Beamforming Techniques for Single-Beam MIMO Interference Networks

David A. Schmidt, Wolfgang Utschick, and Michael L. Honig

***Abstract*— We consider the joint optimization of beamformers and linear receivers in a MIMO interference network. Each transmitter transmits a single beam corresponding to a rank-one precoder. When the number of users K is greater than the number of antennas at each terminal N, the maximum degrees of freedom is achieved via spatial interference alignment. Interference alignment is feasible for up to $K = 2N-1$ users, in which case there is a finite number of solutions to the alignment conditions. This number of solutions increases rapidly with N, and the solutions depend only on the cross-channel coefficients (i. e., they are independent of the direct channels). To maximize the achievable sum rate at high SNRs we therefore wish to select an aligned solution which is best matched to the direct channels. We evaluate the performance of this scheme for large K and N, assuming that the solution is the best out of a random subset of aligned solutions. We then compare numerically this performance with the performance of previously proposed numerical (e. g., forward-backward) techniques for optimizing beams, and a new technique which tracks the local optimum as the SNR is incrementally increased, similar to a homotopy method for improving convergence properties. We observe that the incremental technique typically achieves better performance than the previously proposed methods.**

I. INTRODUCTION

Determining the sum-rate optimal beamforming strategy in MIMO interference networks is a difficult problem. Not only is it in general impossible to express the optimal strategy in closed form, in many cases there exist multiple local optima, making it even more challenging to find the global optimum. Recently, much research has been devoted to understanding the problem at asymptotically high signal-to-noise ratios (SNRs). In the high-SNR regime it is clear that the interference must be fully separable from the desired signal. This can be achieved by *alignment* of the interfering signals at the receivers in lower-dimensional subspaces [1], [2].

When the number of users and beams is such that the zero-interference conditions form a system of equations with equally many variables and equations, there is only a finite number of beamforming strategies for which the interference is aligned [2]. The challenge of finding the optimal beamforming strategy at high SNR is therefore equivalent to finding the best one among these aligned solutions. Computing and comparing all aligned solutions, however, is a very difficult problem and is computationally infeasible unless the system dimensions are very small, i. e., when there are more than two antennas per node.

Instead, a number of iterative algorithms have been proposed [3], [4], [5], [6] that reliably converge towards a "good" (but not necessarily globally optimal) aligned solution. These algorithms can be numerically compared for different scenarios, but deriving analytical performance results appears to be difficult.

Another possibility is to compute many ($L \gg 1$) different aligned solutions directly from the alignment conditions and then choose the best one. The performance of this approach can be accurately approximated analytically as a function of L [7] assuming that the method for computing the different aligned solutions does not depend on the direct channels.

In this paper we compare the performance of the "Max-of-L Alignment" (MLA) strategy with the iterative methods in [3], [4], [5]. We also introduce another numerical optimization method in which the beamformers are re-optimized as the SNR is incrementally increased. Our results show that this incremental technique performs significantly better than the iterative (forward-backward) algorithms in [3], [4], [5] for large loads (K/N) and high SNRs. Furthermore, the iterative methods perform significantly better than the MLA method even when L is very large (although still less than the total number of aligned solutions). Hence we conclude that a computationally feasible search for the optimal aligned solution must take into account the direct channels.

In the next section we introduce the system model, and in Section III we discuss the behavior at asymptotically high SNRs. In Section IV we summarize the performance approximation proposed in [7], in Section V we present a novel numerical method based on gradually increasing the SNR, and in Section VI we compare our new method with the state of the art and the analytical performance approximations by means of numerical simulations.

II. SYSTEM MODEL

We examine a scenario with K transmitter-receiver pairs, or *users*, where each transmitter and receiver has N antennas. The complex channel gains between the antennas of transmitter j and the antennas of receiver k are the elements of the channel matrix $\boldsymbol{H}_{kj} \in \mathbb{C}^{N\times N}$. Each transmitter k forms the transmitted vector by multiplying its single unit-variance data symbol b_k with its respective beamforming vector $\boldsymbol{v}_k$. For all users k, we require $\|\boldsymbol{v}_k\|_2^2 \leq 1$, corresponding to a unit power constraint.

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D. A. Schmidt and W. Utschick are with the Associate Institute for Signal Processing, Technische Universität München, 80290 Munich, Germany dschmidt@tum.de, utschick@tum.de

M. L. Honig is with the Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60208, USA mh@eecs.northwestern.edu

The channel output at receiver k is

$$\boldsymbol{y}_k = \boldsymbol{H}_{kk}\boldsymbol{v}_k b_k + \sum_{j\neq k} \boldsymbol{H}_{kj}\boldsymbol{v}_j b_j + \boldsymbol{n}_k \tag{1}$$

where $\boldsymbol{n}_k$ is the Gaussian noise with zero mean and covariance matrix $\mathrm{E}[\boldsymbol{n}_k\boldsymbol{n}_k^{\mathrm{H}}] = \sigma^2\mathbf{I}$. We define σ^{-2} as the SNR. Receiver k is interested in decoding only the data stream b_k from $\boldsymbol{y}_k$; the components from all other transmitters $j \neq k$ are interference.

We assume that all users employ random Gaussian codebooks and decode their data stream (treating the interference as additional noise) after applying a linear receive filter $\boldsymbol{g}_k^{\mathrm{H}}$. Without loss of generality, we can assume $\|\boldsymbol{g}_k\|_2^2 = 1$ for all users k. The following rate is achievable by user k:

$$R_k = \log\left(1 + \frac{|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_{kk}\boldsymbol{v}_k|^2}{\sum_{j\neq k}|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_{kj}\boldsymbol{v}_j|^2 + \sigma^2}\right) \tag{2}$$

$$= \log\left(1 + \boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{H}_{kk}^{\mathrm{H}}\left(\sum_{j\neq k}\boldsymbol{H}_{kj}\boldsymbol{v}_j\boldsymbol{v}_j^{\mathrm{H}}\boldsymbol{H}_{kj}^{\mathrm{H}} + \sigma^2\mathbf{I}\right)^{-1}\boldsymbol{H}_{kk}\boldsymbol{v}_k\right) \tag{3}$$

where (2) holds for any unit-norm receive filter $\boldsymbol{g}_k$ and (3) holds for the optimal receive filter vector.

The overall goal is to optimize the sum of all users' rates:

$$\max_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_K} \sum_{k=1}^{K} R_k \quad \text{s.t.:} \quad \|\boldsymbol{v}_k\|_2^2 \leq 1 \;\; \forall k \in \{1,\ldots,K\}. \tag{4}$$

By taking the derivative of the Lagrangian function w.r.t. $\boldsymbol{v}_k$, we obtain the following necessary conditions for local and global optimality:

$$\boldsymbol{A}_k \cdot \boldsymbol{v}_k = \lambda_k \cdot \boldsymbol{v}_k \qquad \forall k \in \{1,\ldots,K\} \tag{5}$$

where

$$\boldsymbol{A}_k = \frac{1}{1+\gamma_k}\boldsymbol{H}_{kk}^{\mathrm{H}}\boldsymbol{X}_k^{-1}\boldsymbol{H}_{kk} - \sum_{j\neq k}\frac{1}{1+\gamma_j}\boldsymbol{H}_{jk}^{\mathrm{H}}\boldsymbol{X}_j^{-1}\boldsymbol{H}_{jj}\boldsymbol{v}_j\boldsymbol{v}_j^{\mathrm{H}}\boldsymbol{H}_{jj}^{\mathrm{H}}\boldsymbol{X}_j^{-1}\boldsymbol{H}_{jk} \tag{6}$$

and

$$\boldsymbol{X}_k = \sum_{j\neq k}\boldsymbol{H}_{kj}\boldsymbol{v}_j\boldsymbol{v}_j^{\mathrm{H}}\boldsymbol{H}_{kj}^{\mathrm{H}} + \sigma^2\mathbf{I} \tag{7}$$

is the interference-plus-noise covariance matrix of receiver k and

$$\gamma_k = \boldsymbol{v}_k^{\mathrm{H}}\boldsymbol{H}_{kk}^{\mathrm{H}}\boldsymbol{X}_k^{-1}\boldsymbol{H}_{kk}\boldsymbol{v}_k \tag{8}$$

is the signal-to-interference-and-noise ratio (SINR) at receiver k. In addition to the norm constraint on the beamformers $\boldsymbol{v}_k$, we have

$$\lambda_k \geq 0 \quad \text{and} \quad \lambda_k(\|\boldsymbol{v}_k\|_2^2 - 1) = 0 \qquad \forall k \in \{1,\ldots,K\}. \tag{9}$$

Note that even though (5) has the form of a simple eigenvector problem, each matrix $\boldsymbol{A}_k$ depends on all beamforming vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_K$ and a non-trivial solution in closed form cannot be readily obtained. Furthermore, numerical experiments show that there are in general multiple sets of beamforming vectors that fulfill these conditions, a subset of which are local optimizers of (4).

Unless K and N are small, it is generally computationally infeasible to find the globally optimal solution to (4) in acceptable time. Instead, we must rely on iterative algorithms to find "good" local optima that are in general not globally optimal.

III. OPTIMALITY AT HIGH SNR

Additional properties of the solution to problem (4) can be given for the regime of asymptotically high SNR, i. e., for $\sigma^{-2} \to \infty$. From (2), we observe that for a given set of beamformers and receive filters R_k approaches either a constant or a constant plus $\log\sigma^{-2}$ for the case that the sum interference power term is non-zero or zero, respectively. Therefore, the sum rate is

$$\sum_{k=1}^{K} R_k = s \cdot \log\sigma^{-2} + r + \mathrm{o}(1) \tag{10}$$

as $\sigma^{-2} \to \infty$, where s is the number of users with zero interference power after the receive filter and r is a constant; s and r can be interpreted as the slope and the y-axis intercept (or offset) of the high-SNR asymptote of the sum rate plotted versus SNR in dB.

If all K users are interference-free, i. e., if

$$\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_{kj}\boldsymbol{v}_j = 0 \qquad \forall(k,j) \in \{1,\ldots,K\}^2 \quad \text{with} \quad k \neq j, \tag{11}$$

then from (2) it follows that the slope is $s = K$ and the sum-rate offset is

$$r = \sum_{k=1}^{K} \log|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_{kk}\boldsymbol{v}_k|^2. \tag{12}$$

Clearly, for high enough SNRs, the slope s dominates the sum rate, i. e., to solve (4) at high SNRs it is necessary to maximize the number of receivers that do not experience any interference. In [2] it was shown that if the channel coefficients $\boldsymbol{H}_{kj}$ are drawn from a continuous distribution, it is possible to fulfill (11) if $K \leq 2N - 1$ almost surely. Conversely, if $K > 2N - 1$ the conditions almost surely cannot be simultaneously fulfilled.

The feasibility argument in [2] is based on counting the number of equations and variables and showing independence of the coefficients in the resulting system of polynomial equations. In particular, when $K = 2N - 1$, the number of equations equals the number of variables, and by the same argument the zero-interference conditions almost surely have a finite number of isolated solutions. This number of solutions is known to be two for a system with $N = 2$ and $K = 3$, and 216 for the next larger (fully loaded) system with $N = 3$ and $K = 5$. This number becomes increasingly difficult to determine for larger systems, as it involves the computation of *mixed volumes* (Bernshtein's Theorem, cf. [2]). An upper bound on the number of solutions is given by Bézout's Theorem, which grows as $\mathrm{O}\big(c^{N^2+\alpha N}\big)$ where $c > 1$ and $\alpha \in \mathbb{R}$ are constants.

In the following we focus on systems with $K = 2N - 1$ and assume that the appropriate number of users has been determined in a previous step of user selection or scheduling. For (11) to be fulfilled, $K - 1 = 2N - 2$ interfering beams must be orthogonal to every receive filter $\boldsymbol{g}_k \in \mathbb{C}^N$, i. e., they may only occupy an $(N - 1)$-dimensional subspace at the kth receiver. This property is referred to as *interference alignment*.

Note that the direct channels $\boldsymbol{H}_{kk}$ do not appear in the system of equations (11). Consequently, while all solutions to (11) have the same slope s, they can differ in the offset r, which is defined solely by the direct channels $\boldsymbol{H}_{kk}$, cf. (12). Therefore, the solution to the optimization problem (4) at asymptotically high SNR is the particular zero-interference (or aligned) solution that results in the highest value of r.

Computing different aligned solutions, let alone finding all aligned solutions, is generally not possible in closed form. Finding the roots of the zero-interference equations is computationally demanding and feasible only for very small networks. In constrast, iterative algorithms have been proposed that converge to a set of beamformers and receive filters that fulfills the zero-interference conditions in reasonable time, e. g., the "minimum leakage" algorithm in [3]. Here we consider two approaches to determining a good, but not globally optimal, high-SNR strategy:

1) We compute many different zero-interference solutions by running an iterative algorithm until convergence from many different initial conditions; we then compare the corresponding sum-rate offsets r for the resulting beamformer/receiver sets and choose the best one. Assuming the offsets are statistically independent and the number of sampled solutions L (as well as the system dimension N) is reasonably large, we are able to approximate the expected value of r averaged over channel realizations. This analysis was presented in [7] and the approximation is summarized in the following section. We refer to this approach as "Max-of-L Alignment" (MLA).
2) We run a numerical algorithm that is designed to maximize the sum rate (or a related objective) directly. Many such algorithms have been proposed [3], [4], [5], often with suitability for distributed implementation in mind. As opposed to the preceding strategy, their performance is difficult to analyze and we must rely on numerical comparisons. In Section V we present a new numerical algorithm based on gradually incrementing the SNR.

In Section VI we numerically compare the sum rates for these different approaches using the analytical approximation for the first MLA method.

IV. PERFORMANCE OF THE MLA SOLUTION

In this section we examine the statistical properties of r in (12) for an i. i. d. channel model. If the number of summands K is large, we can use the central limit theorem to assume a Gaussian distribution, and argue that taking the best out of L independently chosen aligned solutions can be modeled by taking the maximum out of L independent realizations of this Gaussian distribution. A result from extreme statistic then gives us an approximation for the average performance of the MLA strategy.

We assume that the elements of all matrices $\boldsymbol{H}_{jk}$ are drawn independently from a unit-variance complex Gaussian distribution. Because the zero-interference conditions do not depend on the direct channels $\boldsymbol{H}_{kk}$, we fix the aligned solution as an arbitrary set of unit-norm beamformers and receivers $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_K)$ and then randomly draw the direct channels $\boldsymbol{H}_{kk}$. The resulting desired signal powers $|\boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H}_{kk} \boldsymbol{v}_k|^2$ have an exponential distribution, and the mean and variance of $\log|\boldsymbol{g}_k^{\mathrm{H}} \boldsymbol{H}_{kk} \boldsymbol{v}_k|^2$ are $-\gamma$ and $\pi^2/6$, respectively, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant [7].

Consequently, if the number of independent summands of r, i. e., the number of users K, is large, r is approximately Gaussian distributed with mean $-K\gamma = -(2N - 1)\gamma$ and variance $K\pi^2/6 = (2N - 1)\pi^2/6$, cf. (12). Note that this is the distribution of r over random channel realizations, assuming that for each channel realization one out of the finite number of beamformer/receiver sets that fulfill (11) is chosen at random.

Instead of taking one random aligned solution per channel realization, however, we would like to compare L different random solutions for the same channel realization and take the one with the highest value of r. It is argued in [7] that if the dimensionality N of the channel matrices is large, it is irrelevant whether we compare L random beamformer/receiver sets for the same channel realization or L random beamformer/receiver sets for different random channel realizations (assuming that the aligned beamformers/receivers are isotropically distributed), as the singular spectra of large random matrices are approximately deterministic.

Therefore, for large N the offset r for the MLA solution can be approximated as taking the maximum out of L realizations of a Gaussian random variable with mean $-(2N-1)\gamma$ and variance $(2N - 1)\pi^2/6$. From extreme statistics it is known that for large L the maximum of L realizations of a Gaussian random variable has a shifted and scaled Gumbel distribution. Specifically, for our Gaussian approximation of the rate offset r the mean of the resulting Gumbel distribution approaches

$$\mathrm{E}\left[r_{\max}\right] \approx -(2N - 1)\gamma + \pi\sqrt{(2N - 1)/6}(\ell + \gamma/\ell) \quad (13)$$

as L becomes large, where $\ell = Q^{-1}(1/L)$ and $Q^{-1}(\cdot)$ is the inverse of the Q-function [7], [8].

In practice, an MLA solution can be obtained by running the "minimum leakage" algorithm proposed in [3] starting from L independent initializations (e. g., chosen from an isotropic distribution). The iterative updates in this algorithm do not depend on the direct channels $\boldsymbol{H}_{kk}$ and the only objective is to fulfill the zero-interference conditions. We emphasize that the previous analysis does not apply to the "maximum SINR" algorithm in [3] or to the algorithms proposed in [4], [5] since the updates in these algorithms depend on the direct channels.

Although this analysis relies on both N and L being asymptotically large, it yields a good prediction of the simulated performance even for moderate system dimensions, as is demonstrated in Section VI. We note, however, that if L grows much faster than N, e. g., $L = c^{N^2}$, the approximations can become inaccurate, as the maximum out of L realizations is so far right on the tail of the probability density function that the Gaussian approximation is inaccurate. In particular, this analysis cannot be used to predict the performance obtained by taking the best out of *all* of the aligned solutions, since the number of aligned solutions may grow as c^{N^2}.

V. INCREMENTAL-SNR OPTIMIZATION

A few algorithms have been proposed that are directly aimed at maximizing the sum rate or a related criterion. The "maximum SINR" algorithm in [3] is based on repeatedly exchanging the roles of transmitters and receivers and after each switch updating all receive filters to maximize the SINR. A different global SINR utility function is defined in [4] and transmitters and receivers are alternatingly updated to maximize the ratio of the network-wide sum signal power to the network-wide sum interference plus noise power. A similar algorithm aimed at minimizing the weighted sum mean square error and adapting the weights in order to find a local optimum of the sum rate was proposed in [5]. In [6], the sum rate is optimized directly over the covariance matrices instead of the beams.[1] Analytical performance evaluation of these algorithms seems difficult; they can, however, be compared numerically for a given channel model.

In contrast to the preceding approaches, here we propose an approach which attempts to track a local optimum as the SNR is incrementally increased from zero. (Equivalently, all transmitters increase their power simultaneously.) This is motivated by the observation that for low SNR the interference can be ignored, so that the optimal beam $\boldsymbol{v}_k$ is the eigenvector of $\boldsymbol{H}_{kk}^{\mathrm{H}}\boldsymbol{H}_{kk}$ corresponding to the maximum eigenvalue. (This also follows from the conditions (5)–(8).) It is furthermore motivated by the intuition that the best high-SNR solution will be in some sense "close" to the low-SNR optimizer.

This intuition is illustrated in Fig. 1: when the SNR is low, the optimal strategy is easily determined and depends on the direct channels. For sufficiently high SNR, on the other hand, there are many different locally optimal solutions. For instance, all of the isolated interference-free beamformer/receiver sets correspond to local optima, but other local optima may also exist, corresponding to solutions where the interference is zero only for some users. (This is supported by the observation that gradient algorithms that follow the steepest ascent of the sum rate occasionally fail to end up at an interference aligned solution at high SNR.) The locations of the local optima are determined by the cross channels.

[1]The covariance-based algorithm in [6] can result in more than one beam per user being active and is therefore not included in the following comparison.

While the (almost) aligned local optima all have close to zero interference at high SNR, they can differ significantly in the resulting sum rate. In terms of the approximation (10), they all have the same slope s, but differ in the offset r. The offset r is determined only by the direct channels, cf. (12), and is highest when the beamformers and receivers are matched to the direct channels, i. e., in the vicinity of the low-SNR optimizer.

Based on these considerations, we propose the following procedure for determining a good local optimum at any SNR: we begin with very low SNR, i. e., a high value of σ^2, and initialize the beamformers with the beams matched to the direct channels. Then we successively increase the SNR, i. e., decrease σ^2, and after each incremental change of the SNR iteratively update the beamformers until they have settled on a nearby local optimum. We formulate a fixed point updating procedure based on the necessary conditions for local optimality (5). Specifically, we keep $\boldsymbol{A}_1, \dots, \boldsymbol{A}_K$ fixed using the beamformers resulting from the previous iteration, and then update the beamformers $\boldsymbol{v}_1, \dots, \boldsymbol{v}_K$ to be the principal eigenvectors of $\boldsymbol{A}_1, \dots, \boldsymbol{A}_K$.[2]

A distributed implementation is possible provided that each transmitter k knows $\boldsymbol{H}_{jk}$ for all $j \in \{1, \dots, K\}$. After each update, the receivers measure the new interference-plus-noise covariance matrix $\boldsymbol{X}_k$ and compute the vector

$$\boldsymbol{t}_k = \frac{1}{\sqrt{1+\gamma_k}} \boldsymbol{X}_k^{-1} \boldsymbol{H}_{kk} \boldsymbol{v}_k. \tag{14}$$

This vector is then communicated to all transmitters. Transmitter k can now compute the new matrix $\boldsymbol{A}_k$ as

$$\boldsymbol{A}_k = \frac{1}{1+\gamma_k} \boldsymbol{H}_{kk}^{\mathrm{H}} \boldsymbol{X}_k^{-1} \boldsymbol{H}_{kk} - \sum_{j \neq k} \boldsymbol{H}_{jk}^{\mathrm{H}} \boldsymbol{t}_j \boldsymbol{t}_j^{\mathrm{H}} \boldsymbol{H}_{jk} \tag{15}$$

and perform the update of $\boldsymbol{v}_k$ on its own. We observe that this update is analogous to the best-response update performed by transmitter k in the *distributed interference pricing* algorithm [9], [10], [11], [12]. The complete incremental-SNR algorithm is summarized in Table I.

For satisfactory results it is critical to fine-tune the convergence threshold and the size of the SNR increments used in the algorithm. In our implementation, we assumed convergence of the beamformers to be complete when the sum of the Euclidean norms of the changes over all beamformers $\boldsymbol{v}_k$ was below 10^{-4}. The SNR in dB $10\log_{10}\sigma^{-2}$ was incremented by 4 until the target SNR was reached.

The concept of gradually transforming a problem from a more managable form into its original, difficult form, is also known as a "homotopy method". It is used in the numerics literature, e. g., to improve the convergence properties of the Newton method for finding zeros of systems of nonlinear equations [13].

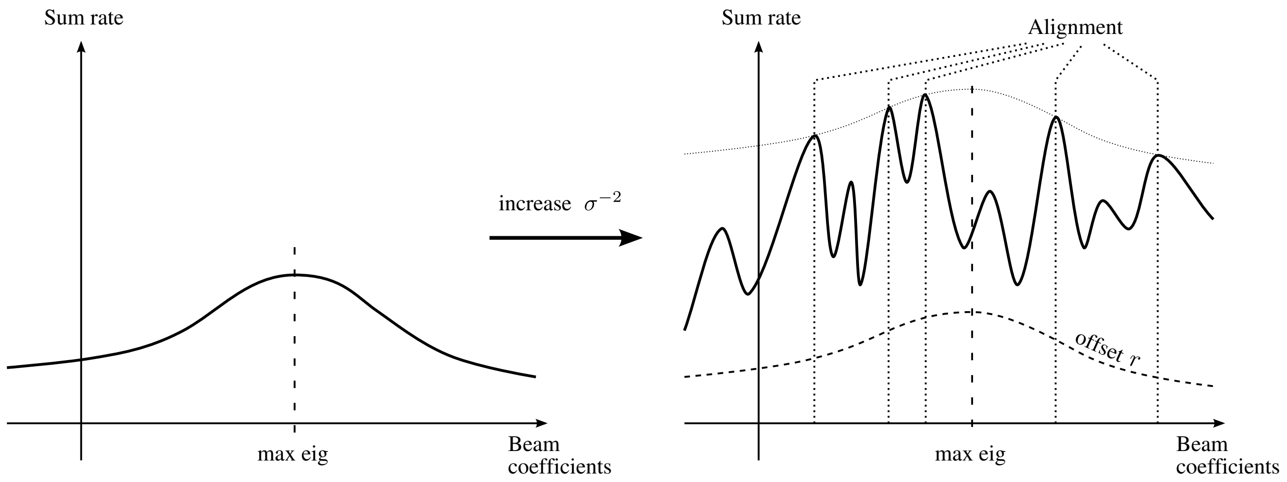

Fig. 1. For low SNR (left), the optimal beamforming strategy is easily found and consists of using the principal eigenvectors of the Gramians of the direct channels. At high SNR (right), there are many local optimizers which depend only on the cross channels. The best aligned solution is the one with the highest high-SNR offset r, which depends only on the direct channels. The expression for the offset r (12) reaches its maximum for the same principal eigenvectors that maximize the sum rate at low SNR. We are therefore interested in finding aligned solutions that are close to the low-SNR optimal strategy.

TABLE I
INCREMENTAL-SNR ALGORITHM

```
for k in {1, ..., K} do
    initialize v_k as principal eigenvector of H_kk^H H_kk
end for
initialize σ^{-2} close to zero
while σ^{-2} < target-SNR do
    repeat
        for all k do
            compute t_k from (14)
        end for
        for all k do
            compute A_k from (15)
            update v_k as principal eigenvector of A_k
        end for
    until convergence of all v_k
    increase σ^{-2}
end while
```

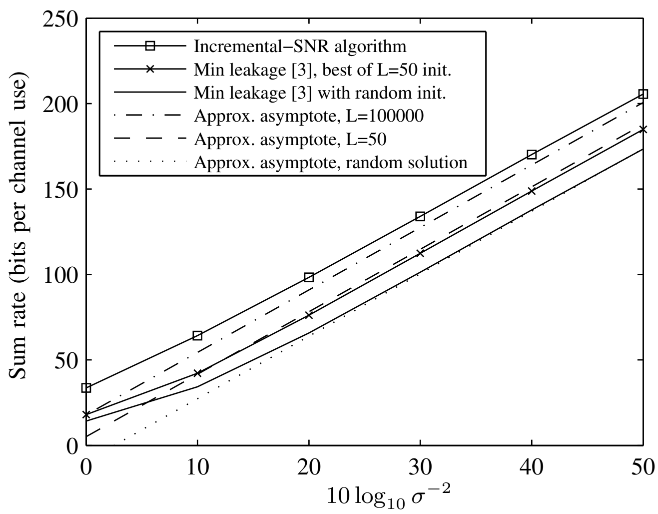

Fig. 2. Comparison of the average sum rate over the SNR of the incremental-SNR algorithm with the MLA strategy for a system with $K = 11$ users and $N = 6$ antennas at each terminal

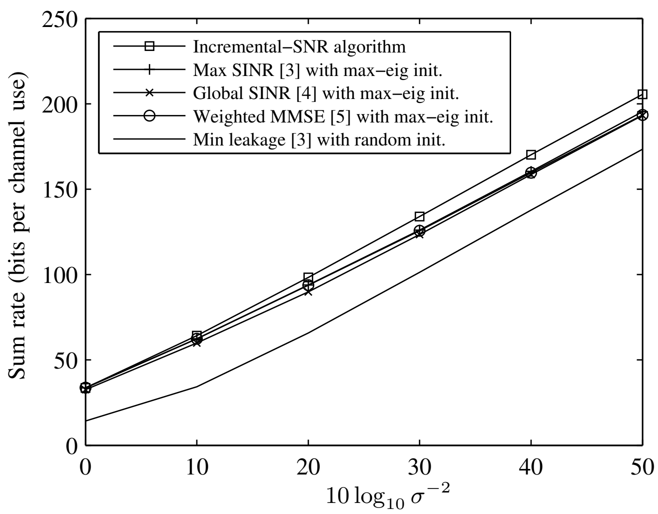

Fig. 3. Comparison of the average sum rate over the SNR of the incremental-SNR algorithm with previously proposed algorithms for a system with $K = 11$ users and $N = 6$ antennas at each terminal

VI. NUMERICAL RESULTS

For the results in Fig. 2 we numerically simulated the incremental-SNR algorithm along with the MLA strategy for a scenario with $N = 6$ antennas at each terminal and $K = 2N-1 = 11$ users. The sum rate is averaged over 50 channel realizations. Also included are the approximated asymptotes for the MLA method with $L = 1$, $L = 50$, and $L = 100\,000$ using the results from Section IV. In Fig. 3 we compare the performance of the incremental-SNR algorithm with that of the algorithms from [3], [4], [5] for the same scenario.

[2]Note that by taking the principal eigenvector regardless of the sign of the highest eigenvalue, we are effectively replacing the inequality power constraints by equality power constraints. This has little effect if the number of users selected initially is feasible, i. e., if $K \leq 2N - 1$. Alternately, the update could be done such that the new beamformer is set to zero if the highest eigenvalue is negative.

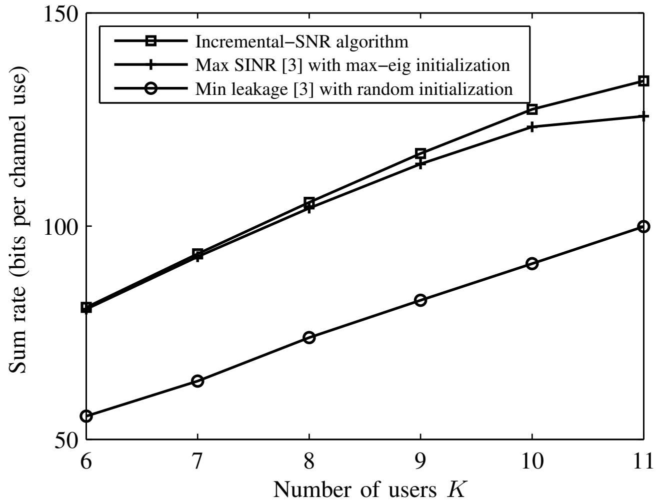

Fig. 4. Average sum rate over number of users K for a system with $N = 6$ antennas at each terminal and an SNR of 30 dB

The proposed incremental-SNR algorithm clearly finds the best aligned solution on average (although this is not true for all channel realizations). Also, the analytical results for the MLA method are reasonably accurate, even for this moderate system size. Assuming that the approximation for $L = 100\,000$ predicts the performance equally well, we can conclude from Fig. 2 that a huge number of aligned solutions must be sampled in order to find one that is as good as the one obtained by the incremental-SNR method for this particular scenario.

In Fig. 4 we show the effect of decreasing the number of users, again for a system with $N = 6$ antennas. It can be seen that the advantage of the incremental-SNR method is largest when the system is fully loaded with $K = 2N - 1 = 11$ users. Also, we note that for the "minimum leakage" algorithm, which converges to a random aligned solution, the average sum rate is proportional to the number of users, since adding a user does not harm the other users as long as zero interference is still feasible. For the incremental-SNR and maximum SINR methods this is not the case.

VII. CONCLUSION

In this work we examined and compared two approaches to obtaining a set of beamformers that maximizes the sum rate in a single-beam MIMO interference network at high SNRs. The first approach relies on comparing a large number of randomly obtained interference aligned solutions and can be analyzed by applying results from the theory of large random matrices and extreme statistics. The second approach is to design a numerical search algorithm that finds a good aligned solution. We proposed a procedure that tracks a locally optimal solution while incrementing the SNR. The performance of this algorithm is significantly better than that of previously proposed algorithms. Furthermore, our simulations suggest that the second approach clearly outperforms the MLA method unless the number of sampled solutions is extremely large.

Finding the average sum rate offset for the best aligned solution, however, remains an open problem. It is not known whether the numerical methods discussed in this work yield significantly suboptimal solutions or whether they come close to the performance of the global optimum at high SNRs.

REFERENCES

[1] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[2] C. M. Yetis, T. Gou, S. A. Jafar, and A. H. Kayran, "On feasibility of interference alignment in MIMO interference networks," *IEEE Trans. Signal Processing*, vol. 58, no. 9, pp. 4771–4782, Sept. 2010.

[3] K. Gomadam, V. R. Cadambe, and S. A. Jafar, "Approaching the capacity of wireless networks through distributed interference alignment," in *Proc. IEEE Global Commun. Conf. (GLOBECOM) 2008*, New Orleans, LA, Nov. 2008.

[4] S. W. Peters and R. W. Heath, Jr., "Cooperative algorithms for MIMO interference channels," arXiv.org, Feb. 2010. [Online]. Available: http://arxiv.org/abs/1002.0424v1

[5] D. A. Schmidt, C. Shi, R. A. Berry, M. L. Honig, and W. Utschick, "Minimum mean squared error interference alignment," in *Proc. 43rd Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, Nov. 2009.

[6] M. Razaviyayn, M. S. Boroujeni, and Z.-Q. Luo, "Linear transceiver design for interference alignment: Complexity and computation," in *Proc. IEEE 11th Int. Workshop Signal Process. Adv. Wireless Commun. (SPAWC)*, Marrakech, Morocco, June 2010.

[7] D. A. Schmidt, W. Utschick, and M. L. Honig, "Large system performance of interference alignment in single-beam MIMO networks," accepted for presentation at *IEEE Global Commun. Conf. (GLOBECOM) 2010*, Dec. 2010.

[8] H. A. David, *Order Statistics*, 2nd ed. New York, NY: John Wiley & Sons, Inc., 1981.

[9] J. Huang, R. A. Berry, and M. L. Honig, "Distributed interference compensation for wireless networks," *IEEE J. Select. Areas Commun.*, vol. 24, no. 5, pp. 1074–1084, May 2006.

[10] D. A. Schmidt, A. Gründinger, W. Utschick, and M. L. Honig, "Distributed precoder optimization for interfering MISO channels," in *Proc. Int. ITG Workshop Smart Antennas 2008*, Darmstadt, Germany, Feb. 2008.

[11] C. Shi, D. A. Schmidt, R. A. Berry, M. L. Honig, and W. Utschick, "Distributed interference pricing for the MIMO interference channel," in *Proc. IEEE Int. Conf. Commun. 2009*, Dresden, Germany, June 2009.

[12] D. A. Schmidt, C. Shi, R. A. Berry, M. L. Honig, and W. Utschick, "Distributed resource allocation schemes," *IEEE Signal Processing Mag.*, vol. 26, no. 5, pp. 53–63, Sept. 2009.

[13] E. L. Allgower and K. Georg, *Introduction to Numerical Continuation Methods*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2003.