

# Investment and Competition in Unlicensed Spectrum

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**Abstract**—Adding unlicensed spectrum, such as the recent opening of the television white spaces in the US, has the potential to benefit customers by increasing competition, but may also increase congestion. In an earlier paper, we studied this using models for price competition in congestible networks. Here we extend that model to include the investment costs of providers. With such costs, we characterize the equilibria of the resulting network investment and pricing game and show that in some cases the equilibrium is not increased competition but rather investment by a single monopolist.

## I. INTRODUCTION

The FCC has recently opened up TV white space spectrum as unlicensed spectrum [1]. This can potentially stimulate new innovation in both technology and new wireless service business models. There is no cost for acquiring such unlicensed spectrum as by design it is open for any service provider (SP) to freely use; hence, unlicensed spectrum will substantially lower the entrance cost for providing wireless services. However, exploiting such spectrum does require investment, such as technology and/or infrastructure. Furthermore, there will be congestion effects for each SP, caused by increasing customer demand and/or interference from the other SPs operating in the same frequency bands. To reduce such congestion cost, SPs can invest in better technology and/or more infrastructure. Therefore, it is of interest to investigate competing SPs' investment and pricing behavior.

In our previous work, [2] and [3], we adopted a framework for price competition of congestible resources developed in the operations and economics literature [4]–[7]. According to this framework, in a competitive equilibrium the delivered price, consisting of the price paid to an SP plus a congestion cost, is the same across all SPs. A key assumption in [2] and [3] is that each SP in the unlicensed band experiences a congestion cost that depends on the total traffic in that band (of all of the SPs). This is motivated by the likelihood that customers sharing the unlicensed band within a given region will interfere with each other even if they are associated with different SPs. In [2], we studied the social welfare obtained by adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent SPs. It has been shown that the resulting equilibrium price in the unlicensed spectrum is zero. As a result, the profit from the unlicensed spectrum

for each SP is zero.<sup>1</sup> Moreover, equilibrium profits for the incumbent SPs with licensed bands may decrease with the additional unlicensed spectrum and the overall social welfare may also decrease, depending on the amount of unlicensed bandwidth introduced. On the other hand, [3] showed that consumer surplus is always non-decreasing with additional unlicensed spectrum for homogeneous customers. Extending the results to a heterogeneous pool of customers who have different tradeoffs in service price and congestion cost, [3] also showed that both social welfare and consumer surplus may decrease with additional unlicensed spectrum.

The models in [2] and [3] did not include any cost for SPs to invest in the unlicensed spectrum. Since the profit from the unlicensed spectrum is zero, it is not clear that SPs would have any incentive to invest in the first place. In this paper, we incorporate investment, in terms of “effort” to reduce congestion cost, into each SP’s strategy. Intuitively, this can allow additional freedom in which each SP may differentiate its service from the other SPs’, and so can provide SPs with more incentives to enter the market via unlicensed spectrum.

We focus on investment and price competition for services in the unlicensed spectrum alone.<sup>2</sup> We extend the model in [2] such that investment can reduce congestion costs. In particular, the investment is explicitly assumed to reduce the congestion caused by a SP’s own traffic plus the traffic of the other SPs’. We then model the investment and pricing dynamics among SPs as a multi-stage game where SPs make decisions on investments in the first stage, and then compete on prices in the second stage. This model is valid for irreversible investment, as is usually the case in the wireless services industry where investment often corresponds to the deployment of additional infrastructure or better technologies. Using the solution concept of subgame-perfect equilibrium, we show that, for the model of interest, the equilibrium of the game, if one exists, corresponds to a monopoly scenario.

The notion of “investment” in this paper is similar to “capacity” or “quantity” in the industrial organizations literature. To this end, price and capacity competition between firms has

<sup>1</sup>If sunk cost is considered, then the equilibrium price will fall to be equal to the sunk cost and SPs’ profit is still zero.

<sup>2</sup>Such scenario is reasonable if the customers in the unlicensed band are from a different group from incumbents’ customers and SPs in the unlicensed band are entrants (who do not have proprietary spectrum). Namely, the service market in the unlicensed spectrum forms a separate market.

been extensively studied, such as in the Edgeworth-Bertrand game [8] and models that built on top of it [9]–[11]. In these models, if firms set quantities and prices sequentially, a pure strategy equilibrium may fail to exist, in which case mixed strategy equilibria are studied. On the other hand, if firms choose to set quantities and prices simultaneously, then a pure strategy equilibrium does not exist. Congestion externalities are not explicitly considered in these work. When congestion cost is present, [7] explored the conditions for a pure strategy Nash equilibrium to exist when firms set price and investment simultaneously. [12] and [13] also consider firms that set capacity or investment and price sequentially in congestible markets in a similar spirit to our model. However, the preceding papers assume that each firm has access to an exclusive resource and the congestion suffered by its customers depends only on the number of customers consuming that resource. Here, the service of each firm is provided through a non-exclusive resource that models unlicensed spectrum. Thus the congestion suffered by customers of each firm also depends on other firms' customers. We shall see that this assumption changes the outcome compared to where firms have exclusive resources.

## II. THE MODEL

Our model is an extension of [2]. The economy consists of *service providers* (SPs) and *customers*. The SPs set investment and prices for their services. The customers respond according to both the service prices and congestion costs of the SPs. We only consider the market associated with the unlicensed spectrum. We next describe the models for SPs and customers in details.

### Service Providers:

Suppose there are  $N$  SPs in the market, denoted by the set  $\mathcal{N}$ . A service provider chooses investment in the unlicensed band  $I$  and the corresponding service price  $p$  for its service. The congestion cost experienced by each SP depends on the mass of customers served by each of the other SPs. Moreover, the congestion cost experienced by the customers of an individual SP  $i$  also depends on its investment  $I_i$  in the unlicensed band. Let  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  be a vector of customer mass of the SPs. The congestion experienced by SP  $i$  in the unlicensed band is denoted by the function  $\tilde{g}(\mathbf{x}, I_i)$ . For the congestion function, we make the following assumptions: if  $I_i = 0$ , then  $x_i = 0$ ,  $\tilde{g}(\mathbf{x}, I_i) = \infty$ , and  $\lim_{I_i \rightarrow 0} x_i \tilde{g}(\mathbf{x}, I_i) = 0$ .

In general, an increase in investment of a SP reduces the congestion it experiences, and hence allows it to raise its service price (due to better quality of service) or admit more customers. On the other hand, there is a cost of investment such as deploying more access points or upgrading existing technology. Thus the trade-off for the SPs is between the revenue earned by serving customers and the cost of investment. Furthermore, investment of the SPs can take on many different forms such as infrastructure and/or technology and it may have different effects on a SP's own traffic and the traffic of other SPs.

In this paper, we consider a particular stylized model of investment: one SP's investment reduces the congestion effects caused by all customers in the unlicensed band. That is SP  $i$ 's congestion cost is given by

$$\tilde{g}(\mathbf{x}, \mathbf{I}) = g\left(\frac{\sum_{i \in \mathcal{N}} x_i}{I_i}\right) \quad (1)$$

with investment  $I_i$ . For example, this model can apply where SPs invest in base station sectorization in a cellular-type system, with  $I_i$  indicating the number of sectors in a cell.

If SP  $i$  sets investment  $I_i > 0$ , we refer to it as being *active*. On the other hand, SP  $i$  is *inactive* if it chooses not to invest, i.e.,  $I_i = 0$ . We assume that if a SP's investment is zero, then its profit is also zero.

### Customers:

As in many other models in the network literature [14], we assume a unit mass of homogeneous customers. Customers choose a SP based on the *delivered price*, which is the sum of the price announced by a SP and the congestion cost it experiences when served by that SP. For SP  $i$ 's customers, the delivered price is  $p_i + g(\frac{\mathbf{x}}{I_i})$ .

The demand for services is governed by a downward sloping demand function  $D(p)$  with the inverse function  $P(q)$ . Customers always choose service from the SP with the lowest delivered price. When facing the same delivered price from multiple SPs, customers are assumed to randomly choose one of the SPs. Thus SPs with the same delivered price will draw the same customer mass in equilibrium.

Given investment and price vectors of every SP,  $(\mathbf{I}, \mathbf{p})$ , the non-negative demand vector  $\mathbf{x}$  induced by  $(\mathbf{I}, \mathbf{p})$  satisfies

$$\begin{aligned} p_i + g\left(\frac{\mathbf{x}}{I_i}\right) &= \min_{i \in \mathcal{N}} p_i + g\left(\frac{\mathbf{x}}{I_i}\right) \quad \text{for } i \in \mathcal{N} \text{ with } x_i > 0 \\ p_i + g\left(\frac{\mathbf{x}}{I_i}\right) &= P(X) \quad \text{for } i \in \mathcal{N} \text{ with } x_i > 0 \\ p_i + g\left(\frac{\mathbf{x}}{I_i}\right) &\geq P(X) \quad \forall i \in \mathcal{N} \end{aligned} \quad (2)$$

where  $X = \sum_{i \in \mathcal{N}} x_i$  is the total customer mass in the whole market. In other words, the demand for each SP is such that no customer can lower the delivered price by switching SPs. It is shown in [3] that such induced demand always exists and is unique for a given investment and price vector.

Furthermore, we make the following assumptions on the congestion cost function,  $g(\cdot)$ , and the demand function,  $P(\cdot)$ .

**Assumption 1:** Assume  $g(x)$  and  $xg(x)$  are strictly increasing and convex;  $P(x)$  is concave decreasing and  $xP(x)$  is strictly concave. In addition, assume  $g(0) = 0$  and there exists  $Q > 0$  such that  $P(Q) = 0$ .

### The Two-stage Investment-price Game:

Given the investment, denoted by  $\mathbf{I}_{-i}$  and prices, denoted by  $\mathbf{p}_{-i}$  set by the other SPs and the demand vector  $\mathbf{x}$  satisfying (2), if  $I_i > 0$  then SP  $i$ 's profit is given by

$$\Pi_i(I_i, \mathbf{I}_{-i}, p_i, \mathbf{p}_{-i}, \mathbf{x}) = p_i x_i - c I_i \quad (3)$$

where the first term represents revenue  $i$  collects from its customers and the second term represents investment cost

which is assumed to be proportional to  $I_i$  with constant marginal investment cost  $c$ . In general, different SPs may have different investment costs. Our focus in this paper is on cases with symmetric investment cost and many of our results can be generalized to asymmetric investment costs. Further,  $\Pi_i = 0$  if  $I_i = 0$ . The objective of each SP is to set investment and price to maximize its profit.

As previously discussed, it is reasonable to assume irreversible investment for wireless service markets. Therefore, the dynamic among different SPs is similar to the capacity-price game in [11]. We refer to the dynamic in our model as the *investment-price game*. In this game,  $N$  SPs move first and simultaneously set investment. Having observed the investment  $I_i \forall i \in \mathcal{N}$  set in the first stage, SPs then announce their service prices  $p_i$  simultaneously in the second stage. Finally, customers choose SPs based on the delivered price given by (2).

The solution concept we use for this sequential game is *subgame perfect equilibrium* (SPE) [15]. Moreover, we only consider pure strategy SPE.

Since the investment set in the first stage is observed by all SPs, every investment vector  $\mathbf{I} = (I_1, \dots, I_N)$  then defines a proper subgame, which we refer to as the *pricing game*. Subgame perfection asserts that the continuation equilibrium strategies constitute a Nash equilibrium in each subgame. The entire game consists of two subgames: a pricing subgame where SPs compete on prices for given investment levels and an investment subgame where SPs set investment levels expecting prices according to the outcome of the pricing subgame.

Thus for each investment subgame (for an investment vector), we define a continuation *price equilibrium*.

**Definition 1:** Given an investment vector  $\mathbf{I}$ , the price and demand vector  $(\mathbf{p}(\mathbf{I}), \mathbf{x}(\mathbf{I}))$  is a pure strategy *price equilibrium* in the pricing subgame for investment vector  $\mathbf{I}$  if  $\mathbf{x}(\mathbf{I})$  satisfies (2) and no SP can improve its profit by changing prices.

We next define the SPE of the entire two-stage game and refer to it as the *investment-price equilibrium*.

**Definition 2:** A tuple  $(\mathbf{I}^e, \mathbf{p}(\mathbf{I}^e), \mathbf{x}(\mathbf{I}^e))$  is an investment-price equilibrium if  $(\mathbf{p}(\mathbf{I}^e), \mathbf{x}(\mathbf{I}^e))$  forms a price equilibrium in the pricing subgame for  $\mathbf{I}^e$  and no SP can improve its profit by changing investment.

### Social Welfare and Social Optimality:

**Definition 3:** Suppose  $\mathbf{I}$  is an investment vector and there exists an induced price equilibrium  $(\mathbf{p}(\mathbf{I}), \mathbf{x}(\mathbf{I}))$ . Then, the *social welfare* is defined as

$$SW = \int_0^X P(q) dq - \sum_{i \in \mathcal{N}} x_i g\left(\frac{\mathbf{x}}{I_i}\right) - \sum_{i \in \mathcal{N}} cI_i, \quad (4)$$

where  $X = \sum_{i \in \mathcal{N}} x_i$ , and the corresponding *consumer surplus* is defined as

$$CS = \int_0^X P(q) - P(X) dq. \quad (5)$$

Finally, the problem for a social planner is to allocate investment  $(I_i)$  and customers  $(x_i)$  to maximize the social

welfare in (4). We call this solution the *socially optimal solution*.

### Limitations

There are a few limitations of our model. First, we do not consider incumbent SPs with proprietary spectrum. Thus our model is valid when all of the SPs are new entrants. Second, we only study a particular form of investment, which is the same for every SP, in terms of its impact on the congestion cost. Other types of investment may yield different equilibrium outcomes. In addition, the assumption of linear investment costs is another simplification. Third, this model essentially assumes that every active SP uses the whole unlicensed spectrum. In practice, it may be possible that a SP transmits over a part of the spectrum (e.g., certain channels in the TV white space case). In such scenario, a SP's interference only affects the SPs transmitting in the same frequencies, which complicates the formulation and analysis of the resulting games.

## III. EQUILIBRIUM

In this section, we characterize the equilibrium structure of the investment-price game. We first examine the investment and price choice for monopoly scenarios. Then, we study oligopoly scenarios with multiple SPs with symmetric investment costs,  $c$ .

### A. Monopoly Scenario

Suppose there is a monopoly SP in the unlicensed band<sup>3</sup>, then the SP will jointly optimize its profit over investment  $I \geq 0$  and price  $p \geq 0$ , which results in the following problem.

$$\begin{aligned} \max_{I, p, x} \quad & \Pi = px - cI & (\mathbf{P}_M) \\ \text{s.t.} \quad & p + g\left(\frac{x}{I}\right) = P(x) \\ & 0 \leq x \leq Q, \quad p \geq 0, \quad I \geq 0. \end{aligned}$$

Substituting  $p = P(x) - g\left(\frac{x}{I}\right)$  into the objective,  $\mathbf{P}_M$  can be rewritten as

$$\begin{aligned} \max_{I, x} \quad & \Pi = xP(x) - x \underbrace{\left(g\left(\frac{x}{I}\right) + c\frac{I}{x}\right)}_{\Delta(x/I)} & (\mathbf{P}'_M) \\ \text{s.t.} \quad & x \geq 0, \quad I \geq 0. \end{aligned}$$

In the objective of  $\mathbf{P}'_M$ , the first term represents the monopoly's revenue collected from the customers, while the second term represents the costs of congestion and investment. Define  $\Delta(t) = g(t) + c\frac{1}{t}$ , such that the cost term can be written as  $x\Delta(x/I)$ , as marked in  $\mathbf{P}'_M$ . Note that given a fixed  $x > 0$ ,  $\Delta(x/I)$  is the only place where  $I$  enters into the objective and it is strictly convex in  $I$  with Assumption 1. Thus, to solve  $\mathbf{P}'_M$ , one can first minimize  $\Delta(t)$  over  $t \geq 0$ , where  $t = x/I$ . Let the solution to this minimization be  $\xi$ ; this must be an interior point that satisfies  $g'(\xi)\xi^2 = c$ . Let the corresponding

<sup>3</sup>The unlicensed band essentially becomes the monopolist SP's proprietary band in this scenario.

objective (minimum marginal cost) be  $\Delta_c^*$ . Since the solution to the problem of minimizing  $\Delta(t)$  for  $t > 0$  must be an interior solution, one can apply the envelope theorem [15] to this problem and show that the optimal objective  $\Delta_c^*$  is increasing in  $c$ . As a result, given any  $x > 0$ , the corresponding optimal investment is given by  $I^*(x) = x/\xi$ . Moreover, the objective of  $P'_M$  becomes  $\Pi = xP(x) - \Delta_c^*x$ . This is a strictly concave objective. Thus, it can be maximized by solving the first order condition,  $\frac{d}{dx}xP(x) = \Delta_c^*$ . We then have the following lemma about the monopolist's behavior.

**Lemma 1:** Suppose Assumption 1 holds. If  $P(0) \leq \Delta_c^*$ , then the monopoly SP does not invest and serves no customers. Otherwise, if  $P(0) > \Delta_c^*$ , then the monopoly SP serves  $x^M$  customers with investment  $I^M = x^M/\xi$ , where  $x^M$  satisfies

$$\left. \frac{d}{dx}xP(x) \right|_{x=x^M} = \Delta_c^*.$$

Note that from Assumption 1,  $\frac{d}{dx}xP(x) \leq P(0)$  for all  $x \in [0, Q]$ . We make the following assumption from hereafter so that a monopoly will always enter the market.

**Assumption 2:**  $P(0) > \Delta_c^*$ .

### B. Oligopoly Scenario

Next we turn to models with multiple SPs. Before addressing the two-stage investment-price game, we briefly consider a game in which the SPs make investment and price decisions *simultaneously*. Then the resulting investment-price game has no pure strategy Nash equilibrium. Such scenarios are referred as *reversible* investment (see [7]).

**Lemma 2:** No pure strategy Nash equilibrium exists for scenarios with reversible investment.

Intuitively, this is because for any investment and price vector, a SP has the incentive to deviate by increasing its investment. This will reduce the congestion of the SP, and hence its delivered price. Lowering the delivered price will attract all of the customers and lead to a positive improvement in the SP's profit. A formal proof follows by simply extending the proof of Theorem 1 in [2], thus is omitted.

As noted previously, for wireless services, a more reasonable model is to assume that the investment of SPs is *not* reversible, where our model of two-stage investment-price competition can be applied. We first characterize the price equilibrium structure, based on which we then study the SPE of the entire investment-price game. Before stating the results on the price equilibrium, we first make the following assumption on the customer response to SPs' prices, in addition to (2).

**Assumption 3:** Any SP  $j$  with  $p_j = 0$  will receive  $x_j = 0$  whenever its delivered price is the same as that of some other SP  $i$  with  $p_i > 0$ .

The purpose of Assumption 3 is to guarantee the existence of a pure strategy price equilibrium. We shall see that this assumption does not essentially change the outcome of the investment-price game, but rather guarantees the existence of a pure strategy equilibrium path, and hence the existence of a pure strategy SPE.

Lemma 3 characterizes the price equilibrium for two SPs. The result can be easily generalized to multiple SPs.

**Lemma 3:** Let  $I_1 > 0$  and  $I_2 > 0$  be the investment of the two SPs. Given Assumption 3, a unique Nash equilibrium of the corresponding pricing subgame always exists. Let  $(p_1^e, x_1^e)$  and  $(p_2^e, x_2^e)$  be the price equilibrium. Then the price equilibrium must satisfy the following

- (i) If  $I_1 = I_2$ , then  $p_1^e = 0$ ,  $x_1^e > 0$  and  $p_2^e = 0$ ,  $x_2^e > 0$ .
- (ii) If  $I_1 > I_2 > 0$ , then  $p_1^e > 0$ ,  $x_1^e > 0$  and  $p_2^e = x_2^e = 0$ .

*Proof:* If  $I_1 = I_2$ , then the congestion costs of the two providers are the same. Thus the SP with lower price will serve all of the customers. Therefore, it can be seen that  $p_1^e = 0$  and  $p_2^e = 0$  is the only equilibrium price vector, and the two SPs equally share a positive mass of customers according to (2). Thus (i) is proved.

We next prove (ii). Since  $I_1 > I_2$ , we have  $g(\frac{X}{I_1}) < g(\frac{X}{I_2})$  for all  $X > 0$ . First, the price equilibrium cannot be such that  $p_1 = 0$  and  $p_2 > 0$ . Otherwise, SP 1's delivered price would then be strictly lower and thus SP 1 would serve all of the customers. As a result, both SPs would have zero profit, and hence SP 1 would have the incentive to increase its price while SP 2 would have the incentive to lower its price.

Second, suppose we have a price equilibrium with  $p_1 > 0$ ,  $x_1 > 0$  and  $p_2 > 0$ ,  $x_2 > 0$ .<sup>4</sup> Then by the condition in (2), we must have  $p_1 + g(\frac{X}{I_1}) = p_2 + g(\frac{X}{I_2})$  where  $X = x_1 + x_2$  denotes the total customer mass. Suppose SP 1 reduces its price by  $\epsilon > 0$  to  $p'_1 = p_1 - \epsilon$  and the resulting total customer mass is  $X'$ . To satisfy (2), we must have  $p'_1 + g(\frac{X'}{I_1}) = P(X') \leq p_2 + g(\frac{X'}{I_2})$  and  $X' \geq X$ . Thus, SP 1 will attract all of the customers (including SP 2's), which leads to a profitable deviation. Therefore, it can be seen that the only possible equilibrium prices are  $p_1^e > 0$  and  $p_2^e = 0$ , which means  $x_2^e = 0$  from Assumption 3. It is then follows that SP 1's equilibrium price is given by maximizing  $p_1x_1$  subject to the constraints

$$p_1 + g\left(\frac{X}{I_1}\right) = P(X) \quad \text{and} \quad g\left(\frac{X}{I_2}\right) \geq P(X). \quad (6)$$

For  $I_1 > I_2$ , this optimization will have a solution  $p_1^e > 0$  from which SP 1 will not want to deviate. Moreover, SP 2 can also not deviate since increasing  $p_2^e = 0$  will not attract any customers.<sup>5</sup> ■

Note that if there are multiple SPs, applying Lemma 3 to every pair of SPs implies that only the SP with largest investment will have strictly positive price and customer mass in the price equilibrium. If there are more than one such SPs with the largest investment, then every SP's price will be zero at price equilibrium and the SPs with the largest investment will equally share a positive amount of customers.

Therefore, there can be at most one active SP in any subgame-perfect equilibrium of the investment-price game.

<sup>4</sup>Note that in any equilibrium for this case, neither  $x_1$  nor  $x_2$  can be zero. If so, the corresponding SP would have the incentive to lower its price.

<sup>5</sup>Assumption 3 is necessary, or else when  $g(X/I_2) = P(X)$ , SP 2 would have  $x_2^e > 0$  and SP 1 could attract all of the customers by reducing  $p_1^e$  by an arbitrarily small amount, preventing a pure strategy equilibrium from existing.

This is because otherwise the SPs with smaller investment would have zero price in the price equilibrium by Lemma 3, and hence negative overall profit. Thus the SPs with smaller investment can be strictly better off by investing zero (being inactive) resulting in zero profit. Therefore, our focus next is on scenarios with a single active SP. Apparently, the SP will behave like a monopolist, whose investment and price are given by  $P_M$ . To show that this is indeed an equilibrium, one needs to show that given the monopoly's investment and price, the other SPs do not have the incentive to enter the market, i.e., entering the market will result in negative profit.

Suppose SP 2 enters the market after SP 1 has made the investment of  $I_M$ . Then, according to Lemma 3, SP 2 must invest more than SP 1, i.e.,  $I_2 > I_M$  to achieve a possible positive price. Considering SP 2's price specified by Lemma 3, we can write down SP 2's profit maximization problem as

$$\begin{aligned} \max_{I_2, p_2, x_2} \quad & \Pi_2 = p_2 x_2 - c I_2 \\ \text{s.t.} \quad & p_2 + g\left(\frac{x_2}{I_2}\right) = P(x_2) \leq g\left(\frac{x_2}{I_M}\right) \\ & 0 \leq x_2 \leq Q, \quad p_2 \geq 0, \quad I_2 \geq I_M, \end{aligned} \quad (\text{P2})$$

where the first constraint is due to the fact that after entering, SP 1 will charge zero price and serve no customers (see (6)). This can be similarly rewritten as

$$\begin{aligned} \max_{I_2, x_2} \quad & \Pi_2 = x_2 P(x_2) - x_2 \left( g\left(\frac{x_2}{I_2}\right) + c \frac{I_2}{x_2} \right) \\ \text{s.t.} \quad & P(x_2) \leq g\left(\frac{x_2}{I_M}\right) \\ & x_2 \geq 0, \quad I_2 \geq I_M. \end{aligned} \quad (\text{P2}')$$

Note that the first order condition w.r.t.  $x$  in  $\text{P}'_M$  implies that  $P(x_M) \geq g\left(\frac{x_M}{I_M}\right)$ . Thus,  $x_2 \geq x_M$  must hold to satisfy the first constraint in  $\text{P2}'$ . Again, SP 2 will set  $\frac{x_2}{I_2} = \xi$  to minimize congestion and investment costs, where  $\xi$  is the minimizer of  $\Delta(t)$  as defined in Section III-A. Note that this is feasible since if  $x_2 \geq x_M$ , then the resulting  $I_2$  will also satisfy the second constraint in  $\text{P2}'$ . Thus the profit of SP 2 becomes  $\Pi_2 = x_2(P(x_2) - \Delta_c^*)$ . Therefore, SP 2 will enter the market only when there exists  $x_2 > 0$  such that  $P(x_2) \geq \Delta_c^*$  and  $P(x_2) \leq g\left(\frac{x_2}{I_M}\right)$ .

Note that  $P(\cdot)$  is strictly decreasing and  $g(\cdot)$  is strictly increasing in  $x_2$ . Let  $\underline{x}$  be such that  $P(\underline{x}) = g\left(\frac{\underline{x}}{I_M}\right)$ , then SP 2 will *not* enter the market, i.e., having a monopoly is the equilibrium of the investment-price game if

$$P(\underline{x}) \leq \Delta_c^*. \quad (\text{C1})$$

**Theorem 1:** Suppose Condition C1 is satisfied. Then the investment-price game has a subgame-perfect equilibrium, in which exactly one SP is active with the monopoly investment and monopoly price.

Generally, we observe that the condition in (C1) requires that the curvature of  $g(\cdot)$  is "small enough" relative to that of  $P(\cdot)$ 's. Fig. 1 shows an example. Initially, consider  $P(x)$  and  $g(x/I_M)$  as shown in solid lines, where  $I_M$  and the corresponding  $\Delta_c^*$  is given by Lemma 1. There is no pure

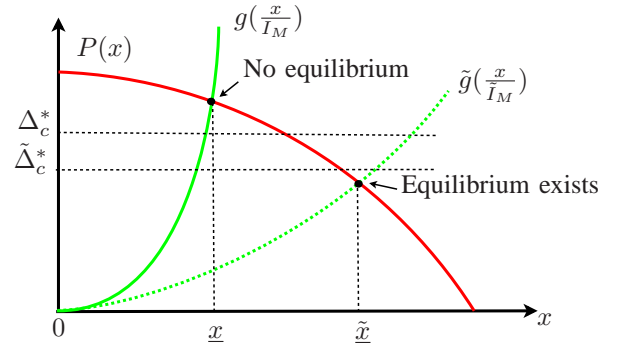


Fig. 1. Illustration of the relation between  $P(\cdot)$  and  $g(\cdot)$  and the existence of an equilibrium.

strategy SPE in this case because  $P(x)$  is above  $\Delta_c^*$  at  $\underline{x}$ , i.e., (C1) is violated. Also in Fig. 1,  $\tilde{g}(\cdot)$  with less curvature than  $g(\cdot)$  is shown as a dashed line. For the  $\tilde{g}(\cdot)$  in this case, it can be shown that  $\tilde{I}_M \geq I_M$ . Thus we have  $\tilde{x} \geq \underline{x}$ , which may lead to (C1). However, it also can be shown that  $\tilde{\Delta}_c^* \leq \Delta_c^*$ . Therefore, if  $\tilde{g}(\cdot)$  yields a large enough  $\tilde{I}_M$  such that  $P(\tilde{x}) \leq \tilde{\Delta}_c^*$ , as shown in Fig. 1, then having a monopoly is indeed the SPE.

In particular, if  $P(x)$  and  $g(x)$  are power functions, we can capture the preceding observation through the following corollary.

**Corollary 1:** If  $P(x) = T - bx^\alpha$  and  $g(x) = ax^\beta$  where  $\alpha \geq 1$  and  $\beta \geq 1$ , then the investment-price game has a unique SPE if  $\beta \leq \alpha$ . Furthermore, there is a single SP in the equilibrium who behaves like a monopolist.

Note that the preceding discussion is for fixed unlicensed spectrum bandwidth. If more bandwidth can reduce the curvature of the congestion function  $g(\cdot)$  (such as decrease the value of  $\beta$  in Corollary 1), then large enough unlicensed spectrum bandwidth can guarantee the existence of a pure strategy SPE of the investment-price game. We will discuss other benefits of greater unlicensed spectrum bandwidth in Section IV.

#### IV. SOCIAL WELFARE AND CONSUMER SURPLUS

According to Theorem 1, the equilibrium of the investment-price game corresponds to monopoly, when one exists. Therefore, we focus on the social welfare and consumer surplus of the monopoly scenario and compare them with socially optimal solutions. Furthermore, we will also examine how social welfare and consumer surplus change if additional bandwidth is assign as unlicensed spectrum.

To find the optimal solution, one needs to optimize the social welfare defined in (4). The following lemma specifies a property of the socially optimal solution

**Lemma 4:** There is a single SP in the socially optimal solution.

Intuitively, this is because the marginal congestion cost  $g(\cdot)$  is a function of the total customer mass and pooling the investment and allocating it to a single SP yields the minimum marginal congestion cost. Formally, Lemma 4 can be proved by showing any solution involving more than one SP can not be a local optima to the social welfare maximization problem.

By Lemma 4, the socially optimal problem becomes

$$\max_{I,x} \int_0^x P(q) dq - (xg(\frac{x}{I}) + cI) \quad \text{s.t. } x, I \geq 0. \quad (\text{Po})$$

For concave and decreasing  $P(\cdot)$ , the monopoly solution is obviously suboptimal in terms of social welfare. Thus, we focus on comparing the investment and customer masses between the monopoly scenario and the socially optimal solution.

**Proposition 1:** Suppose Assumption 1 holds. Let  $I^M$  and  $I^*$  be the monopoly's investment and the socially optimal investment, respectively. Let  $x^M$  and  $x^*$  be the corresponding customer masses. Then  $I^* \geq I^M$  and  $x^* \geq x^M$ .

*Proof:* Comparing  $P'_M$  to Po, one can see that both objectives have the same cost terms (congestion plus investment cost), which can be written as  $x\Delta(x/I)$  as in Section III-A. As a result, the objective of Po becomes  $\int_0^x P(q) dq - x\Delta_c^*$  and the optimal  $x^*$  must satisfy  $P(x^*) = \Delta_c^*$ . Similarly, the objective of  $P'_M$  becomes  $\Pi = xP(x) - x\Delta_c^*$  and the optimal  $x^M$  must satisfy  $P(x^M) + x^M P'(x^M) = \Delta_c^*$ . Since  $P'(x) < 0$  for all  $x > 0$ , we conclude that  $x^* \geq x^M$ , and  $I^* \geq I^M$  follows since  $\frac{x^*}{I^*} = \frac{x^M}{I^M} = \xi$ . ■

Consider a family of congestion cost functions  $g_B(\cdot)$ , which are decreasing functions of the bandwidth of the unlicensed spectrum,  $B$ . We next study how social welfare and consumer surplus change with  $B$ . Given Assumption 1, we have the following theorem.

**Theorem 2:** In the SPE of the investment-price game, social welfare and consumer surplus are non-decreasing in the bandwidth of the unlicensed spectrum.

*Proof:* By Theorem 1, if a SPE exists, it must be the monopoly scenario, where the investment ( $I^M(B)$ ), price ( $p^M(B)$ ) and customers ( $x^M(B)$ ) correspond to the solutions to  $P_M$  with bandwidth  $B$ . Let the resulting consumer surplus and the monopoly profit be  $CS^M(B)$  and  $\Pi^M(B)$ , respectively.

First, we show that  $\Pi(B') \geq \Pi(B)$  for any  $B' \geq B$ . Let  $p' = p^M(B) + g_B(\frac{x^M(B)}{I^M(B)}) - g_{B'}(\frac{x^M(B)}{I^M(B)})$ . Note that  $(I^M(B), p', x^M(B))$  is not only a feasible solution to  $P_M$  with bandwidth  $B'$ , but also gives more revenue. Since the monopoly is optimizing its revenue for  $B'$ , we must have  $\Pi^M(B) \leq p' x^M(B) - cI^M(B) \leq \Pi^M(B')$ .

Next, we study consumer surplus ( $CS^M$ ) which is increasing in the number of customers served by the monopoly,  $x^M(B)$ , by (5). Therefore, to show that consumer surplus is increasing in  $B$ , we only need to show that  $x^M(B)$  is increasing in  $B$ .

Similarly, we have  $\Delta_B(t) = g_B(t) + c\frac{1}{t}$ , such that the objective in  $P'_M$  containing  $I$  can be expressed as  $\Delta_B(x/I)$  and  $\Delta_B(x/I)$  is strictly convex in  $I$  for fixed  $x$ . Again, one can first minimize  $\Delta_B(x/I)$ , and express the resulting solution as  $I_B^*(x) = x/\xi_B$  and the optimal objective value as  $\Delta_{B,c}^*$ . By the envelope theorem [15], it can be shown that  $\Delta_{B,c}^*$  is decreasing in  $B$ . Since  $xP(x)$  is strictly concave by Assumption 1,  $x^M(B)$ , which is the solution to  $\frac{d}{dx} xP(x) = \Delta_{B,c}^*$ , must be non-decreasing in  $B$ .

Finally, since both consumer surplus and monopoly's profit (producer surplus) are non-decreasing in  $B$ , social welfare must also be non-decreasing in  $B$ . ■

The result in Theorem 2 is different from that in [2] which showed that social welfare may decrease in  $B$  for many choices of demand and congestion functions. Note that this is due to two major differences between the model in this paper and that in [2]. First, the model in [2] assumes that any investment cost is already sunk for each SP. Second, our model here does not include incumbent SPs who have proprietary spectrum in the market. We conjecture that both of these differences are needed to get decreasing welfare and consumer surplus.

## V. CONCLUSION

In this paper we studied a model of investment and price competition in unlicensed spectrum with congestion effects. By studying a sequential game formulation, we showed that the only possible equilibria are for a single monopolist to emerge in spite of the lower barriers to entry, which is equivalent to assigning the spectrum as licensed. This suggests that simply opening spectrum may not lead to significant increase in competition.

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