by an incident field other than a plane wave. However, the uniform high-frequency solution (1) is valid only for broadside plane-wave incidence.

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An Application of the WKJB Technique to the On-Surface Radiation Condition

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***Abstract*—The on-surface radiation condition (OSRC) method and the WKBJ method are used to derive an analytic formula for the surface currents on a two-dimensional perfectly conducting convex target. The currents are induced by an incident TE-polarized plane wave. The case of a circular cylinder is used to demonstrate the usefulness of the combined methods. It is shown that a two-term expansion yields good results for the surface currents and excellent results for the ensuing bistatic radar cross section.**

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I. INTRODUCTION

Recently, Kriegsmann *et al.* have introduced a new method for solving scattering problems for two-dimensional convex cylinders [1]. By applying a radiation boundary condition on the surface of the scatterer (OSRC) they obtained a simple analytic expression for the surface current when the incident wave was transverse magnetic (TM) polarized. When the incident wave was TE polarized, the method yielded an ordinary differential equation for the surface current. This equation contains variable coefficients which depend upon the geometry of the cylinder and the nature of the incident wave. In general, it cannot be solved exactly.

In this communication, we shall derive an approximate solution to this differential equation by using the WKBJ technique [2]. The motivation for such an approximation is twofold. First, it affords an accurate analytical approximation to the surface current without recourse to the numerical solution of a boundary value problem for arbitrary convex shapes. Secondly, and perhaps more importantly, a recent work by Jones [3] suggests that the approximate "surface current" for a three-dimensional convex acoustic target (hard) satisfies a second-order partial differential equation on the target's surface. We believe that a similar situation will occur when the OSRC method is extended to handle three-dimensional electromagnetic scattering problems. It seems plausible that the method presented herein could be extended to handle such situations.

The remainder of this work is organized in three additional sections. In Section II the scattering problem is formulated, and the OSRC method is used to derive an ordinary differential equation for the surface current. An approximate solution to this equation is deduced by the WKBJ method in Section III. Finally, in Section IV the results for a circular cylinder are presented.

II. FORMULATION

We shall consider a transverse electric plane wave illuminating a two-dimensional perfectly conducting convex cylinder. The incident wave, propagating at an angle α with respect to the $-x$ axis, is given by

$$\vec{H}_{\text{inc}} = U_{\text{inc}} e^{-j\omega t} \hat{z} \qquad U_{\text{inc}} = e^{jk(x \cos \alpha - y \sin \alpha)} \tag{1}$$

where the unit vector $\hat{z}$ is parallel to the axis of the cylinder. The parameter ω is the frequency of the incident wave, $k = \omega a/c$, c is the speed of light in free space, and a is a characteristic dimension of the cylinder's cross section. The variables x and y are the corresponding dimensionless Cartesian coordinates in the plane orthogonal to $\hat{z}$. They are scaled with respect to the length a.

The scattered magnetic field $\vec{H}_s$ is given by

$$\vec{H}_s(\vec{x}) = U(\vec{x}) e^{-j\omega t} \hat{z} \tag{2a}$$

$$U(\vec{x}) = \int_c J(\vec{x}) \frac{\partial G}{\partial \nu'} (\vec{x}, \vec{x}')\, ds' \tag{2b}$$

where C represents the boundary of the cylinder's cross section, $\partial/\partial\nu'$ denotes an outward normal derivative on C, and G is the free-space Green's function given by

$$G(\vec{x}\,|\,\vec{x}') = \frac{j}{4} H_0^{(1)}(kR) \tag{2c}$$

$$R=|\vec{x}-\vec{x}'|=\sqrt{(x-x')^2+(y-y')^2}. \tag{2d}$$

The vectors $\vec{x}$ and $\vec{x}'$ appearing above are just normalized (x, y) and (x', y'), respectively. The tangential surface current J appearing in (2b) is related to U_{inc} and U by

$$J(\vec{x})=-[U_{\text{inc}}(\vec{x})+U(\vec{x})]. \tag{3}$$

The tangential current is unknown, because U is not prescribed for the TE polarization. The OSRC method provides a means of generating an approximation to U in terms of known geometric quantities and U_{inc}. The motivation for this method, and its complete description are explained in [1]. Here, we present a single second-order approximation which is the one most often used in practice. It is

$$\frac{d^2}{ds^2}U+A(s)U=B(s)\frac{\partial U_{\text{inc}}}{\partial \nu} \tag{4a}$$

$$A(s)=2k^2+3jk\zeta(s)-\frac{3}{4}\zeta^2(s) \tag{4b}$$

$$B(s)=2j(k+j\zeta(s)) \tag{4c}$$

where $\zeta(s)$ is the curvature of the cylinder's surface at s and d^2/ds^2 is the second derivative with respect to the arclength of C. Combining (3), (4), and the definition of U_{inc} from (1), we find that J satisfies

$$\frac{d^2J}{ds^2}+A(s)J=-F(s)e^{jk\phi} \tag{5a}$$

$$F(s)=k^2\left\{2-\left(\frac{d\phi}{ds}\right)^2-2a(s)\right\}+jk\left\{\frac{d^2\phi}{ds^2}+3\zeta(s)-2a(s)\zeta(s)\right\}-\frac{3}{4}\zeta^2(s) \tag{5b}$$

$$a(s)=\hat{n}(s)\cdot(\cos\alpha,\ -\sin\alpha) \tag{5c}$$

$$\phi(s)=\vec{x}_0(s)\cdot(\cos\alpha,\ -\sin\alpha) \tag{5d}$$

where $\vec{x}_0(s)$ is the vector representation of the curve C. In addition to satisfying (5) J must also be periodic, i.e.,

$$J(s+L)=J(s),\ \frac{dJ}{ds}(s+L)=\frac{dJ}{ds}(s), \qquad 0\le s\le L \tag{6}$$

where L is the length of C. Thus the OSRC method has reduced the determination of the surface current to the problem of solving an ordinary second-order linear differential equation with variable coefficients and periodic boundary conditions.

We note here that the coefficient $A(s)$ in (5a) has a nonzero complex component. Thus the homogeneous solution of (5), (6), i.e., $F=0$ in (5a), has only the zero solution. From this we deduce that (5), (6) has a unique solution [4].

III. WKBJ Analysis

The actual computation of the surface current J which satisfies (5), (6) is impossible to perform analytically for an arbitrary convex cylinder. In general, it must be done numerically. However, it is quite easy to obtain a WKBJ approximation of J which yields an analytic formula.

According to this procedure we express J as

$$J(s)=V(s,k)e^{jk\phi(s)} \tag{7a}$$

where the amplitude $V(s, k)$ has the asymptotic expansion

$$V(s,k)\sim\sum_{n=0}^{\infty}V_n(s)k^{-n}. \tag{7b}$$

Since the phase $\phi(s)$ defined in (5d) satisfies (6), the function J will too, as long as the amplitudes $V_n(s)$ also satisfy these conditions. Inserting (7) into (5) and equating to zero the coefficients of the powers of k, we deduce an infinite number of algebraic equations which sequentially determine the $V_n(s)$. The first two amplitudes, which suffice for our purpose here, are given by

$$V_0(s)=-1+\frac{a(s)}{D(s)} \tag{8a}$$

$$V_1(s)=-j\left\{\frac{\ddot{\phi}+(\ddot{\phi}+3\zeta)V_0+2\dot{\phi}\dot{V}_0-\zeta(2a-3)}{2D}\right\} \tag{8b}$$

$$D=1-\frac{1}{2}\dot{\phi}^2 \tag{8c}$$

where the dots denote differential with respect to the arclength s. We note that the denominator D does not vanish because $\dot{\phi}$ is the projection of the unit tangent vector onto $(\cos\alpha,\ -\sin\alpha)$ and is thus less than one in modulus. We also observe that V_0 and V_1 satisfy (6) because the curvature $\zeta(s)$ and $\phi(s)$ are periodic functions.

Inserting the first two terms of (7b) into (7a) we formally deduce that

$$J\sim\left[V_0+\frac{1}{k}V_1+O(1/k^2)\right]e^{jk\phi(s)} \tag{9}$$

where $O(1/k^2)$ represents the remaining terms. This is the WKBJ approximation of the periodic solution of (5), (6).

The approximate surface current given by (9) can be inserted into (2) to determine the scattered field. This expression simplifies in the far field, $r \gg 1$, to

$$U\sim A(\Theta,k)\frac{e^{jkr}}{\sqrt{r}} \tag{10a}$$

$$A(\Theta,k)=-\sqrt{\frac{k}{8\pi}}\,e^{-j\pi/4}\int_C\left[V_0(s)+\frac{1}{k}V_1(s)\right]e^{j\psi(s)}\cos\delta(s)\,ds \tag{10b}$$

where $\psi(s)=-\vec{x}_0(s)\cdot\hat{r}+\phi(s)$, $\cos\delta=\hat{n}\cdot\hat{r}$, $\hat{n}$ is the unit normal of C at s, and $\hat{r}=(\cos\Theta,\ \sin\Theta)$ is the unit vector in the observation direction.

IV. Example: The Circular Cylinder

In this example C is a circle of unit radius so that $\zeta=1$ in all the preceding formulas. Without loss of generality, the angle α defined in (1) is set to zero in the subsequent equations. The exact boundary value problem for the Helmholtz equation can be solved exactly using a Fourier series representation. In Fig. 1 we have graphed the results predicted by (9) versus the Fourier series solution for the $k=5$ circular cylinder. Thirty terms were taken in the Fourier series to obtain an accurate answer. As can be seen in this diagram, the results given by (9) are quite close to the exact answer. Similarly, Fig. 2 shows our results for the $k=10$ circular cylinder. Here again 30 terms were used in the partial sum to insure accuracy. The agreement between (9) and the exact solution is even better than before: this is to be expected since the WKBJ method is a high-frequency approximation.

Fig. 3 compares the bistatic radar cross section predicted by (10b) for a $k=5$ circular cylinder verses the exact answer computed by a

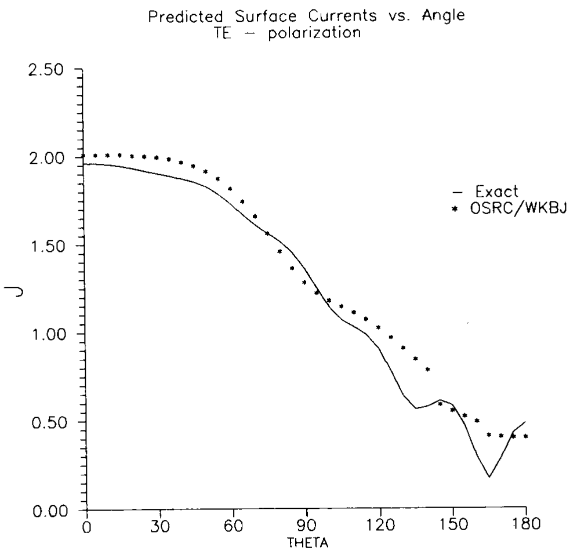

Fig. 1. OSRC predicted surface currents using a two-term asymptotic expansion for $k = 5$ circular cylinder.

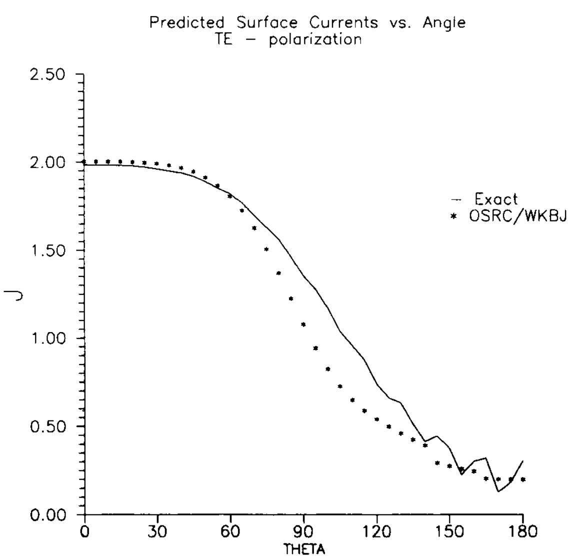

Fig. 2. OSRC predicted surface currents using a two-term asymptotic expansion for $k = 10$ circular cylinder.

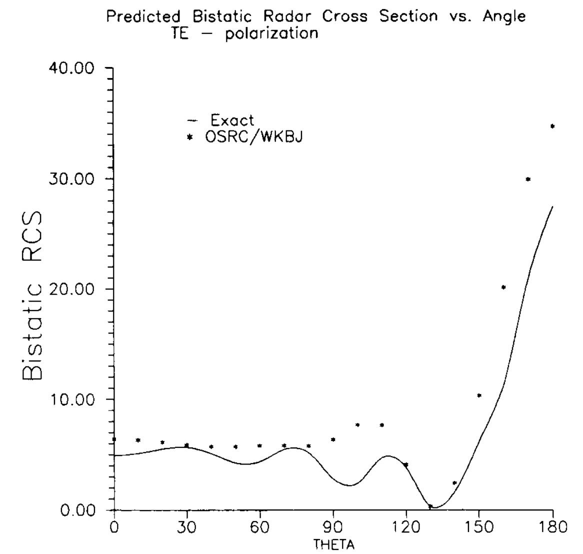

Fig. 3. Predicted RCS for $k = 5$ circular cylinder (RCS is scaled with respect to k).

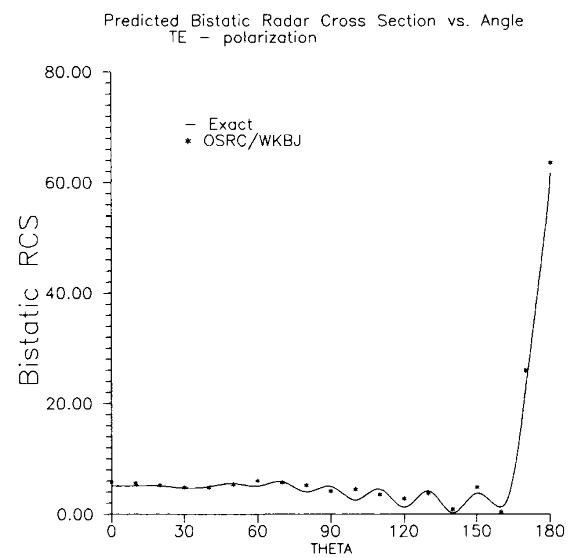

Fig. 4. Predicted RCS for $k = 10$ circular cylinder (RCS is scaled with respect to k).

Fourier series. Fig. 4 shows the bistatic radar cross section for a $k = 10$ circular cylinder. We can see that the agreement between the predicted and the exact RCS is very good over the entire range of angles, and as before, the error is even smaller for the larger cylinder. This is to be expected since the integration process tends to remove small errors introduced by the asymptotic expansion. The most significant errors are in the deep shadow where the phase of our approximate currents differs from the exact answer. This is not a deficiency in the WKBJ method but rather the OSRC approximation.

In conclusion we see that the asymptotic expansion (9) does a good job of estimating the surface current over a wide range of frequencies while even better agreements can be seen in the bistatic radar cross section results. Thus, for convex objects being illuminated by a TE polarized wave, the combination of the OSRC and WKBJ methods provides a powerful tool for analyzing scattering problems.

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