The Price of Free Spectrum to Heterogeneous Users

Thành Nguyen*, Hang Zhou*, Randall A. Berry*, Michael L. Honig * and Rakesh Vohra†
*EECS Department
Northwestern University, Evanston, IL 60208
{hang.zhou, thanh, rberry, mh}@eecs.northwestern.edu
†CMS-EMS, Kellogg School of Management
Northwestern University, Evanston, IL 60208
r-vohra@kellogg.northwestern.edu

***Abstract*—Adding unlicensed spectrum, such as the recent opening of the television white spaces in the US, has the potential to benefit customers by increasing competition, but may also increase congestion. In an earlier paper, we presented a model for studying the effects of adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent service providers. Assuming that customers choose providers based the sum of announced price and a congestion cost, it was shown that the social welfare can decrease with additional spectrum. Here we extend this work in two key ways. First, in the earlier work all customers traded-off congestion costs with announced prices in the same way. Here, we consider a heterogeneous pool of customers who may have different trade-offs. Second, the earlier work focused on the overall welfare including both that of service providers and customers. Here we characterize customer surplus as well as total welfare. In particular we show that with homogeneous customers, customer welfare is non-decreasing, while with heterogeneous ones it can decrease.**

I. Introduction

The Federal Communications Commission has recently announced a new policy for the use of spectrum assigned to vacant broadcast television channels, or *white space* [1]. The ruling states that this white space can be used as an unlicensed spectrum *commons*, analogous to the model for spectrum usage associated with WiFi devices.[1] While this change in policy potentially allows valuable unoccupied spectrum to be put to productive use, it can also influence the market for services in existing licensed bands.

More specifically, introducing unlicensed bandwidth alongside service providers (SP) with licensed bands has several consequences. First, it increases the supply of spectrum and lowers the costs to entrants seeking to offer wireless services. The increased competition among SPs should lower prices to customers of wireless services. However, this benefit is offset by increased traffic and associated interference, which produces lower Quality of Service (QoS) (e.g., smaller throughputs and/or larger delays).

In previous work we have studied the effects of adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent wireless Service Providers (SPs) [2]. A key assumption is that each SP in the unlicensed band experiences a congestion cost that depends on the *total* traffic in that band (due to both incumbents and new entrants). This is motivated by the likelihood that secondary users sharing white space within a given region will interfere with each other even if they are associated with different SPs. In contrast, the congestion cost in a licensed band is due only to the traffic of the SP holding the license. A model was presented based on the framework for price competition in markets for congestible resources developed in the operations and economics literature [3]–[7]. It was shown in [2] that for the model considered, adding white space to an existing allocation of licensed spectrum can actually *decrease* total (customer plus SP) welfare.

An assumption in [2] is that the customers are *homogeneous*, meaning that they have the same value trade-off between price paid for the service and congestion cost. In practice, this is unlikely to be true, and it might be expected that adding white space may help to differentiate the market and allocate white space to those customers that prefer to trade-off more congestion for a lower price.[2]

Here we consider an analogous model as in [2], but with *heterogeneous* customers. Specifically, we assume two user groups (”high-QoS” and ”low-QoS”) with different price-congestion trade-offs. Our model shows that with additional unlicensed spectrum, SP profits always decrease (for both homogeneous and heterogeneous customers). More interesting is that with heterogeneous customers adding unlicensed spectrum can also decrease *customer* surplus. Thus, additional unlicensed spectrum can reduce aggregate welfare for producers and customers in the market. In contrast, with homogeneous customers customer surplus always increases.

The intuition is that to maximize revenue the SPs may increase the price to shift customers to the unlicensed band. This applies to both homogeneous and heterogeneous customers; however, with heterogeneous customers this can produce a discontinuous change in demand. Specifically, we show that as the amount of white space increases, an SP has an incentive to switch from one equilibrium in which both high- and low-

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[1] An additional constraint for TV white space is that unlicensed transmitters must not interfere with reception of active broadcast television channels.

[2] Unlicensed spectrum may also encourage new types of wireless services that allow for offerings that better match customer needs [8].

QoS customers are served, to another equilibrium in which a smaller number of high-QoS customers are served. This shifts more low-QoS customers to the white space increasing congestion there. Hence when this switch happens, the customer surplus decreases along with SP surplus. Furthermore, the surplus is strictly smaller than without the additional white space.

We also show that the customer surplus can be a complicated function of the amount of white space added. There can be many break points between which the customer surplus increases, decreases, or stays the same. In contrast, with homogeneous customers the customer surplus is nondecreasing with the amount of white space, and the total surplus has one local (global) minimum. Our results suggest that adding new spectrum as a commons to existing allocations for exclusive use may not be optimal in terms of social welfare.

Structure of the Paper: The remainder of the paper is organized as follows. We describe the model formally in the next section. In Section III we discuss the impact of unlicensed spectrum on customer surplus in both heterogeneous and homogeneous user models. Section IV characterizes social welfare as a function of the amount of additional unlicensed spectrum. Technical proofs are given in the appendix.

II. The Model

Our model is an extension of the standard model of network pricing games with congestion effects [9]. The economy consists of *service providers* (SPs) and two classes of *customers* with different sensitivity to congestion costs. In particular, we assume there are two classes of customers with high and low sensitivity to congestion cost who are referred as *high customers* (h) and *low customers* (l), respectively hereafter. SPs have their own licensed bands. The investment costs for acquiring these (e.g., via government auctions) as well as the infrastructure for exploiting them are considered sunk at this stage. Besides the licensed bands, there is also an unlicensed band available for use by all SPs. There is no cost for acquiring such a band as by design it is open for any provider to freely use. Exploiting such spectrum does require investment in infrastructure, which we again consider sunk at this stage.[3] Examples of such unlicensed bands are the WiFi band and the TV white spaces.

Service Providers

Let $\mathcal{N}$ be a set of N SPs. Each SP has her own licensed band and may also use the unlicensed band. An entrant in the unlicensed band can be modeled as an SP with a licensed band of capacity zero.

The SPs compete for all classes of customers by simultaneously choosing prices in the unlicensed band and/or their own licensed bands. SPs are assumed to set a flat price for all classes of customers and serve all customers who accept their posted price. Suppose a SP i sets prices p_i for service in its licensed band and p_i^w in the unlicensed band and serves $x_i = x_i^h + x_i^l$ and $x_i^w = x_i^{wh} + x_i^{wl}$ customers, respectively. The superscripts h and l denote high customers and low customers respectively. Then, i's profit π_i is given by $\pi_i = p_i x_i + p_i^w x_i^w$.

There is a congestion externality suffered by customers served by SPs. This externality is perceived differently by the two different classes of customers. Details on how the customers perceive congestion externalities are given in the next section.

If SP i's licensed band serves a mass $x_i = x_i^h + x_i^l$ of customers, then each customer served in this band experiences a *congestion cost* $l_i(x_i)$, where the nature of this loss will depend on the bandwidth of the licensed band and the technology deployed by SP i. Congestion suffered in the unlicensed band, however, is a function of the *total* mass of customers served in the unlicensed band. Specifically, if $x_i^w = x_i^{wh} + x_i^{wl}$ is the mass of customers served by SP i in the unlicensed band, the congestion suffered by each customer served in the unlicensed band is $g(X^w)$ where $X^w = \sum_{i\in\mathcal{N}} x_i^w$ is the total number of all classes of customers served in the unlicensed spectrum.[4] The congestion cost $g(X^w)$ also depends on the bandwidth of the unlicensed band. In Section IV we consider the case where l_i is fixed and g is varied according to the available bandwidth of the unlicensed spectrum.

Customers

Customers choose an SP based on the *delivered price*, which is the weighted sum of the price announced by an SP and the congestion cost she experiences when served by that SP. For SP i's high customers, the delivered price in her licensed band is $p_i + \lambda_h l_i(x_i)$ and the delivered price in the unlicensed band is $p_i^w + \lambda_h g(X^w)$ where $\lambda_h > 0$ represents the high sensitivity to congestion cost of this customer. The delivered price for a low customer in SP i's licensed and unlicensed band can be defined correspondingly as $p_i + \lambda_l l_i(x_i)$ and $p_i^w + \lambda_l g(X^w)$ where $\lambda_l > 0$ represents the low sensitivity to congestion cost of this customer. Naturally, we assume $\lambda_h > \lambda_l$.

The demand for services from the two classes is governed by two downward sloping demand functions $D_h(p)$ and $D_l(p)$ with the inverse function $P_h(q)$ and $P_l(q)$, respectively. Both classes of customers always choose services from the SPs with the lowest delivered prices defined above. When facing the same delivered price from multiple SPs, customers are assumed to randomly choose one of the SPs. Thus SPs with the same delivered price will draw the same customer mass in equilibrium.

Pricing Game and Nash Equilibrium

We consider a game in which SPs move first and simultaneously announce prices. Then, customers choose SPs based

[3] Moreover, these costs for a provider who also operates in licensed spectrum may be reduced as the provider could reuse parts of the licensed infrastructure.

[4] w stands for "white space". For the white space case, the FCC allows any type of user to transmit as long as no substantial interference is caused to primary spectrum users (TV broadcasters and licensed wireless microphone users) [1]. There are constraints on transmit power, antenna height and power spectral density. However, there are no additional constraints to reduce the interference among different white space users. Moreover, due to the propagation characteristics of the corresponding radio frequencies, such interference effects will be greater than in the WiFi band.

on the delivered price.

Given a price vector $(\mathbf{p}, \mathbf{p}^w)$ the non-negative demand vector $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ induced by $(\mathbf{p}, \mathbf{p}^w)$ must satisfy in the licensed bands:

$$\begin{aligned} p_i + \lambda_h l_i(x_i) &= P_h(Q_h) \quad &\text{if } x_i^h > 0 \\ p_i + \lambda_h l_i(x_i) &> P_h(Q_h) \quad &\text{if } x_i^h = 0 \\ p_i + \lambda_l l_i(x_i) &= P_l(Q_l) \quad &\text{if } x_i^l > 0 \\ p_i + \lambda_l l_i(x_i) &> P_l(Q_l) \quad &\text{if } x_i^l = 0 \end{aligned} \tag{1}$$

and in the unlicensed bands:

$$\begin{aligned} p_i^w + \lambda_h g(X^w) &= P_h(Q_h) \quad &\text{if } x_i^h > 0 \\ p_i^w + \lambda_h g(X^w) &> P_h(Q_h) \quad &\text{if } x_i^h = 0 \\ p_i^w + \lambda_l g(X^w) &= P_l(Q_l) \quad &\text{if } x_i^l > 0 \\ p_i^w + \lambda_l g(X^w) &> P_l(Q_l) \quad &\text{if } x_i^l = 0 \end{aligned} \tag{2}$$

where $Q_h = \sum_i (x_i^h + x_i^{wh})$ is the total number of high customers served in the market, and $Q_l = \sum_i (x_i^l + x_i^{wl})$. In other words, the demand for each SP is such that no customer can lower the delivered price she pays by switching SPs.

Remark: It can be shown given a price vector, the corresponding demand vector satisfying the above conditions always exists and is the solution to some convex program. This result is summarized in Appendix F.

DEFINITION 1: *A pair* $(\mathbf{p}, \mathbf{p^w})$ *and* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *is a pure strategy Nash equilibrium if* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *satisfies equation (1) and (2) given* $(\mathbf{p}, \mathbf{p^w})$*, and no SP can improve her profit by changing prices.*

In the most general case of demand function and latency, Nash equilibrium might not exist. The main goal of this paper is to study the impact of unlicensed spectrum on existing market. We will consider special cases of demand curve such that the game has a unique equilibrium.

Social Welfare and Customer Surplus

Next, we define the *social welfare* in this market, which represents the total surplus of both producers (SPs) and customers and *customer surplus*, which is the welfare gained from the consumption of the service by the consumers alone.

DEFINITION 2: *Suppose* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *is the demand vector induced by some price vector* $(\mathbf{p}, \mathbf{p^w})$ *according to (1) and (2). Then* social welfare *is given by*

$$\begin{aligned} SW = &\int_0^{Q_h} P_h(q)dq + \int_0^{Q_l} P_l(q)dq - \sum_{i\in\mathcal{N}} \lambda_h x_i^h l_i(x_i) \\ &- \sum_{i\in\mathcal{N}} \lambda_l x_i^l l_i(x_i) - \lambda_h g(X^w)X^{wh} - \lambda_l g(X^w)X^{wl} \end{aligned} \tag{3}$$

where $X^w = X^{wh} + X^{wl}$*,* X^{wh} *and* X^{wl} *are the number of high and low customers in the unlicensed band respectively.* $Q_h = \sum_{i\in\mathcal{N}} x_i^h + X^{wh}$ *and* $Q_l = \sum_{i\in\mathcal{N}} x_i^l + X^{wl}$ *are the total number of high and low customers served in the market.*

DEFINITION 3: *Suppose for some price and demand vectors,* W_h^d *and* W_l^d *are the resulting delivered price for high class and low class customers, respectively. Then the customer surplus is given by*

$$CS = \int_0^{Q_h} (P_h(q) - W_h^d)dq + \int_0^{Q_l} (P_l(q) - W_l^d)dq \tag{4}$$

where Q_h *and* Q_l *are defined the same as in Definition 2.*

Fig. 1 provides an example of an equilibrium for *homogeneous* customers. There are two SPs and unlicensed spectrum in the pricing game. The mesh area on the top represents the customer surplus, while the shared area within the red contour represents the social welfare. Note that social welfare is the sum of customer surplus and revenue of all SPs.

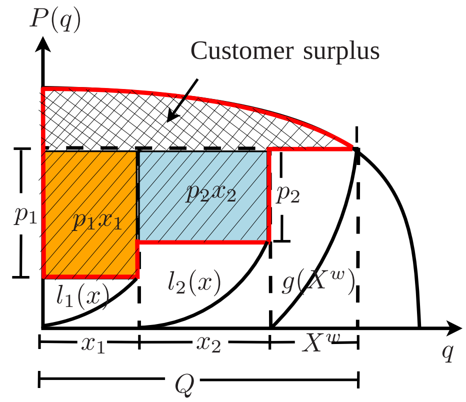

Fig. 1. Illustration of pricing game with two SPs and unlicensed spectrum.

Equilibrium Price in Unlicensed Spectrum

We first characterize equilibrium prices in the unlicensed band. The result serves as a building block for the analysis in the succeeding sections. Let $\mathbf{p}^*$ and $(\mathbf{x^h}^*, \mathbf{x^l}^*)$ denote the Nash equilibrium price vector and demand vector in the licensed bands, while $\mathbf{p}^{w*}$ and $(\mathbf{x^{wh}}^*, \mathbf{x^{wl}}^*)$ denote the corresponding equilibrium prices and demands in the unlicensed band.

Lemma 1: *If* $(\mathbf{p}^*, \mathbf{p}^{w*})$ *and* $(\mathbf{x^h}^*, \mathbf{x^l}^*, \mathbf{x^{wh}}^*, \mathbf{x^{wl}}^*)$ *form an NE, then* $\mathbf{p}^{w*} = \mathbf{0}$*.*

Intuitively, this is because all of the customers in the unlicensed spectrum experience the same (weighted) congestion costs. Thus a SP with positive price in the unlicensed band always has a strictly profitable deviation of decreasing her price. Therefore, the equilibrium price in the unlicensed band must be zero for both classes of customers. The proof of Lemma 1 is a simple extension of Theorem 1 in [2], and thus omitted.

Lemma 1 suggests that in the unlicensed band, competition will force the prices to be zero and SPs will earn zero profit in that band. Competition in the unlicensed band can also reduce prices in the licensed band and decrease SPs' profits there. We shall see examples of this in later sections. Moreover, since all SPs have zero profit in the unlicensed band, the volume of customers each SP serves there does not affect her profit or the equilibrium. Thus, we only focus on the total customer mass of each class X^{wh} and X^{wh} in the unlicensed spectrum hereafter.

III. Customer Surplus with Additional Unlicensed Spectrum

In this section we analyze the change in customer surplus when additional unlicensed spectrum is added to an existing market of wireless services offered in licensed spectrum. We show that when customers are homogeneous, that is, $\lambda_h = \lambda_l$ then adding additional unlicensed spectrum can only make the delivered price decrease, which indicates an increase in customer surplus. However, important difference emerges with a heterogeneous model: as the capacity of unlicensed spectrum changes, the incumbent SP may switch the class(es) of customers she is serving to maximize her revenue. In particular, suppose that the incumbent SP serves both high and low class customers with no unlicensed band (to maximize her revenue), then intuitively, for an unlicensed band with small enough capacity, the incumbent will keep serving both classes of customers. While for an unlicensed band with large enough capacity, the incumbent will choose to raise the price and serve only high customers, causing a drop in customer surplus. This effect is clearly absent in the homogeneous model and is a very robust prediction. In particular, we show that the effect described above is *always* present in heterogeneous models for general demand curves and latency functions.

More formally the results in this section are stated in Theorem 1 and Theorem 2 below.

THEOREM 1: *Consider an incumbent SP with licensed spectrum in a homogeneous model, i.e. $\lambda_l = \lambda_h$. Let D_0, D_1 be the delivered prices at equilibrium before and after unlicensed spectrum is introduced, respectively, then $D_0 > D_1$.*

We note that Theorem 1 is not trivial and proved by contradiction. If $D_0 < D_1$ is assumed, then by using the convexity of latency functions we can contradict the fact that D_0 is the delivered price at which the incumbent maximizes her revenue. The formal proof of Theorem 1 is given in Appendix A.

The following theorem describes a drop in customer surplus in the case of heterogeneous users.

THEOREM 2: *Consider a heterogeneous model with a single incumbent that serves a mixture of different customer types before unlicensed spectrum is introduced. Then there exists a $C_0 > 0$ such that when the bandwidth of unlicensed is increased from C_0^- to C_0^+, the incumbent will increase her price discontinuously to serve high customers exclusively which causes a drop in the customer surplus.*

Moreover, there exists a range of parameters such that when C_0^+ bandwidth of unlicensed spectrum is introduced the customer surplus is strictly smaller than when there was only licensed spectrum.

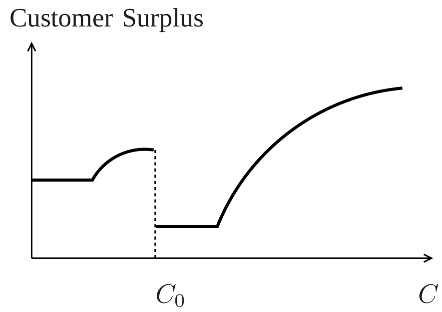

Fig. 2. Customer surplus as a function of unlicensed band's capacity

Figure 2 illustrates the customer surplus as a function of unlicensed spectrum's bandwidth. Numerical examples are given in Appendix E. These examples show that in fact at C_0^+ social surplus can be as low as 0, while before licensed spectrum is introduced customer surplus is positive. Intuitively this phenomenon can be described as follow.

Consider the case where before unlicensed spectrum is introduced, the provider charged a price p that serves both types of customers. In particular all high customers and a fraction of the low customers are served at this price. Let R be the total revenue of this scenario. Note that the provider can also charge a high enough price p_H such that at this price only high customers use the service. We assume the resulting revenue under p_H is $R_H < R$.

Now, some amount C of unlicensed spectrum is introduced to the economy, which creates competition with the provider. As a result the optimal revenue R_C will decrease in C. Observe that when C is small the provider will also change the price p by a small amount such that at this price both type of users are still served, but when C is large enough the provider will have an incentive to suddenly increase the price and in many cases she could raise p up to p_H to eliminate low type customers and still obtain a revenue of R_H from the high types. When such price raise occurs, high customers need to pay a higher price thus their surplus is dropped. On the other hand, the lower customers now need to use the unlicensed band, which will be highly congested. As a result, the customer surplus decreases.

However, to show the general result in Theorem 2 we need to prove that there is always a C_0 where the SP switches the class of targeting customers and when she does so customer surplus drops. The formal proof is rather interesting and is given in Appendix B.

Our results in this section suggest that congestion is a very important and distinctive element of spectrum market. Our results can be seen as an contrast with markets without congestion, where a market segmentation that emerges as a result of competition usually indicates improvement in efficiency or customer surplus.

IV. Social Welfare with Additional Unlicensed Spectrum

In this section we analyze the social welfare as a function of unlicensed spectrum bandwidth. It has been shown in our previous work [2], that even in a homogeneous model social welfare may worsen as additional unlicensed spectrum is introduced. This phenomenon is reminiscent to the famous Braess's paradox in congestion games [10]. However, the difference here is that the paradox is caused by the service providers rather than by the users.

As shown in Section III, in the heterogeneous model, an increase in unlicensed spectrum bandwidth not only causes a decrease in the incumbent's revenue, but can also reduce the customer surplus. Thus, clearly additional unlicensed spectrum can also decrease the social welfare in the heterogeneous model.

In the homogeneous model, although social welfare is not a monotonic function of unlicensed spectrum bandwidth, it changes in a rather simple manner. It has only a single local (global) minimum. We will show that with heterogeneous customers, the social welfare as a function of unlicensed band capacity can be much more complex.

We again consider a simple scenario, where a single incumbent (monopoly) operates on a licensed band before the unlicensed band is open. One or more entrants can then enter the market using the unlicensed band only. The incumbent can also offer services on the unlicensed band. A particular "box" demand function is assumed for simplicity in this section. A "box" demand function corresponds to a Q mass of customers with a common valuation of W for receiving services. This corresponds to $P(q)$ being a constant W for $0 \le q \le Q$ and then dropping to zero for $q \ge Q$. Customers with such a demand function choose an SP as long as its delivered price is at most W. Note that a box demand function is uniquely determined by the tuple (W, Q). We will use this type of demand for two classes of customers with demand (W_h, Q_h) and (W_l, Q_l). Moreover, it is also reasonable to assume $W_h > W_l$ and $Q_h < Q_l$, i.e., the high class customers have high valuations for the service and there are more low class customers than high class customers.

We first observe that if the incumbent is serving both classes of customers in her licensed band, then the there will only be low class customers in the unlicensed band. On the contrary, if there are both classes of customers in the unlicensed band at equilibrium, then there can only be high-class customers in the licensed band. This result is summarized in Lemma 2.

Let $x^h \ge 0$ and $x^l \ge 0$ be the number of high and low class of customers in the incumbent's licensed spectrum. Similarly, let $X^{wh} \ge 0$ and $X^{wl} \ge 0$ be the corresponding numbers of customers in the unlicensed band.

Lemma 2: (i) *If $x^h > 0$ and $x^l > 0$, then $X^{wh} = 0$ and $X^{wl} \ge 0$.*
(ii) *If $X^{wh} > 0$ and $X^{wl} > 0$, then $x^l = 0$ and $x^h \ge 0$.*

This lemma also suggests that incumbent's licensed spectrum and the unlicensed spectrum cannot serve both type of customers at the same time. The proof of this lemma is given in Appendix C. Note that Lemma 2 holds for markets with arbitrary number of incumbent SPs and general forms of congestion cost functions.

Further, we assume congestion costs are linear for both incumbent's licensed band and the unlicensed band. The incumbent operates on the licensed band with the congestion cost

$$l(x) = bx, \quad \text{where } b > 0.$$

The bandwidth of the unlicensed band is $C \ge 0$. The congestion cost in the unlicensed band is

$$g(x) = \alpha_C x.$$

Here we assume that α_C is some parameter decreasing in C; and when no unlicensed spectrum is open then $\alpha_0 = \infty$.

As mentioned in preceding section, how social welfare changes with the capacity of unlicensed band depends on the initial state of the market, i.e., which class(es) of customers are served by the incumbent without unlicensed band. To be specific, there are three possible scenarios with two classes of customers :
i) the incumbent initially serves high class only,
ii) the incumbent initially serves low class only and
iii) the incumbent initially serves mixture of high and low class.

In this section, we focus on Scenario **iii)**. It can be shown that similar results and insights can also be obtained for Scenario **i)** and Scenario **ii)**. Thus, we assume that W_h, W_l, Q_h, Q_l λ_h and λ_l are chosen such that Scenario **iii)** is true.[5]

THEOREM 3: *Assume*

$$\frac{W_h}{\lambda_h} < \frac{W_l}{\lambda_l} \text{ and } Q_h < \min\left\{\frac{W_h}{2b\lambda_h}, \frac{W_h - W_l}{b(\lambda_h - \lambda_l)}\right\}. \tag{5}$$

Consider an incumbent SP with licensed spectrum that serves all of the Q_h high customers and $x_{l0} < Q_l$ low class customers in the absence of unlicensed spectrum. If unlicensed spectrum with capacity C is added, then the social welfare at a NE, which is denoted by $SW(C)$, can be described as follows. There exist $0 \le C_1 \le C_2' \le C_2 \le C_3 \le C_4 \le \infty$ such that

$$SW(C) \text{ is } \begin{cases} \text{constant} & \text{for } 0 \le C \le C_1 \\ \text{monotone decreasing} & \text{for } C_1 \le C \le C_2' \\ \text{non-decreasing} & \text{for } C_2 \le C \le C_3 \\ \text{monotone decreasing} & \text{for } C_3 \le C \le C_4 \\ \text{monotone increasing} & \text{for } C \ge C_4 \end{cases}$$

An illustration of $SW(C)$ is shown in Fig. 3.

Remark: The assumptions in (5) are not essential for our results, but they focus our attention on one specific scenario and simplify the analysis.

[5] It can be shown that this is always feasible.

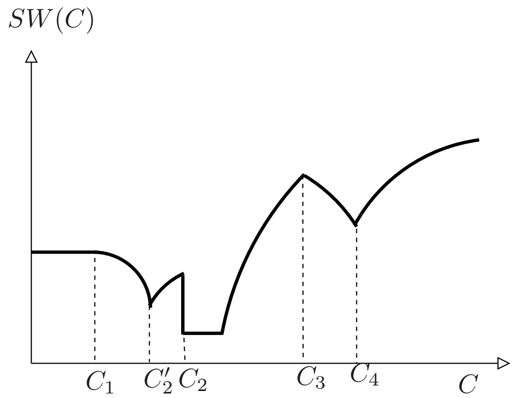

Fig. 3. Social welfare as a function of unlicensed band's capacity

Sketch of the Proof: The formal proof of this theorem is given in Appendix D. The main idea is sketched as follows. Before unlicensed spectrum is added, the monopoly set price p_0 and served Q_h high class customers and $x_{l0} < Q_l$ low class customers. We refer this as the *initial state* of the game.

First of all, it can be shown that there exists $C_1 \geq 0$ such that when $C \leq C_1$, the quality of service in the unlicensed band is not good enough to impact the monopoly or the initial state. Since the customer surplus is shown to be zero in this case (Section III), social welfare remains the same.

By Lemma 2, the monopoly and the unlicensed spectrum cannot serve both classes of customers simultaneously. Then there are three possible scenarios with additional unlicensed spectrum:
1) the monopoly serves a mixture of both classes and the unlicensed band serves low class,
2) the monopoly serves high class and the unlicensed band serves low class
3) the monopoly serves high class and the unlicensed band serves mixture of high and low classes.

Because of Theorem 1 and the fact that the monopoly is the only strategic agent in the game, the NEs of the game can be described as solutions to the monopoly's revenue maximization problems. By comparing the optimizations for each of the three scenarios listed above, it can be shown that there exist $C_2 > 0$ and $C_3 > 0$ such that for $C \leq C_2$, the monopoly maximizes her revenue in Scenario **1)** while for $C \geq C_3$ the monopoly maximizes her revenue in Scenario **3)** and finally Scenario **2)** is optimal for $C \in [C_2, C_3]$.

Now, the line of C has been divided into three interesting regions: $[C_1, C_2]$, $[C_2, C_3]$ and $[C_3, \infty]$.

For C in $[C_1, C_2]$ and $[C_3, \infty]$, it can be shown that the game corresponding to each scenario can be transformed to either equivalent or special cases of the game in the model with homogeneous customers. Thus the result for homogeneous customers (Theorem 3 in [2]) can be applied to characterize $SW(C)$ and show the existence of C_2' and C_4. Note that it is possible that $C_2' \geq C_2$ in which case we simply define $C_2' := C_2$.

Finally, for $C \in [C_2, C_3]$, it can be shown that the monopoly serves only high customers at a relatively high price and the unlicensed spectrum has no impact on the monopoly. Therefore, increasing capacity will improve the congestion in the unlicensed band, hence the social welfare is nondecreasing.

Remark: Theorem 3 shows that the social welfare, unlike that in the model with homogeneous customers, can be very complex. Moreover, it suggests that as long as there is competition between the services in the licensed and unlicensed bands on a certain class of customers, there can be a decrease of social welfare with increasing additional unlicensed spectrum.

A numerical example supporting Theorem 3 is given in Appendix E. Moreover, a description of the impact of different unlicensed band capacity, such as incumbent's price and revenue and social welfare is given along with the example.

V. Conclusion

Motivated by the recent FCC ruling on television white space, we have studied a model for adding unlicensed spectrum to a market for wireless services in which incumbents have licensed spectrum. We found that with heterogeneous customers if the amount of unlicensed spectrum is not sufficient, then adding unlicensed spectrum can decrease both the provider's revenue and customer surplus. This effect is due to the assumption that any SP can use the unlicensed spectrum for free to compete with other SP's having licensed bands. This suggests that in similar settings a better policy may be one that restricts the direct competition with existing SPs by allowing only specific services to use the white space or by allocating the additional spectrum as licensed spectrum.

Our work examines a particular scenario within the broader area of spectrum market design. There one of the main questions is how to assign, or design mechanisms for assigning spectrum according to a mix of licensed, unlicensed, and dynamic models (markets) in which SPs can trade spectrum assets over relatively short time periods. Approaches to solving this general problem will require richer models of interaction among SPs and customers as well as more refined models for the interaction among spectrum assets.

Appendix

A. Proof of Theorem 1

We will prove this theorem by contradiction. First, we introduce the following notations.

Consider a general situation, let D be the delivered price. Given such delivered price, let a be the number of customers in unlicensed band and x, p be the number of customer and the price of the licensed band such that the resulting delivered price is D. See Figure 4.

In particular, we will consider the following three pricing situations: 1. Before unlicensed spectrum is introduced and the monopoly maximizes the revenue. In this case let D_0 be the delivered price. Because unlicensed spectrum was not introduced $a_0 = 0$ and thus $D_0 = l(x_0) + p_0$. 2. After a fixed amount of unlicensed spectrum is introduced and the provider charge a price to maximize revenue with the competition with unlicensed spectrum. In this scenarios we denote the parameters by D_1, p_1, x_1, a_1. Lastly, We consider the situation where the provider charges a price p_3 so that under the competition with unlicensed band, the delivered price is $D_2 = D_0$. In such case, let x_2, a_2 be the number of customers in licensed and unlicensed band, respectively.

We will assume that $D_1 > D_0 = D_2$ and show that $p_2 x_2 > p_1 x_1$, which is a contradiction because the provider is assumed to maximize revenue at p_1.

To see this, we first observe that because $D_1 > D_2$ and $l_w(a_1) = D_1; l_w(a_2) = D_2$, therefore $a_2 < a_1$. Second, consider the function

$$R(x) = (D_2 - l(x))x = (D_0 - l(x))x.$$

Because $l(x)$ is a convex function, $R(x)$ is concave and it is assume that in scenario 1 (before unlicensed spectrum is introduced) the provider achieves maximum revenue of $R(x_0)$. Therefore, $R(x)$ is an increasing function from 0 to x_0. Thus, we have

$$R(x_2) > R(x_2 + a_2 - a_1) = R(x_3).$$

Here $x_3 = x_2 + a_2 - a_1 < x_2$. Thus $x_3 + a_1 = x_2 + a_2$. Now, because $D_2 = D_0 < D_1$ we have

$$x_2 + a_2 > x_1 + a_1.$$

Furthermore, because $x_3 + a_1 = x_2 + a_2 > x_1 + a_1$ we have $x_3 > x_1$.

Now consider the difference

$$R(x_3) - R(x_0)$$

As seen in Figure 4, this is the difference between area A and B, which is

$$R(x_3) - R(x_0) = x_3(l(x_3 + a_1) - l(x_3)) - a_1(D_0 - l(x_3 + a_1)).$$

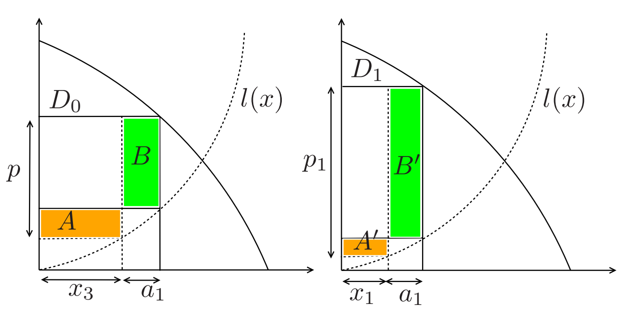

Fig. 4. Social welfare as a function of unlicensed band's capacity

Similarly, consider the difference between A' and B' in Figure 4 we have

$$p_1 x_1 - (D_1 - l(x_1 + a_1))(x_1 + a_1) =$$

$$x_1(l(x_1 + a_1) - l(x_1)) - a_1(D_1 - l(x_1 + a_1)).$$

Now because $x_3 > x_1$ and $l(x)$ is convex, we have

$$x_3(l(x_3 + a_1) - l(x_3)) > x_1(l(x_1 + a_1) - l(x_1))$$

Furthermore, we also have

$$a_1(D_0 - l(x_3 + a_1)) < a_1(D_1 - l(x_1 + a_1))$$

because $D_0 < D_1$ and $x_3 + a_1 > x_1 + a_1$. Thus, we have

$$R(x_3) - R(x_0) > p_1 x_1 - (D_1 - l(x_1 + a_1))(x_1 + a_1)$$

Moreover, $(D_1 - l(x_1 + a_1))(x_1 + a_1)$ corresponds to the revenue of the provider would achieve before unlicensed band is introduced if he charge the price $D_1 - l(x_1 + a_1)$ thus, $R(x_0) > (D_1 - l(x_1 + a_1))(x_1 + a_1)$. Therefore, we have

$$R(x_3) > p_1 x_1,$$

which is a contradiction to the fact that the provider optimizes his revenue.

B. Proof of Theorem 2

We first need to prove the existence of C_0 where SP wants to change the class of targeting customers. To see this, we will show that when the unlicensed spectrum bandwidth is large enough, the optimal revenue is obtained when the SP only serves high customers. Thus, because of the assumption that before unlicensed spectrum is introduced the SP serves a mixture of the two types, there must be a C_0 bandwidth of unlicensed spectrum at which the SP switches from serving both types to high customer.

Given $\delta > 0$ let x_δ be the optimal point of

$$R_\delta = \max_x \{x \cdot (\delta - l(x))\}. \tag{6}$$

R_δ is the shaded area in Figure 5.

Clearly, $l(x_\delta) < \delta$. We will choose δ small enough such that the inverse demand for the high customers at x_δ is at least $\lambda_h \delta$. That is

$$P_h(x_\delta) \geq \lambda_h \delta.$$

Note that because $P_h(x)$ is decreasing, $l(x)$ is increasing and $\lambda_h l(0) < P_h(0)$ thus, we can always choose such $\delta > 0$.

Now consider a situation where there is only unlicensed spectrum and there are only low customers. Let C_δ be a value such that if the bandwidth of unlicensed spectrum is C_δ, then the the latency of the unlicensed band is δ. We will show that in the setting with the incumbent SP when $C = C_\delta$ the optimal revenue of the incumbent is obtained by serving high customer only.

To see this, observe that when serving both types of customers the delivered price for lower customers cannot be higher than $\lambda_l \delta$. This is true because we know that when serving both types of customers, the users on unlicensed spectrum is the lower types and because we have C_δ bandwidth of unlicensed spectrum the congestion on unlicensed is at most δ. Therefore, the optimal revenue that the incumbent SP can obtain while serving both types can be at most

$$\max_x \{x \cdot \lambda_l(\delta - l(x))\} = \lambda_l R_\delta,$$

where R_δ is defined in (6).

However, if the SP charges the price $p = \lambda_h(\delta - l(x_\delta))$, then the equilibrium is that no low type customers uses the licensed spectrum and x_δ high customer uses the service in the licensed band. This is true because the congestion of unlicensed band is $g(x) = \delta$, the delivered price for low customers on licensed band is

$$\lambda_l l(x_\delta) + \lambda_h(\delta - l(x_\delta)) > \lambda_l l(x_\delta) + \lambda_l(\delta - l(x_\delta)) = \lambda_l \delta.$$

Thus, no lower customer would choose to use licensed band. On the other hand the delivered price for high customers is $\lambda_l \delta$ therefore no high customer would use unlicensed band either.

Now, in this case, the SP's revenue is

$$p \cdot x_\delta = \lambda_h R_\delta > \lambda_l R_\delta.$$

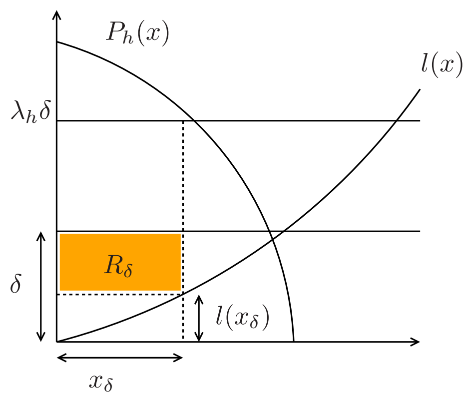

Fig. 5. Improving revenue by targeting only high typed customers

This shows that when $C = C_\delta$ the incumbent SP only serves high customers. Therefore, there exists a $0 < C_0 < C_\delta$ such that if the unlicensed bandwidth is increased from C_0^- to C_0^+, the incumbent has incentive to switch the class of customers and target only high types.

Consider this transition, when $C = C_0^+$ the unlicensed band is open to all low typed customers and they do not have other choice, thus the delivered price for lower customers must be non-decreasing compared with when $C = C_0^-$. Let x_h, x_l be the number of customers of high and low types in licensed band and X^w be the number of customer on unlicensed band at $C = C_0^-$. We have

$$\lambda_l l(x_h + x_l) + p = \lambda_l g(W^w).$$

Thus, the delivered price for the high customers at that time is

$$\lambda_h l(x_h + x_l) + p < \frac{\lambda_h}{\lambda_l}(\lambda_l l(x_h + x_l) + p) = \lambda_h g(W^w).$$

This means that high customers strictly prefer the licensed band to the unlicensed one.

Now, at $C = C_0^+$ the the quality the unlicensed band has worsen. Therefore, the incumbent also has incentive to raise the delivered price for high customers. This shows that the delivered price for low types is non decreasing and the delivered price for high types increases discontinuously, which concludes the proof.

C. Proof of Lemma 2

We prove the lemma in the general setting with arbitrary number of SPs in the set $\mathcal{N}$. Moreover, assume the general congestion function associated with each SP is $l_i(\cdot)$. Let $x_i = x_i^h + x_i^l$ and $X^w = X^{wh} + X^{wl}$.

First, we show that it is NOT possible that some SP i and the unlicensed band can serve both classes of customers at the same time. If so, by (1) and (2) we have

$$p_i + \lambda_h l_i(x_i) = \lambda_h g(X^w)$$
$$p_i + \lambda_l l_i(x_i) = \lambda_l g(X^w)$$

But since $frac p_i + \lambda_h l_i(x_i) p_i + \lambda_l l_i(x_i) < \frac{\lambda_h}{\lambda_l}$ given $\lambda_h > \lambda_l$, the equalities cannot hold.

Next, we prove **(i)** suppose there exists some $i \in \mathcal{N}$ with price p_i such that $x_i^h > 0$ and $x_i^l > 0$, then we must have

$$p_i + \lambda_h l_i(x_i) = W_h^d$$
$$p_i + \lambda_l l_i(x_i) = W_l^d$$

according to (1) where W_h^d and W_l^d are the equalized delivered prices for high and low classes customers respectively. If $X^{wh} > 0$, then by (2), we have

$$\lambda_h g(X^w) = W_h^d.$$

Since

$$\frac{\lambda_h g(X^w)}{W_l} = \frac{W_h}{W_l} = \frac{p_i + \lambda_h l_i(x_i)}{p_i + \lambda_l l_i(x_i)} \leq \frac{\lambda_h}{\lambda_l},$$

we have $\lambda_l g(X^w) < W_l$. This implies $X^{wl} > 0$, which contradicts the first result. Therefore $X^{wh} = 0$ if $x_i^h > 0$ and $x_i^l > 0$.

Finally, we prove **(ii)**. Suppose $X^{wh} > 0$ and $X^{wl} > 0$. If there exists some $i \in \mathcal{N}$ such that $x_i^l > 0$, then it can be similarly shown that $x_i^h > 0$ which implies a contradiction.

D. Proof of Theorem 3

To prove this Theorem, we first give the following lemma.

Lemma 3: *Let X^{Wh^*} and X^{Wl^*} be the number of high and low class customers served at equilibrium in the unlicensed band of capacity C. If*

$$\frac{W_h}{\lambda_h} < \frac{W_l}{\lambda_l}$$

and congestion costs are linear, then for all unlicensed band capacity $C \geq 0$, if $X^{Wh^*} > 0$, then $X^{Wl^*} = Q_l$.

The proof of this lemma is attached at the end of the section. Lemma 3 implies that given the inequality, as the capacity of unlicensed band increases, low-class customers will join the service in the unlicensed band first and followed by the high-class customers.

Suppose before unlicensed spectrum is added, monopoly set price p_0 and served Q_h high class customers and $x_{l0} < Q_l$ low class customers. We refer this as the *initial state* of the game. Since $x_{l0} < Q_l$, it is clear that $W_l^d = p_0 + \lambda_l b(Q_h + x_{l0}) = W_l$.

First of all, let C_1 be such that $\lambda_l g(Q_l - x_{l0}) = \lambda_l \alpha_{C_1}(Q_l - x_{l0}) = W_l$. It can be seen that for $C \leq C_1$, it must be ture that $\lambda_l \alpha_C(X^w) = W_l$ and $X^w \leq Q_l - x_{l0}$. Thus the unlicensed band has no impact on the monopoly and the initial state holds. Since in this case the unlicensed band does not improve customers surplus, social welfare remains constant.

By Lemma 2, the monopoly and the unlicensed spectrum cannot serve both classes of customers simultaneously. Then there are three possible scenarios with additional unlicensed spectrum:

1) the monopoly serves mixture of both classes and the unlicensed band serves low class,

2) the monopoly serves high class and the unlicensed band serves low class

3) the monopoly serves high class and the unlicensed band serves mixture of high and low classes.

Because of Theorem 1 and the fact that the monopoly is the only strategic agent in the game, the NE of the game can be described as solutions to the monopoly's revenue maximization problems. Given the initial state, we have the following optimization problem (P1) for Scenario **1**

$$\max_{p,x_l,X^w} \; r_1(C) = p(Q_h + x_l) \tag{P1}$$

$$\text{S.t.} \; p + \lambda_h b(Q_h + x_l) \leq W_h \tag{7}$$

$$p + \lambda_l b(Q_h + x_l) = \lambda_l \alpha_C X^w \leq W_l$$

$$x_l + X^w \leq Q_l - x_{l0} \text{ and } p, x_l, X^w \geq 0$$

where $r_1(C)$ denotes the optimal revenue the monopoly achieved with C in Scenario **1**. The problem for Scenarios **2** (P2) and **3** (P3) can be similarly written as

$$\max_{p,x,X^w} \; r_2(C) = px \tag{P2}$$

$$\text{S.t.} \; p + \lambda_h bx \leq W_h$$

$$\lambda_l \alpha_C X^w \leq W_l \tag{8}$$

$$p + \lambda_l bx \geq \lambda_l \alpha_C X^w \tag{9}$$

$$\lambda_h \alpha_C X^w \geq p + \lambda_h bx \tag{10}$$

$$p \geq 0, \; 0 \leq x \leq Q_h \text{ and } 0 \leq X^w \leq Q_l$$

and

$$\max_{p,x,X^w} \; r_3(C) = px \tag{P3}$$

$$\text{S.t.} \; p + \lambda_h bx = \lambda_h \alpha_C X^w \leq W_h$$

$$p + \lambda_l bx \geq \lambda_l \alpha_C X^w$$

$$\lambda_l \alpha_C X^w \leq W_l$$

$$x \leq Q_h \text{ and } X^w \leq Q_h + Q_l \text{ and } p, x, X^w \geq 0.$$

Claim 1: There exist $C_2 \leq C_3$ such that

$$r_1(C) = \max\{r_1(C), r_2(C), r_3(C)\} \text{ for } C \leq C_2$$

$$r_2(C) = \max\{r_1(C), r_2(C), r_3(C)\} \text{ for } C_2 \leq C \leq C_3$$

$$r_3(C) = \max\{r_1(C), r_2(C), r_3(C)\} \text{ for } C \geq C_3.$$

To prove this claim, we first compare Problems (P2) and (P3). Let C_3 be such that $\alpha_{C_3} = \frac{W_h}{\lambda_h Q_l}$. Now we examine (P2). Because of the assumptions on the initial state and (5), the constraints in (9) and(10) must be loose for $C < C_3$. Therefore, $r_2(C)$ is essentially independent of C, i.e. $r_2(C)$ is constant for $C < C_3$. On the other hand, when $C \geq C_3$, it can be seen that constraint (8) will be loose since

$$\lambda_l \alpha_C X^w \leq \lambda_l \alpha_C Q_l \leq \lambda_l \frac{W_h}{\lambda_h} < W_l,$$

where the last inequality is due to (5). Then it can be seen that Problem (P2) and (P3) only differ in the constraint of X^w such that $(P3)$ has a strictly large feasible region that $P(2)$. Therefore we have $r_3(C) \geq r_2(C)$ for $C \geq C_3]$.

Next we compare Problems (P1) and (P2). The initial state implies that $r_1(C_1 + \epsilon) > r_2(C_1 + \epsilon)$ for small enough $\epsilon > 0$. Further, it can be seen through simply algebra that $r_1(C)$ is non-increasing in C. Also, $r_2(C)$ is constant for $C \leq C_3$. Thus $r_1(C)$ and $r_2(C)$ must intersect at some point denoted by C_2. Moreover, by the assumption on the initial state, we have $C_2 \leq C_3$. Therefore, the claim is proved. (See Fig. 6 for a numerical example of the monopoly's revenue.)

Finally, we examine the monotonicity of $SW(C)$ in $[C_1, C_2]$, $[C_2, C_3]$ and $[C_3, \infty]$.

For $C \in [C_1, C_2]$, by Claim 1, NE is corresponding to the solution of Problem (P1). It can be seen that (5) implies that the constraint in (7) is always loose for $C \leq C_2$. Therefore, by setting $\hat{x} = Q_h + x_l$ and rewriting the last constraint as $\hat{x} + X^w \leq Q_h + Q_l - x_{l0}$, the problem can be transformed into a problem corresponding to a game with homogeneous customers. Then we apply Theorem 3 in [2] and conclude that the social welfare of low class customers will first monotonically decrease and then increase.

For $C \in [C_1, C_2]$, we examine Problem (P2). Since the monopoly's revenue is independent of C, the social welfare coming from serving high class customers stays constant. However, increasing C will decrease the left-hand-side of constraint (8), thus possibly reduce the delivered price for the low class customers as will increase the social welfare from the low customers. Thus, the overall social welfare $SW(C)$ is non-decreasing.

For $C \in [C_3, \infty]$, it can be seen that $X^w = Q_l + X^{wh}$. Then Problem (P3) can be transformed into a problem corresponding to a special case of the homogeneous model with congestion function in the unlicensed band as $\hat{g}(x) = \alpha_C(Q_l + x)$. It can be shown through simple algebra that Theorem 3 in [2] can be extended to such case. Thus the social welfare from high customers can be shown to decrease first until some C_4, then increase monotonically. Note that the social welfare from the low customers may increase for $C \geq C_3$. However, it can be seen that the s in (5) ensures that the decrease of high class customers' social welfare exceeds the increase of social welfare of the low class customers. Therefore, we conclude that SW(C) will first decrease until some C_4 and then increase monotonically.

This completes the proof of Theorem 3.

Proof of Lemma 3: Let x be the number of customers served in the licensed band and $X^{w*} = X^{wh^*} + X^{wl^*}$.

Suppose $X^{wh^*} > 0$, then we have $\lambda_l g(X^{w*}) \leq W_h$. We prove this lemma for three possible equilibrium scenarios.

Scenario 1: The Incumbent's licensed band serves high class customers at equilibrium. To show such we need to show that $\lambda_l g(X^{w*}) \leq W_l$. This is ture since

$$\lambda_l g(X^{w*}) \leq \frac{\lambda_l}{\lambda_h} \lambda_h g(X^{w*}) \leq \frac{\lambda_l}{\lambda_h} W_h \leq W_l.$$

Scenario 2: The Incumbent's licensed band serves both high and low class customers at equilibrium. Then the result is automatic by Lemma 2.

Scenario 3: The Incumbent's licensed band serves low class customers at equilibrium with price p, which implies $\lambda_h g(X^{w*}) \leq p + \lambda_h l(x^*)$. Now, we have

$$\lambda_l g(X^{w*}) \leq \frac{\lambda_l}{\lambda_h} \lambda_h g(X^{w*}) \leq \frac{\lambda_l}{\lambda_h} (p + \lambda_h l(x^*)) \leq p + \lambda_l l(x^*).$$

This implies that the unlicensed band serves all of the low customers, i.e. $X^{wl^*} = Q_l$. ∎

E. Numerical Example

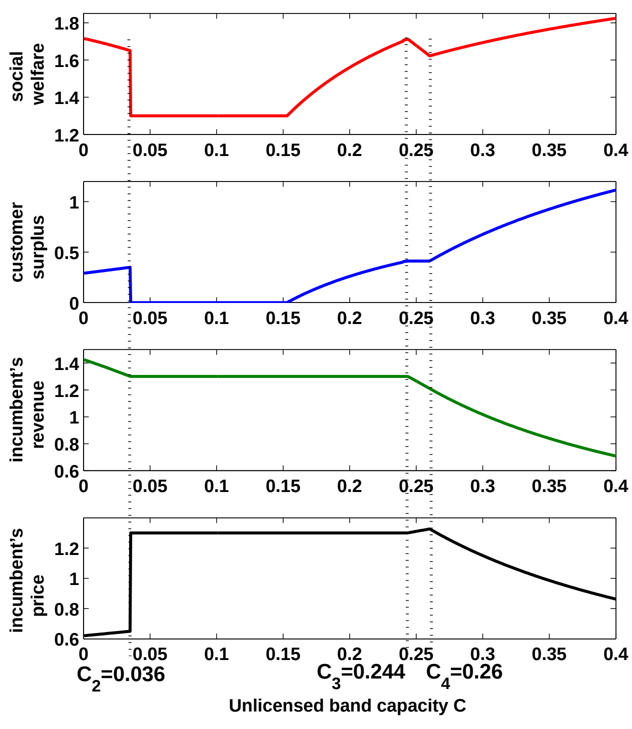

Fig. 6. An example where the incumbent served both classes initially

Let $W_h = 1.6$, $Q_h = 1$, $W_l = 0.85$ and $Q_l = 1.3$, $\lambda_h = 0.4$ and $\lambda_l = 0.1$. Set $l(x) = x$ and $g(x) = x/C$. Without unlicensed band, it can be shown that the incumbent SP would set price $p_0 = 0.62$ to serve all of low class customers to maximize her revenue. The numerical results of social welfare, customer surplus and incumbent's price and revenue with unlicensed band are shown in Fig. 6.

Remark: In the numerical example, we found the social welfare, as a function of the capacity C in the unlicensed band, is not monotone. As shown in Fig. 6, there are two regions of C, $[0, C_2]$ and $[C_3, C_4]$ where social welfare decreases. Note that $C_2' > C_2$ for the parameters in this example, so we define $C_2' := C_2$. Also, since the incumbent served all of both classes of customers, C_1 in Theorem 3 does not exist. Thus we set $C_1 := 0$.

Comparing social welfare and incumbent's price in Fig. 6, we find that the two regions where social welfare decreases are also where incumbent's price rises. This phenomenon is reminiscent to that in the model with homogeneous customers and can be explained in a similar way. Namely, the incumbent may benefit from raising her price since this may reduce the congestion in her licensed band and worsen the quality of service in the unlicensed band.

In particular, there are three stages as C increases.

Facing the competition from the service in the unlicensed band as C increases, the incumbent will eventually "retreat" from serving low class and suddenly increase her price to serve high class customers only to gain higher revenue. This corresponds to the jump in the monopoly's price and the drop in social welfare and customer surplus in Fig. 6 at C_2.

Thus the first stage corresponds to $C \in [0, C_2]$. In this stage, the service in the unlicensed band and that of the incumbent's licensed band will be competing on *low* class customers while the incumbent still serves all of the high class customers. Here we have the sam observation that social welfare decreases as a result of the rise of the monopoly's price[6] and congestion in the unlicensed band.

The second stage is when $C \in [C_2, C_3]$. This is the stage in which the market is sorted. Namely, unlicensed band serves only low class and licensed band serves high class customers. Thus increasing capacity C has no impact on high class customers but improving the congestion in the unlicensed band. Therefore, the social welfare is constant or increasing in this stage.

Finally, when $C \in [C_3, \infty]$, the unlicensed band and licensed band will be competing on the *high* class customers while all of the low customers are being served in the unlicensed band. Similarly, the observation is that the social welfare decreases first as the increase of the monopoly's price [7] until C reaches C_4 and then eventually starts to increase as the quality of service in the unlicensed band improves.

F. Existence of user equilibrium given price vector

Lemma 4: Given a price vector $(\mathbf{p}, \mathbf{p^w})$ the induced non-negative demand vector $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ is the solution to

[6]This is true given the assumptions on the initial state and the assumption in (5). In general, the incumbent may not benefit from increasing her price in this case. This is because higher price may decrease the number of high class customers she is serving thus decrease her revenue.

[7]This is true given the assumptions on the initial state and (5). In general, social welfare may not decrease. This is because although the social welfare from high class customers may decrease due to the increase in the incumbent's price, the social welfare from low class customers always increases as the capacity in the unlicensed band increases.

the following maximization problem

$$\begin{aligned} \max \quad & \int_0^{Q^h} \frac{P_h(q)}{\lambda_h} dq + \int_0^{Q^l} \frac{P_l(q)}{\lambda_l} dq \qquad \text{(Pu)} \\ & - \sum_{i \in \mathcal{N}} \left(\frac{p_i x_i^h + p_i^w x_i^{wh}}{\lambda_h} + \frac{p_i x_i^l + p_i^w x_i^{wl}}{\lambda_l} \right) \\ & - \sum_{i \in \mathcal{N}} \int_0^{x_i^h + x_i^l} l_i(z) dz - \int_0^{X^w} g(z) dz \\ \text{S.t.} \quad & x_i^h,\, x_i^l,\, x_i^{wh},\, x_i^{wl} \geq 0 \forall i \in \mathcal{N} \end{aligned}$$

where $Q_h = \sum_i (x_i^h + x_i^{wh})$, $Q_l = \sum_i (x_i^l + x_i^{wl})$ and $X^w = \sum_i (x_i^{wh} + x_i^{wl})$.

The above lemma can be easily proved using complementary slackness. If the inverse demand functions P_h and P_l are concave and congestion function l_i and g are convex, then it can be seen the Problem (Pu) is a well-defined convex optimization which always have solutions. Furthermore, if the objective function in (Pu) is strictly concave, the there exist unique such demand vector. Otherwise there may be multiple such demand vectors which are equivalent.

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