Introduction to Computational Complexity

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Assignment 2 Due on Friday, Feb 1, 2008

- Boolean Formula satisfiability is NP-complete
- Problem Definition: Set of variables (x,y,z ...), each of which can take values {True,False}, set of operators {AND, OR, NOT}
- SAT = { ϕ| There is some assignment of variables to {True,False} such that ϕ = True}
- Cook-Levin Theorem
  SAT is NP-complete i.e SAT $\in$ NP, and $\forall L \in NP$, L $\leq SAT^p_m$
- SAT is in NP: Nondeterministically guess an assignment and check it
- CNF - Conjunctive Normal Form
  AND of OR of Literal
  Literal is a variable or its negation
  CNF-SAT = { ϕ| $\phi \in$ CNF and $\phi \in$ SAT }
- Let $L \in NP \Rightarrow$ there is some NTM M accepting L in time $|x|^k$ on input x
- Then there exists a Tableau (as described in an earlier lecture) consisting of a list of configurations such that if M(x) accepts, then
  - $conf_o$ is initial configuration
  - $conf_f$ is accepting configuration
  - Transition from $conf_i \rightarrow conf_{i+1}$ must be possible in one application of the δ function
- Total size of the Tableau is $\leq |x|^{2k}$
- Proof Idea: Use boolean variables to describe Tableau and use clauses to check syntax/semantics of Tableau
- conf = { State, Head position, contents of tape }

- States Q = $\{q_0, q_1, q_2, ...q_n\}$, Tape Symbols $\Gamma = \{\gamma_0, \gamma_1, ...\gamma_r\}$

- q_{ij} = TRUE if at time i, we are at state j
  h_{ik} = TRUE if at time i, head is at position k
  t_{ikl} = TRUE if at time i, k^{th} tape cell position contains γ_l
  $0 \leq i \leq |x|^k$, $0 \leq j \leq |Q|$, $1 \leq k \leq |x|^k$, $1 \leq l \leq |\Gamma|$

- Number of variables = O($|x|^{2k}$) $\rightarrow$ polynomial in $|x|$

- We need to ensure that we are in exactly one state at time i
  Atleast 1 state at time $i \rightarrow q_{i0} \bigvee q_{i1} \bigvee ... \bigvee q_{iQ}$
  Atmost 1 state $\rightarrow \neg(q_{ia} \bigwedge q_{ib})$ $\forall a \neq b$
  $\neg q_{ia} \bigvee \neg q_{ib}$, $|Q||Q-1||x|^k$ such clauses

- We need to add similar clauses to ensure that at every time i, the head is exactly in one location

- Also at every time, and at every location k, there is one symbol at tape cell k at time i

- Let t_{ikl} be a variable that is true it at time i, tape cell k contains γ_l
  We can describe the input with the use of above mentioned variables

- Describe initial config: $conf_0$ is initial
  q_{00}, h_{01} must be true

- Describe final config: $conf_f$ is accepting
  $q_{acc} = q_s \rightarrow q_{|x|^k S}$ must be TRUE

- Describe the transition in the tableau from index i to index *i+1*

  - If the head was in position k at time i, then at time i+1, head must be in either k-1 or k+1
    $h_{ik} \rightarrow h_{i+1,k+1} \bigvee h_{i+1,k-1}$
    This is equivalent to $\neg h_{ik} \bigvee h_{i+1,k+1} \bigvee h_{i+1,k-1}$
  - For all other tape cells, the contents must remain unchanged

    $(\neg h_{ik} \bigwedge t_{ikl}) \rightarrow t_{i+1,k,l}$
    which reduces to $h_{ik} \bigvee \neg t_{i,k,l} \bigvee t_{i+1,k,l}$

  - Describe the δ function. Instead of describing all the possible nondeterministic choices that are possible, we describe those transitions that are not possible. So if $\delta(q_a, \gamma_b) \neq (q_{a'}, \gamma_{b'}, R)$, we add the clause

$(q_{ia} \bigwedge h_{ik} \bigwedge t_{i,k,b}) \rightarrow \neg(q_{i+1,a'} \bigwedge t_{i+1,k,b'} \bigwedge h_{i+1,k+1})$ (This can also be written in CNF, since $p \rightarrow q = \neg p \bigvee q$

- We now have a method to describe any NTM M in CNF using a polynomial size reduction method.