beamforming case there is a constant capacity gap per beam between SQ (or TCQ) and the case with prefect CSI at transmitter. With multiple beams the gap is equal to that constant multiplied by the rank of the precoding matrix.

II. MIMO Precoding

Consider a single-user, narrowband wireless MIMO system with N_t transmit antennas and N_r receive antennas. Consider precoding of $K \le \min\{N_r, N_t\}$ independent symbol streams. That is, during any symbol interval, the symbol from the k-th stream, denoted by x_k, is weighted by a complex vector $\hat{\boldsymbol{v}}_k$ of dimension $N_t \times 1$ with power constraint $\|\hat{\boldsymbol{v}}_k\|^2 = 1$, where $\|\cdot\|$ is the vector two-norm. Let $\hat{\boldsymbol{V}} = [\hat{\boldsymbol{v}}_1 \ \dots \ \hat{\boldsymbol{v}}_K]$ be referred to as the precoding matrix. Let $\boldsymbol{H}$ denote the $N_r \times N_t$ matrix of channel coefficients, which are assumed to be independent and identically distributed (*i.i.d.*) with circularly symmetric complex Gaussian (CSCG) distribution of unit variance per complex dimension. Let $\boldsymbol{x} = [x_1 \ \dots \ x_K]^T$. The received signal of dimension $N_r \times 1$ can be expressed as

$$\boldsymbol{y} = \sqrt{\rho}\, \boldsymbol{H}\hat{\boldsymbol{V}}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where ρ denotes the SNR gain of the channel, and $\boldsymbol{n}$ denotes the noise vector consisting of CSCG elements of unit variance per complex dimension.

Assume that the time-average of $|x_k|^2$ does not exceed $1/K$, so that the transmitted vector signal is constrained to unit power. The capacity of the channel is then

$$C(\hat{\boldsymbol{V}}) = \log\det\left(\boldsymbol{I} + \frac{\rho}{K}\boldsymbol{H}\hat{\boldsymbol{V}}\hat{\boldsymbol{V}}^\dagger\boldsymbol{H}^\dagger\right) \tag{2}$$

in bits[1], where $(\cdot)^\dagger$ denotes Hermitian transpose. Suppose KB bits of noiseless, zero-delay feedback is available per coherence block. Let $\mathcal{V} = \{\boldsymbol{V}_1, \dots, \boldsymbol{V}_{2^{KB}}\}$ denote the quantization codebook. With full CSI at the receiver, the precoding matrix is choosen to be $\hat{\boldsymbol{V}} \in \mathcal{V}$ which maximizes the rate $C(\hat{\boldsymbol{V}})$.

To find the optimal precoding matrix, the computational complexity of the brute force search increases exponentially with KB, namely $O(2^{KB})$, if there is no inherent structure in the codebook to simplify the search. Motivated by the prohibitive computational complexity of unstructured codebooks, we propose two types of simple structured codebooks with corresponding encoding schemes of linear complexity, while the codebook performance is nearly optimal in terms of ergodic capacity.

III. Scalar Quantization (SQ) for Beamforming

In the case of single receive antenna ($N_r = 1$), the precoded MIMO system reduces to a multiple-input single-output (MISO) beamforming system. The channel matrix and the precoding matrix reduce to two vectors ($\boldsymbol{h}$, a row vector, and $\hat{\boldsymbol{v}}$ respectively), so that the received signal can be expressed as

$$y = \sqrt{\rho}\boldsymbol{h}\hat{\boldsymbol{v}}x + n. \tag{3}$$

A. The SQ Scheme

The optimal beamformer from the codebook is chosen as

$$\hat{\boldsymbol{v}} = \arg\max_{\boldsymbol{v}_k \in \mathcal{V}} |\boldsymbol{h}\boldsymbol{v}_k|^2. \tag{4}$$

Let $\bar{B} \triangleq B/N_t$, which denotes the number of feedback bits per complex coefficient of the precoding matrix. Suppose $\bar{B} \ge 1$ for a moment. SQ quantizes the channel vector element-by-element, i.e., for each element h_i, choose an element $\hat{v}_i$ from a finite-size scalar reproduction alphabet $\mathcal{A}$ according to some criterion. We choose $\mathcal{A}$ to be a certain set of PSK symbols depending on the number of feedback bits per coefficient (normalized by $1/\sqrt{N_t}$) and the quantization rule is to pick $\hat{v}_i \in \mathcal{A}$, so that the real part of $h_i\hat{v}_i$ is maximized, i.e.,

$$\hat{v}_i = \arg\max_{v \in \mathcal{A}} \mathrm{Re}\{h_i v\}, \ i = 1, \dots, N_t. \tag{5}$$

It is straightforward to see that this criterion is equivalent to choosing $\hat{v}_i$ which is best aligned with the phase of $h_i^\dagger$.

In the case $\bar{B} < 1$, it is impossible to quantize the channel vector element-by-element. To generalize the SQ method to this case, we can combine multiple elements into one element and quantize the sum of multiple elements with SQ. Take for example the case $B = N_t/2$, i.e., $\bar{B} = 1/2$. Let $\hat{v}_i$ be chosen to maximize the real part of $\hat{v}_i(h_{2i-1} + h_{2i})$, i.e.,

$$\hat{v}_i = \mathrm{sgn}(\mathrm{Re}\{h_{2i-1} + h_{2i}\}), \ i = 1, \dots, B, \tag{6}$$

[1] All logarithms are of base 2 and the units of all information measures are bits throughout the paper.

where $\text{sgn}(\cdot)$ takes the sign of a real number. The beam is thus quantized as

$$\hat{\boldsymbol{v}} = \frac{1}{\sqrt{N_t}} [\hat{v}_1 \ \hat{v}_1 \ \hat{v}_2 \ \hat{v}_2 \ \ldots \ \hat{v}_B \ \hat{v}_B]^T \ . \tag{7}$$

Note that for $\bar{B}$ that is neither an integer nor the inverse of an integer, for instance, $1 < \bar{B} < 2$, SQ can be done in the following way: a fraction $(\bar{B} - 1)$ of the channel coefficients are quantized with QPSK symbols and the remaining fraction $(2 - \bar{B})$ are quantized with BPSK symbols.

It should be noted that although we discuss SQ based on the MISO channel vector $\boldsymbol{h}$, it is essentially equivalent to implement SQ (or TCQ to be discussed in section IV) on its phase $\boldsymbol{h}/\|\boldsymbol{h}\|$, which is isotropic over the N_t-dimensional complex unit hyper-sphere.

B. Performance Analysis

The ideal beamformer in this case is the normalized channel vector $\boldsymbol{h}^\dagger/\|\boldsymbol{h}\|$. As pointed out in [3], $\boldsymbol{h}\boldsymbol{h}^\dagger/N_t \to 1$ almost surely as $N_t \to \infty$, so that $\log(1 + \rho \boldsymbol{h}\boldsymbol{h}^\dagger) - \log(\rho N_t) \to 0$. This implies that with perfect CSI at the transmitter, the ergodic channel capacity increases as $\log(\rho N_t)$. To gain more insight into the performance of the proposed structured codebooks, we adopt a similar analytical approach as in [3], which considers the asymptotic rate loss w.r.t. the case with perfect CSI. Similar to the definition in [3], we define the rate loss for SQ with the MPSK symbols I_{sq}^{Δ} as

$$I_{sq}^{\Delta} = \log(\rho N_t) - I_{sq}^{N_t} \tag{8}$$

where $I_{sq}^{N_t}$ is the ergodic capacity of SQ with N_t transmit antennas. It is difficult to quantify the rate loss analytically for finite number of antennas. Some insights can, however, be gained by considering a specific large-system limit where B and N_t tend to infinity with their ratio $\bar{B}$ fixed.

Proposition 1: In the large-system limit, the rate loss due to SQ I_{sq}^{Δ} almost surely converges to a deterministic constant α. For integer-valued $\bar{B}$ ($\bar{B} \geq 1$), it is given by $\alpha = -2\log(\frac{\sqrt{\pi}}{2}\text{sinc}(2^{-\bar{B}}\pi))$, and for $\bar{B}$ that is the inverse of an integer ($\bar{B} < 1$), $\alpha = \log(\pi/\bar{B})$.

Proof: See Appendix A.

The expressions for α are also accurate approximations for arbitrary $\bar{B}$. For finite-size systems, the quantization rule can be modified to take advantage of the differences in the real and imaginary parts. To be specific, two SQs can be done, one to maximize the real parts as in (5), and the other to maximize the imaginary parts. Afterwards, we can choose one based on the quantization rule in (4).

Fig. 1 shows the performance of SQ in terms of ergodic capacity in a MISO beamforming system with different number of feedback bits (obtained by simulation). For comparison, the performance of RVQ (obtained by simulation) is also shown, which can be regarded as essentially the optimal performance for the given feedback. Clearly, the performance degradation due to SQ w.r.t. RVQ is small (less than 1 dB in terms of SNR) in all the cases with different feedback rates. In addition, even with a finite number of transmit antennas ($N_t = 8$), the rate loss is more or less a small constant (independent of the SNR) though it is not equal to (actually less than) the constant given in Proposition 1 derived in the large-system limit. The inconsistency observed is due to the finite size of the system and the fact that we exploit the differences between the real and imaginary parts of the channel. The percentage rate loss is small for moderate to large ρ and vanishes as $\rho \to \infty$.

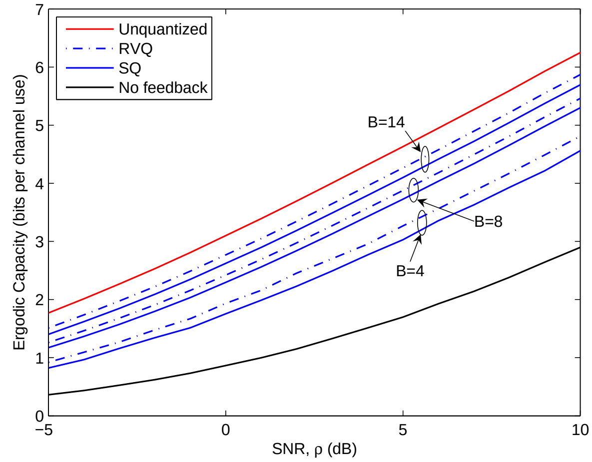

Fig. 1. Capacity of scalar quantization for MISO beamforming system with $N_t = 8$.

IV. Trellis Coded Quantization for Beamforming

A. The TCQ Scheme

By exploiting the duality between digital modulation and source coding, a modified trellis coded modulation [9] technique, namely, TCQ was proposed in [11] for source coding of memoryless and Gauss-Markov sources with mean squared error (MSE) distortion metric. The key feature of the TCQ approach is the use of a structured codebook with an expanded set of quantization levels compared with SQ. The trellis structure then prunes the expanded number of quantization levels down to the quantization rate based on set partitioning.

We present only a simple example that is quite straightforward and intuitive to illustrate the benefits of TCQ compared to SQ. Suppose the quantization/encoding rate is $\bar{B} = 1$. With TCQ, the two PSK symbols per channel coefficient are doubled to four PSK symbols and then partitioned into two subsets, $D_0 = \{1, -1\}$ and $D_1 = \{j, -j\}$. We use the simplest trellis which only has two states $\{0, 1\}$ (one memory register). If in state "0", the output is chosen from D_0 according to the current input, and if in state "1", the output is chosen from D_1 according to the current input. One stage of the corresponding trellis diagram is shown in Fig. 2 (a) with transitions labeled $[m, c]$, where m is the input bit that triggers the transition and c is the output. 2^B binary vectors of length B are passed through the shift register, and all the valid paths in the B-stage trellis diagram form the TCQ codebook. Shown in Fig. 2 (b) is one stage of the trellis diagram used for $\bar{B} = 1/2$ in the numerical results.

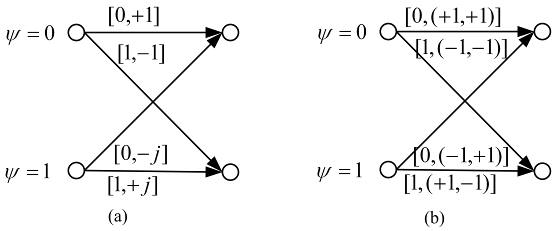

Fig. 2. One stage trellis diagram of TCQ with: (a) $\bar{B}=1$ and (b) $\bar{B}=1/2$. The binary state of the register is denoted by ψ.

If MSE is chosen as the branch metric in the trellis, the Viterbi algorithm is the optimal quantizer/encoder. The quantization rule in that case is to find the minimum Euclidean distance between the received vector and the valid paths in the trellis. And the Euclidean distance can be decomposed as the sum of the metrics in each branch, which is a precondition to applying the Viterbi algorithm. In the beamforming problem, however, the optimal quantization rule in (4) is to find the maximum magnitudes of the inner products between the channel vector and the codewords, which is not an additive metric. Thus, it is suboptimal to perform per-element operation in a trellis at each stage and the Viterbi algorithm cannot be applied with this quantization rule.

Similar to the SQ case, the quantization rule we choose for TCQ is to maximize $\mathrm{Re}\{\boldsymbol{h}\boldsymbol{v}_k\}$ instead of $|\boldsymbol{h}\boldsymbol{v}_k|^2$ despite the fact that it is suboptimal. Since $\mathrm{Re}\{\boldsymbol{h}\boldsymbol{v}_k\}=\frac{1}{2}(\|\boldsymbol{h}\|^2+\|\boldsymbol{v}_k\|^2-\|\boldsymbol{h}^\dagger-\boldsymbol{v}_k\|^2)$, $\|\boldsymbol{h}\|^2$ remains the same for any $\boldsymbol{v}_k$, and $\|\boldsymbol{v}_k\|^2=1$, then maximizing $\mathrm{Re}\{\boldsymbol{h}\boldsymbol{v}_k\}$ is equivalent to minimizing the Euclidean distance between $\boldsymbol{h}^\dagger$ and $\boldsymbol{v}_k$, which is amenable to low-complexity implementation using the Viterbi algorithm.

B. Performance Analysis

A quick observation to the trellis in Fig. 2 (a) is that with MSE as the quantization rule, the performance of TCQ is at least as good as SQ. The reason is that the outputs on the two diverging branches (± 1 or $\pm j$) at each state have opposite signs and the distributions of the real and imaginary parts of the channel coefficient are identical. Similarly, we can get some insight into TCQ by considering its asymptotic rate loss with respect to (w.r.t.) the case with perfect CSI. Define the rate loss of TCQ I_{tcq}^{Δ} as

$$I_{tcq}^{\Delta}\triangleq\log(\rho N_t)-I_{tcq}^{N_t} \tag{9}$$

where $I_{tcq}^{N_t}$ is the ergodic capacity of TCQ with N_t transmit antennas.

Proposition 2: For the trellis example in Fig. 2 (a), in the large-system limit, the rate loss due to TCQ I_{tcq}^{Δ} is almost surely no more than a deterministic constant α, where $\alpha\approx 1.53$ bits per channel use.

Proof: See Appendix B.

Compared with the asymptotic rate loss of SQ, $\log(\pi)\approx 1.65$, TCQ can provide about 0.12 bit gain.

Fig. 3 shows the performance of TCQ in terms of ergodic capacity in a MISO beamforming system with different numbers of feedback bits (obtained by simulation). For comparison, the performance of SQ and RVQ shown in Fig. 1 is also included. One can see that TCQ outperforms SQ and the gap between TCQ and RVQ is negligible.

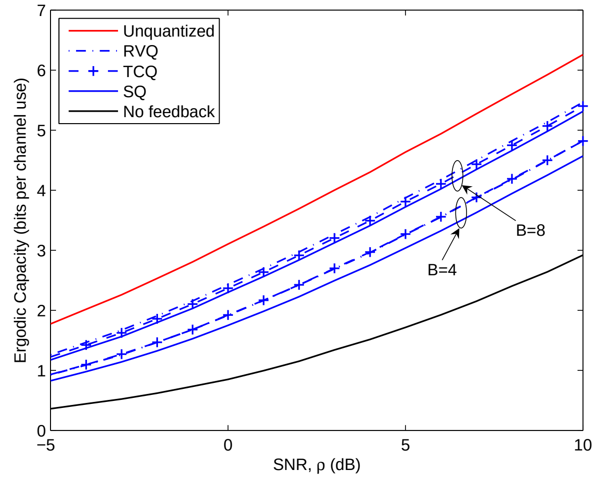

Fig. 3. Capacity of trellis coded quantization for MISO beamforming system with $N_t=8$.

V. Structured Codebook with Arbitrary Rank

In this section we consider the structured codebook of a narrowband single-user precoded MIMO system with multiple symbol streams ($K>1$). Note that K can also be understood as the number of beams as well as the rank of the precoding matrix.

Let the singular value decomposition of $\boldsymbol{H}$ be given by

$$\boldsymbol{H}=\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\dagger, \tag{10}$$

where $\boldsymbol{U}$ has dimension $N_r\times N_r$ and $\boldsymbol{V}$ has dimension $N_t\times N_t$, both are unitary matrices, and $\boldsymbol{D}$ is an $N_r\times N_t$ diagonal matrix. According to [5], the ideal precoder with quantization criterion (2) is $\hat{\boldsymbol{V}}=\bar{\boldsymbol{V}}_K$ where $\bar{\boldsymbol{V}}_K$ is a matrix constructed from the first K columns of $\boldsymbol{V}$, which correspond to its largest singular values.

We propose to quantize each column of $\bar{\boldsymbol{V}}_K$ separately with SQ or TCQ, so that the search complexity can be reduced from $O(2^{KB})$ to $O(KB)$. Regarding the performance of this per-vector based quantization, as in the MISO beamforming case, here we wish to determine the asymptotic rate loss w.r.t. the case with perfect CSI in the large-system limit, in the sense that N_r and K ($K\leq N_r$) are fixed, and N_t and B tend to infinity with fixed ratio $\bar{B}$. Using heuristic arguments, we conclude that the rate loss of per-vector based precoding matrix quantization with multiple transmit rank increases roughly linearly with the number of beams. Intuitively, if the number of transmit antennas is much larger than the number of beams, then the interference caused by one quantized beam to the others is almost negligible even though the beams are

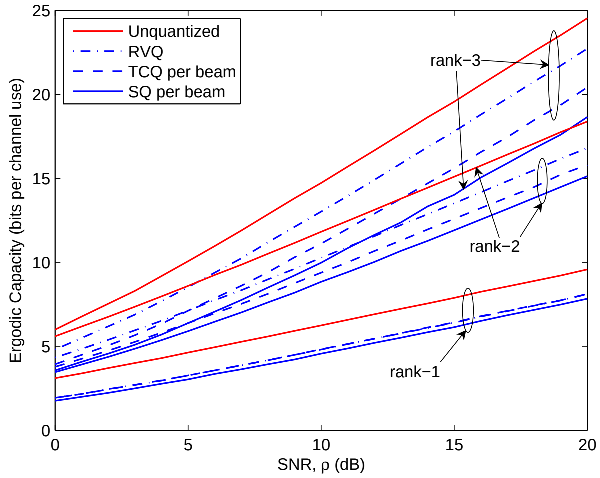

Fig. 4. Capacity of per-vector based SQ and TCQ precoding matrix quantization in a MIMO system with $N_t = 8$, $N_r = 4$, $B = 4$, and $\bar{B} = 1/2$. Rank stands for the number of beams in the precoding matrix.

quantized separately, which does not maintain orthogonality. Thus the rate loss can be approximated by the sum of the rate loss in each beam, so that it increases linearly with the transmit rank.

In Fig. 4, we plot the ergodic capacity of a precoded MIMO system with various number of beams (obtained by simulation). The performance of RVQ (randomly generating 2^{KB} unitary matrices) is also included for comparison. One can see that in the multiple-beam case, SQ or TCQ based per-vector quantization of the precoding matrix also works well and the gap between the unquantized and per-vector based precoding matrix quantization increases roughly linearly with the number of beams. This is also true for the gap between per-vector based SQ (or TCQ) and RVQ.

VI. Conclusions

To overcome the exponential computational complexity of unstructured precoding codebooks, we proposed two simple quantization schemes, namely scalar quantization and trellis coded quantization for MISO beamfoming, which have linear complexity while achieving near optimal performance in terms of ergodic capacity. In particular, to achieve the same ergodic capacity, the SNR gap between SQ and random vector quantization is less than 1 dB, whereas the performance gap between TCQ and RVQ is almost negligible. For precoded MIMO with arbitrary rank, we propose to quantize the beams in the precoding matrix separately using SQ or TCQ. The performance gap w.r.t. the case with full transmitter CSI remains relatively constant over a wide range of SNRs. We conclude that simple scalar quantization and trellis coded quantization may have roles to play in practical precoded MIMO systems.

Appendix

A. Proof of Proposition 1

We prove this proposition in two different cases corresponding to different values of $\bar{B}$.

1) Integer-Valued $\bar{B}$: Denote the beamforming vector as $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{N_t}}[\hat{v}_1 \ \ldots \ \hat{v}_{N_t}]^T$. The rate loss is given by

$$I_{sq}^{\Delta} = \log(\rho N_t) - \log\left(1 + \frac{\rho}{N_t}\left|\sum_{i=1}^{N_t} h_i \hat{v}_i\right|^2\right) \quad (11)$$

$$= -\log\left(\frac{1}{\rho N_t} + \left|\frac{1}{N_t}\sum_{i=1}^{N_t} h_i \hat{v}_i\right|^2\right) \quad (12)$$

$$= -\log\left(\frac{1}{\rho N_t} + \left|\frac{1}{N_t}\sum_{i=1}^{N_t} |h_i| e^{j\theta_i}\right|^2\right). \quad (13)$$

By the distribution of h_i and the quantization rule of SQ, we know that $|h_i|$ follows Rayleigh distribution and θ_i is uniformly distributed in the interval $[-2^{-\bar{B}}\pi, 2^{-\bar{B}}\pi]$. Then, $\mathbb{E}[|h_i|] = \sqrt{\pi}/2$, $\mathbb{E}[\cos\theta_i] = \text{sinc}(2^{-\bar{B}}\pi)$, and $\mathbb{E}[\sin\theta_i] = 0$. Invoking the law of large numbers, the sample mean converges to the distribution mean almost surely, i.e.,

$$\frac{1}{N_t}\sum_{i=1}^{N_t} |h_i| e^{j\theta_i} \to \mathbb{E}[|h_i|(\cos\theta_i + j\sin\theta_i)] = \frac{\sqrt{\pi}}{2}\text{sinc}(2^{-\bar{B}}\pi). \quad (14)$$

Substituting (14) into (13) and let N_t go to infinity, the desired result follows.

In the case that $\bar{B}$ is not an integer, by quantizing the channel coefficients with two different quantization levels, it is easily shown that α can accurately approximate the rate loss.

2) $\bar{B}$ Is the Inverse of an Integer: The rate loss in this case can be derived similarly. Denote $\bar{B} = 1/N$, where N is an integer. We view $h_i^N \triangleq \sum_{l=N(i-1)+1}^{Ni} h_l$ as one element and SQ is done with BPSK symbols. Going through similar derivation as in (11)–(13), we get

$$I_{sq}^{\Delta} = -\log\left(\frac{1}{\rho N_t} + \left|\frac{1}{N_t}\sum_{i=1}^{B} |h_i^N| e^{j\theta_i}\right|^2\right), \quad (15)$$

where $\mathbb{E}[|h_i^N|] = \sqrt{N\pi}/2$, $\mathbb{E}[\cos\theta_i] = 2/\pi$, and $\mathbb{E}[\sin\theta_i] = 0$. Thus, $\frac{1}{N_t}\sum_{i=1}^{B} |h_i^N| e^{j\theta_i}$ converges to $\sqrt{1/\pi N}$ almost surely as $N_t \to \infty$. Then the desired result follows for $\bar{B} = 1/N$. For $\bar{B}$ that is not the inverse of an integer, by combining the channel coefficients with two different ways it is easily shown that the resulting expressions are good approximations.

B. Proof of Proposition 2

To get the exact asymptotic rate loss of TCQ seems quite challenging even with the simple trellis structure in Fig. 2 (a). An upper bound on the asymptotic rate loss can be derived by considering the suboptimal quantization method – sequentially quantizing the channel vector every two channel coefficients. The whole trellis is divided into pieces, each with two stages (since the minimum distance error event spans two stages). In the i-th piece (stage $2i-1$ and $2i$), in state "0", the metric

Re$\{h_{2i-1}v_{2i-1} + h_{2i}v_{2i}\}$ can take the following four values: Re$\{h_{2i-1} + h_{2i}\}$, Re$\{h_{2i-1} - h_{2i}\}$, Re$\{-h_{2i-1} + jh_{2i}\}$, and Re$\{-h_{2i-1} - jh_{2i}\}$. Define X_i as the maximum of these four random variables. Since both the real and imaginary parts of the channel and the trellis are symmetric, in state "1", the same random variable X_i can be used to represent the maximum metric. Then the rate loss of TCQ can be calculated as

$$I_{tcq}^{\Delta} = \log(2\rho N_t) - \log\left(1 + \frac{\rho}{2N_t}\left|\sum_{i=1}^{2N_t} h_i \hat{v}_i\right|^2\right) \tag{16}$$

$$\leq -\log\left(\frac{1}{2\rho N_t} + \left|\frac{1}{2N_t}\sum_{i=1}^{N_t} X_i\right|^2\right) \tag{17}$$

$$\to -\log\left(\frac{\mathbb{E}[X_i]}{2}\right)^2 \text{ a.s., as } N_t \to \infty. \tag{18}$$

Based on the distributions of h_{2i-1} and h_{2i}, it is easily shown that X_i can be simplified to

$$X_i = \max\left\{|Y_1| + |Y_2|, -|Y_1| + |Y_3|\right\}, \tag{19}$$

where Y_l are *i.i.d.* Gaussian with mean 0 and variance $1/2$. We resort to Monte Carlo methods to calculate $\mathbb{E}[X_i]$ and it is given by $\mathbb{E}[X_i] \approx 1.1771$. Therefore, the lower bound of the asymptotic rate loss of TCQ is given by $\alpha \approx 1.53$.

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