The Determination of the Effective Radius of a Filamentary Source in the FDTD Mesh

G. Waldschmidt and A. Taflove, *Fellow, IEEE*

***Abstract*—This paper proposes a rigorous method for determining the effective radius, r_{eff}, of a single axial field component, E_z or H_z, in a two-dimensional (2-D) TM_z or TE_z FDTD grid, respectively. The method is based upon matching FDTD results for a filamentary field source with the analytical Green's function in two dimensions. We find that $r_{\text{eff}} \approx 0.2$ grid cells over a wide range of grid resolutions. Further, our findings vividly demonstrate the nondissipative nature of the Yee algorithm even for very coarse grid resolutions.**

***Index Terms*—Effective radius, FDTD, filamentary source.**

I. INTRODUCTION

TWO-DIMENSIONAL (2-D) TM_z or TE_z FDTD solvers often use a single axial field component (E_z or H_z, respectively) to source a radially outgoing wave. The effective radius, r_{eff}, of such a filamentary source has been subject to some conjecture. Knowledge of r_{eff} can be important in certain three-dimensional modeling problems, for example in calculating the driving-point impedance of an antenna comprised of such a filament, or in specifying the wire gauge that would yield the same fields as the filament.

Accurate, effective subcell models of thin wires [1] are used in FDTD grids to precisely mesh fine geometrical features which are a fraction of a grid cell in size. Such models have often been used in antenna modeling and design [2], [3]. However, this letter does not use a subcell model. Rather, a filamentary hard source is excited and its effective radius r_{eff} is determined. It is shown that $r_{\text{eff}} \approx 0.2h$ where h is the grid resolution.

In order to verify the accuracy of a given technique, the results are commonly compared to a known analytical solution. We discuss one of the most basic tests of this type for the Yee algorithm, namely the response of the space lattice to an impulsive source located at its center. We will focus on the frequency domain Green's function for the 2–D Yee grid, and the generation of comparable FDTD data.

II. GREEN'S FUNCTION

Maxwell's equations in two dimensions yield the following scalar wave equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{1}{c^2}\frac{\delta^2 u}{\delta t^2} \tag{1}$$

where c is the speed of light in the given medium. This equation is valid for either a TM_z or TE_z mode, where u is substituted by E_z or H_z, respectively. Given a Dirac delta function in time as the excitation pulse, the solution can be shown to be the following 2-D time-domain Green's function, G_{2D}

$$u(r,t) = G_{2D}(r,t) = \frac{cU(ct-r)}{2\pi\sqrt{c^2t^2 - r^2}} \tag{2}$$

where r is the radial distance from the source and the unit step function U is defined as

$$U(ct-r) = \begin{matrix} 1, r < ct \\ 0, r > ct \end{matrix}. \tag{3}$$

Note that $G_{2D}(r,t) \to \infty$ at the leading edge of the outgoing wave since $r = ct$. Any numerical solver is unable to model a wave with infinite amplitude. However in the frequency domain this is easily resolved. Therefore it is instructive to analyze the 2-D frequency-domain Green's function

$$u(r,\omega) = G_{2D}(r,\omega) = \frac{jH_0^{(2)}(kr)}{4} \tag{4}$$

where $H_0^{(2)}$ is the Hankel function of the second kind and k is the free space wave number at $\omega = 2\pi f$.

G_{2D} is also unbounded in the frequency domain, but only at $r = 0$ due to the singularity of the Hankel function. This, in fact, does not present a problem since the singularity is naturally avoided. A line source in the FDTD grid has been known to have some measurable radius, an effective radius, although it has never been quantified. The excitation field in the grid radiates from the edges of this filament just as an antenna in a physical system. Therefore, a finite distance exists from the geometrical center of the source to the location where the wave is actually radiating. As a result, the Hankel function is no longer infinite and is in fact easily handled by the FDTD mesh. Numerical experiments discussed below demonstrate that $G_{2D}(r,\omega)$ can be calculated very accurately by an FDTD model which excites a single E_z or H_z field.

III. METHODOLOGY FOR FINDING THE EFFECTIVE SOURCE RADIUS

FDTD frequency-domain data may be gathered by performing a discrete Fourier transform concurrently with time stepping. Alternatively, a time-harmonic source may be excited and the envelope of the response collected after the high frequency transients have dissipated. Using this latter approach, we need only a simple peak detector to acquire the envelope.

Manuscript received February 21, 2000; revised April 19, 2000.

The authors are with the Department of Electrical and Computer Engineering, Northwestern University, Evanston, IL 60208-3118 USA (e-mail: gwald@tesla.ece.nwu.edu).

Publisher Item Identifier S 1051-8207(00)05942-0.

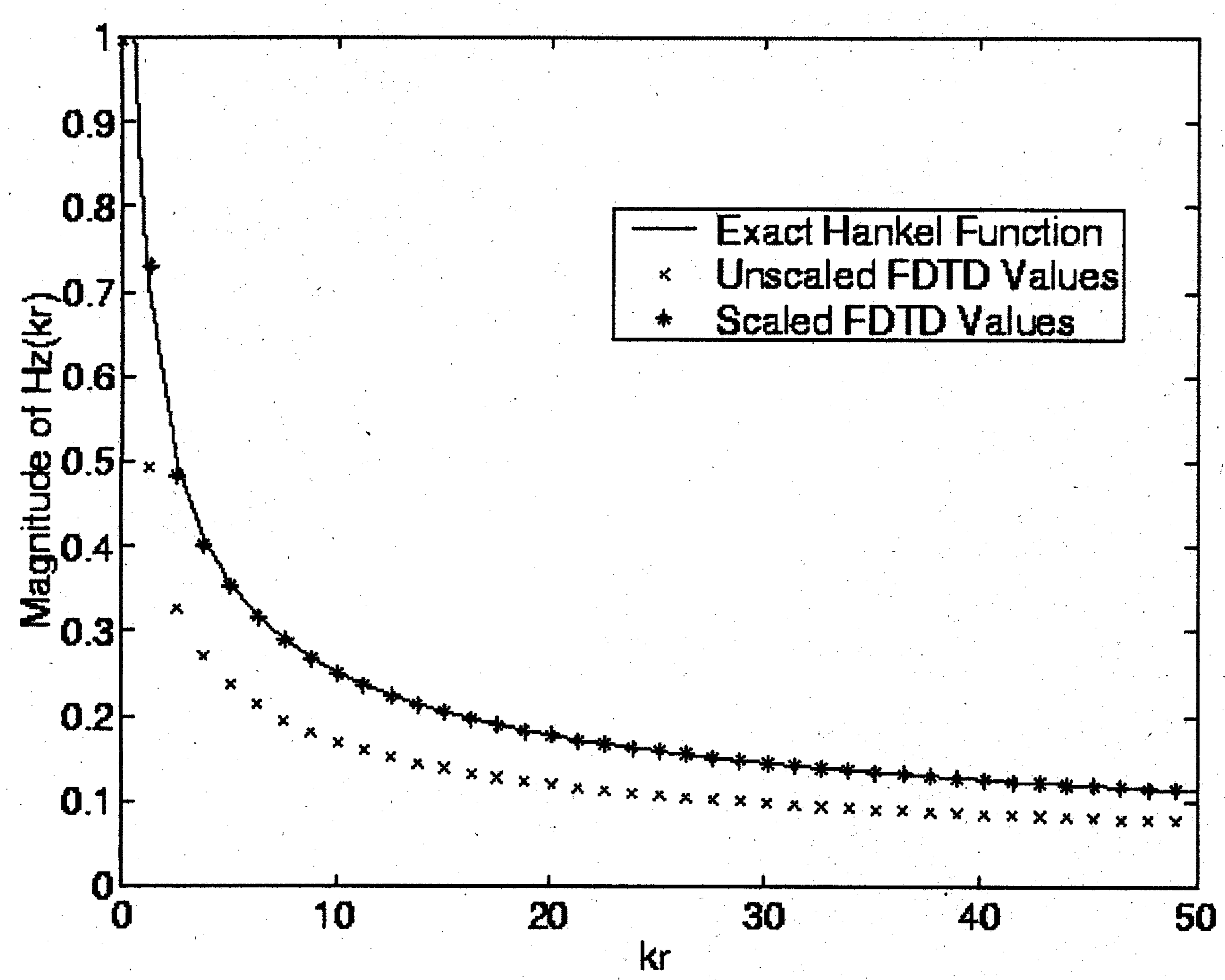

Fig. 1. $|H_z(kr)|$ with scaled and unscaled FDTD values along the x axis for $\lambda_0/5$ grid resolution.

Consider for example the hard-source excitation of exciting a single H_z field component at the center of a 2-D TE$_z$ grid. We assume that the equivalent radius of the excited filamentary source is given by $r_{\text{eff}} = f^*h$ where h is the grid-cell size and f^* is a decimal fraction that we call the effective source radius. A step-by-step process for determining f^* follows:

1) Run the FDTD simulation for a given grid resolution with a unity excitation amplitude of the hard-sourced H_z.
2) Graph the FDTD-calculated sinusoidal steady-state values of H_z versus radial distance from the source.
3) Apply a constant multiplying factor C to the graphed FDTD data to generate the best fit in the L_2 norm sense relative to the analytical frequency-domain Green's function.
4) By our basic assumption concerning the effective source radius of the excited H_z component we determine f^* from the following:

$$C = |G_{2D}(f^*h, \omega)| = \left| \frac{j}{4} H_0^{(2)}(kf^*h) \right|. \tag{5}$$

Upon substituting k, h, and C into (5), we obtain f^* by using a table of Hankel function values generated by MATLAB$^{\text{TM}}$.

IV. Numerical Results

Fig. 1 shows a comparison of the analytical Green's function with both raw and scaled FDTD results for $|H_z(kr)|$ along the x-axis of a coarse grid with resolution $\lambda_0/5$. The finite scaling factor to achieve a best fit between the exact and FDTD results is 1.252. From (5), this yields $f^* = 0.247$.

Fig. 2 is similar to Fig. 1 except that it compares the radial variation of the magnitude of the Green's function with scaled FDTD-calculated $|H_z(kr)|$ values as observed at *all* points within the grid. Our methodology is to scale the FDTD results so that the Green's function forms: (a) the upper envelope of the variation of FDTD values, and (b) the lower envelope of the same variation of values. This yields two results for the multiplying factor C, which by (5) results in two corresponding values of f^*. (In Fig. 2, we show for clarity only the scaled upper envelope of the FDTD values.) These two values bound the range of f^* observed for the particular grid resolution used.

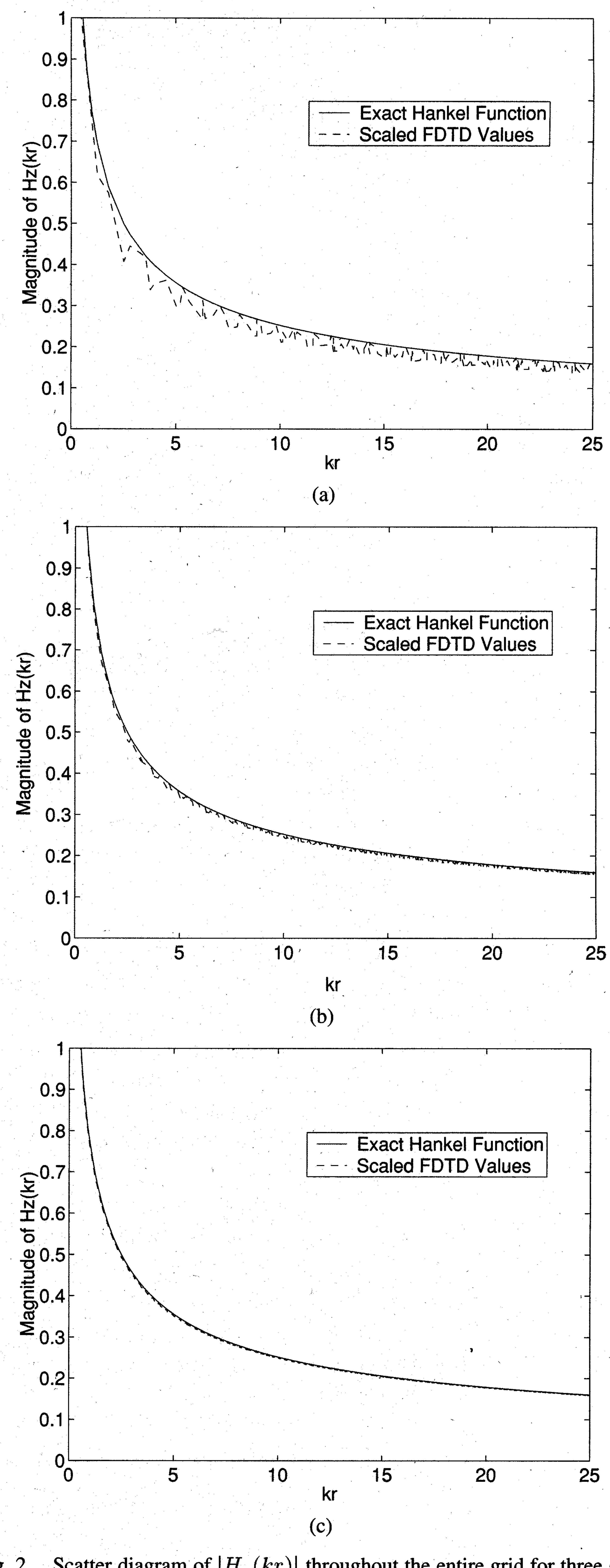

Fig. 2. Scatter diagram of $|H_z(kr)|$ throughout the entire grid for three grid resolutions: (a) $\lambda_0/5$; (b) $\lambda_0/10$; (c) $\lambda_0/20$.

For $\lambda_0/5$ resolution [Fig. 2(a)] there is a significant variation of the FDTD-calculated $|H_z(kr)|$ values throughout the grid which yields a large range of f^* between 0.149 and 0.247. As

TABLE I
EFFECTIVE FILAMENTARY SOURCE RADIUS, f^* (FRACTION OF ONE SPACE CELL) FOR VARIOUS FDTD GRID RESOLUTIONS

Resolution	*Range of* f^*
$\frac{\lambda_0}{5}$	0.149 – 0.247
$\frac{\lambda_0}{10}$	0.187 – 0.211
$\frac{\lambda_0}{20}$	0.200 – 0.207
$\frac{\lambda_0}{40}$	0.206 – 0.209

the resolution improves to $\lambda_0/10$ [Fig. 2(b)], the variation of the FDTD values decreases. This narrows the range of f^* to between 0.187 and 0.211. Refining the resolution further to $\lambda_0/20$ [Fig. 2(c)] causes the variation to markedly diminish, thereby tightening the range of f^* to between 0.200 and 0.207. Continuing in this manner converges the spread of FDTD-calculated $|H_z(kr)|$ values to the Green's function, yielding a converged value for f^* of approximately 0.21. Table I summarizes the convergence properties of f^* for grid resolutions between $\lambda_0/5$ and $\lambda_0/40$.

We see from Table I that $r_{\text{eff}} \approx 0.2$ grid cells over a wide range of resolutions. We make the following additional observation from Fig. 2. Although a variation of FDTD values of $|H_z(kr)|$ exists throughout the grid, for any given cut at a fixed azimuth angle ϕ the agreement between the Green's function and the FDTD values is excellent for an appropriate choice of C. While the coarse-grid FDTD data are known to have significant phase-velocity errors due to numerical dispersion, the amplitude-distribution data show no evidence of either dissipation or incorrect fall-off with r.

Note that the fluctuations observed in the FDTD data seen in Fig. 2 are due to the effectively noncircular shape of the source, i.e., r_{eff} is a function of ϕ, and to the dependence of numerical dispersion on ϕ [4]. As the resolution improves to $\lambda_0/20$, however, the numerical dispersion is greatly reduced and r_{eff} becomes nearly independent of ϕ.

V. CONCLUSION

A filamentary HARD source in a 2-D Yee grid has been shown to have an effective radius of approximately 0.2 grid cells for a wide range of commonly-used grid resolutions. The nondissipative nature of the Yee algorithm was indicated as part of this study.

REFERENCES

[1] A. Taflove, "Local subcell models of fine geometrical features," in *Computational Electrodynamics: The Finite-Difference Time-Domain Method*. Norwood, MA: Artech House, 1995, ch. 10.

[2] M. Douglas, M. Okoniewski, and M. A. Stuchly, "Accurate modeling of thin-wire antennas in the FDTD method," *Microwave Opt. Technol. Lett.*, vol. 21, pp. 261–265, 1999.

[3] S. Watanabe and M. Taki, "An improved FDTD model for the feeding gap of a thin-wire antenna," *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 152–154, 1998.

[4] A. Taflove and S. C. Hagness, "Numerical dispersion and stability," in *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 2nd ed. Norwood, MA: Artech House, 2000, ch. 4.