Uncapacitated Facility Location Problem with Quadratic Flow Costs

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- **Mixed Integer Programming (MIP) Problem:**

$$z \quad = \quad \min \left\{ cx + dy \ : \ (x, y) \in P \right\}$$

where $P = \left\{ (x, y) \in \mathbb{R}^{n1} \times \mathbb{Z}^{n2} \ : \ Ax + Cy \ \geq \ d \ \right\}$

The convex hull of P can be described with linear inequalities

$$z = \min \left\{ cx + dy \ : \ (x, y) \in \mathit{ConvHull}(P) \right\}$$

- **Solution method: Branch-and-cut**
  - *Start with the continuous relaxation of P*

$$P^{rel} = \left\{ (x, y) \in \mathbb{R}^{n1} \times \mathbb{R}^{n2} \ : \ Ax + Cy \ \geq \ d \ \right\}$$

  - *Strengthen it with valid inequalities (cutting)*
  - *Apply enumeration (branching)*

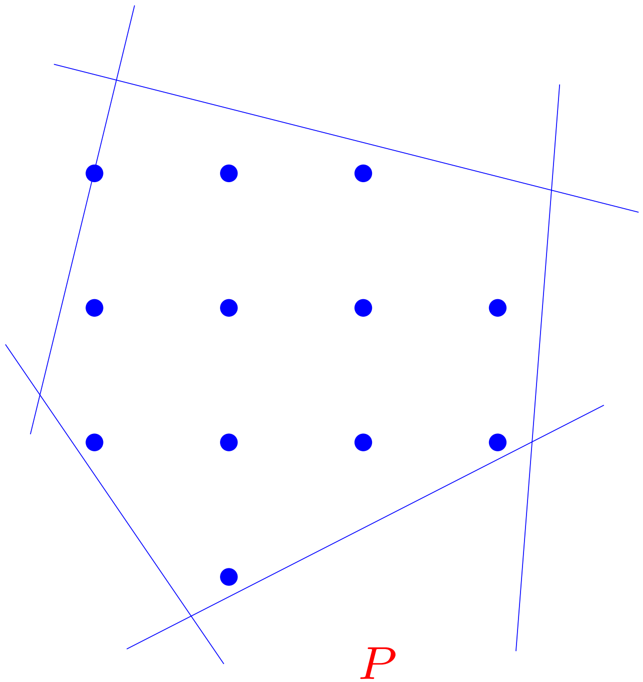
Intro – continued

- **Mildly non-linear MIP Problem:**

$$z = \min \left\{ f(x,y) \; : \; (x,y) \in P \right\}$$

where P is a mixed-integer set and f is a nice convex function

$$z \neq \min \left\{ f(x,y) \; : \; (x,y) \in \textit{ConvHull}(P) \right\}$$



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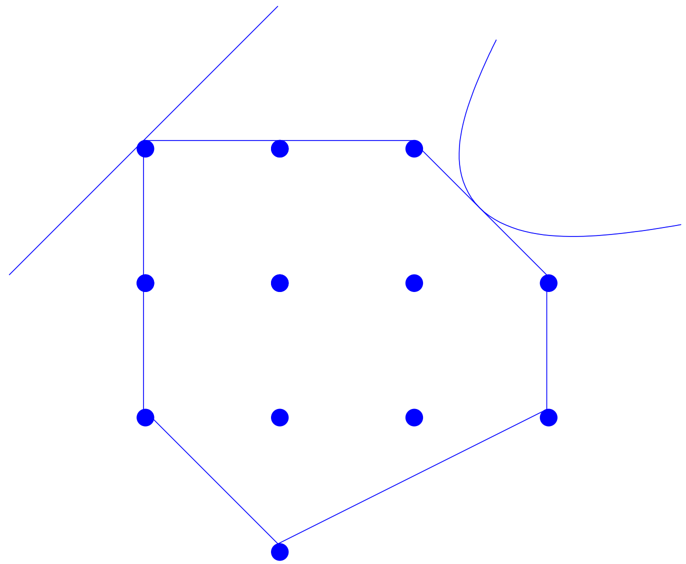


ConvHull(P)

Intro – continued

- **Computationally:**
  - *A well known MIP with linear constraints*
  - *Objective function linear or convex quadratic*
  - *Using branch-and-bound*

	Linear objective		Convex objective	
	nodes	time	nodes	time
weak relaxation	10,616	332.24	45,901	16,697.46
strong relaxation	0	0.17	29,277	21,206.56

Mildly non-linear MIP Problems– feasibility cuts

- *Reformulate the problem*

$$\begin{aligned} z &= \min \left\{ f(x,y) \;:\; (x,y) \in P \right\} \\ &= \min \left\{ \phi \;:\; \phi \geq f(x,y),\; (x,y) \in P \right\} \\ &= \min \left\{ \phi \;:\; (x,y,\phi) \in ConvHull(P') \right\} \end{aligned}$$

where

$$P' = \left\{ (\phi, x, y) \in \mathbb{R} \times \mathbb{R}^{n1} \times \mathbb{Z}^{n2} \;:\; Ax + Cy \;\geq\; d,\; \phi \geq f(x,y) \;\right\}$$

- $ConvHull(P')$ *is not a polyhedral set*
- *Find valid inequalities* $g(\phi, x, y) \leq 0$ *that are*
  - *Satisfied by all points in* P',
  - *but violated by some in the continuous relaxation of* P'

- *Inequalities using optimality conditions (not valid for all feasible points)*
- *Let*

$$
\begin{aligned}
P_y &= \left\{ y \in \mathbb{Z}^{n2} \; : \; \exists (x,y) \in P \right\} \\
\phi(y) &= \min \left\{ f(x,y) \; : \; (x,y) \in P \right\} \\
\gamma(y) &= \left\{ x \in \mathbb{R}^{n1} \; : \; (x,y) \in P, \;\; f(x,y) = \phi(y) \right\}
\end{aligned}
$$

- *Inequality* $g(\phi, x, y) \leq 0$ *is an optimality cut if*

$$g\left(\phi(\bar{y}), \gamma(\bar{y}), \bar{y}\right) \;\leq\; 0$$

for all $\bar{y} \in P_y$.

$$
\begin{aligned}
z &= \min \left\{ \phi \; : \; (x,y,\phi) \in ConvHull(P^*) \right\} \\
&= \min \left\{ \phi \; : \; y \in P_y, \;\; \phi = \phi(y), \;\; x = \gamma(y) \right\}
\end{aligned}
$$

Uncapacitated Facility Location Problem

- **Given**
  - *Facilities with fixed set-up cost*
  - *Warehouses (customers) with given demand*
  - *Linear delivery cost between each facility and warehouse*
- **Choose** *a subset of facilities to open, and* **supply** *the warehouses from open facilities.*

$$z = \min \left\{ \sum_{i \in F} c_i y_i + \sum_{i \in F} \sum_{j \in C} q_{ij} x_{ij} \; : \; (y, x) \in \mathit{UFL}(C, F) \right\}$$

where

$$\mathit{UFL}(C,F) = \Big\{ (y,x) \in \{0,1\}^{|F|} \times \mathbb{R}^{|C||F|} \; : \; \begin{aligned} \sum_{i \in F} x_{ij} &= 1 && \forall j \in C \\ x_{ij} &\leq y_i && \forall i \in F, j \in C \end{aligned} \Big\}$$

A variation on the cost function

Consider non-linear costs on flows

$$z = \min \left\{ \sum_{i \in F} c_i y_i + \sum_{i \in F} \sum_{j \in C} q_{ij} (x_{ij})^d \; : \; (y, x) \in \textit{UFL}(C, F) \right\}$$

where $d > 1$ *measures the level of risk-averseness.*

We first rewrite the problem as

$$\begin{aligned} \min \quad & \sum_{i \in F} c_i y_i + \sum_{i \in C} \phi_i \\ \textit{subject to} \quad & \phi_i - \sum_{i \in F} q_{ij} (x_{ij})^d \geq 0 \quad \forall i \in C \\ & \sum_{i \in F} x_{ij} = 1 \quad \forall i \in C \\ & 0 \leq x_{ij} \leq y_i \quad \forall i \in F, j \in C \\ & y_i \in \{0, 1\} \quad \forall i \in F \end{aligned}$$

Basic properties of optimal solutions

- *For any given* $\bar{y} \in \{0,1\}^{|F|}$, $\bar{y} \neq 0$, *let* $\bar{F} = \{i \in F \ : \ \bar{y}_i = 1\}$.

- *The problem decomposes for each customer* $j \in C$ *to*

$$\phi_j = \min \sum_{i \in \bar{F}} q_{ij}(x_{ij})^d$$

$$\text{subject to} \quad \sum_{i \in \bar{F}} x_{ij} = 1$$

$$0 \leq x_{ij} \leq 1, \quad \forall i \in \bar{F}$$

with optimal solution:

$$d\, q_{ij}\, (x^*_{ij})^{d-1} = \lambda \quad \Rightarrow \quad x^*_{ij} \sim \sqrt[d-1]{1/q_{ij}} \quad \forall i \in \bar{F}$$

and optimal value $z^* = \sum_{i \in \bar{F}} c_i y_i + \sum_{i \in C} \phi^*_j$

Basic properties of optimal solutions – II

As optimal flows satisfy

$$x^*_{ij} = \frac{{q_{ij}}^{-1/d-1}}{\sum_{l \in F} {q_{lj}}^{-1/d-1}\, y^*_l}\, y^*_i$$

for all $i \in F$ $j \in C$. *For a fixed customer* $j \in C$ *we have*

$$\phi^*_j \;=\; \sum_{i \in F} q_{ij} (x^*_{ij})^d = \frac{\sum_{i \in F} q_{ij}\; {q_{ij}}^{-d/d-1}\; y^*_i}{\left(\sum_{l \in F} {q_{lj}}^{-1/d-1}\; y^*_l\right)^d} \;=\; \left(\sum_{l \in F} {q_{lj}}^{-1/d-1}\; y^*_l\right)^{-(d-1)}$$

Note $(y^*_l)^d = y^*_l$ *as* $y_l \in \{0, 1\}$.

When $d = 2$ *we have*

$$\phi^*_j \;=\; \frac{1}{\sum_{l \in F} \frac{1}{q_{lj}}\, y^*_l} \qquad \textit{and} \qquad x^*_{ij} \;=\; \frac{1}{q_{ij}}\, \phi^*_j\; y^*_i$$

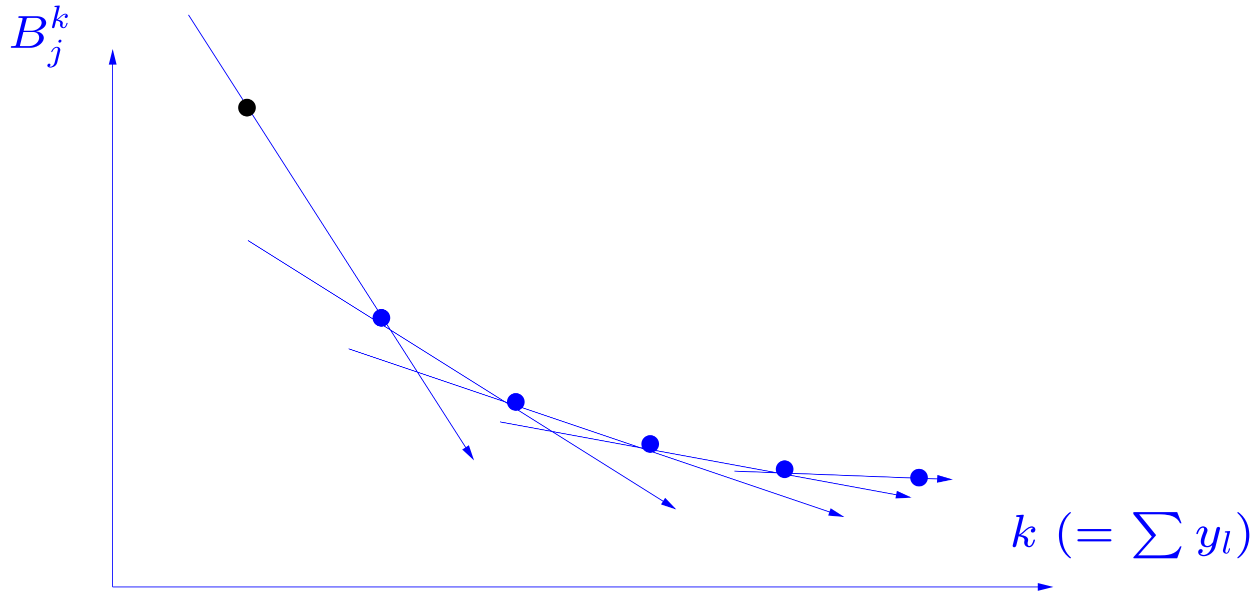
Bounds on flow costs

- *Assume* $\sum y_l^* = k$ *for some* $k \in \mathbb{Z}$ *and let* $j \in C$*.*

  As

$$\phi_j^* = \frac{1}{\sum_{l \in F} \frac{1}{q_{lj}} y_l^*} \quad \text{we have} \quad \phi_j^* \geq \frac{1}{\sum_{l \in F^k} \frac{1}{q_{lj}}} = B_j^k$$

- *Furthermore,* B_j^k *is convex in* k *for all* j *(i.e.* $B_j^k + B_j^{k+2} \geq 2B_j^{k+1}$*)*



Bounds on flow costs - II

Lemma : *For* $j \in C$ *and* $k = 1, 2, \ldots, |F| - 1$, *the* customer-cost lower bound

$$\phi_j \geq B_j^k \left(k + 1 - \sum_{l \in F} y_l \right) + B_j^{k+1} \left(-k + \sum_{l \in F} y_l \right)$$

is valid for all feasible solutions.

Proof :

- *If* $\sum_{l \in F} y_l = k$ *we have* $\phi_j \geq B_j^k$,
- *If* $\sum_{l \in F} y_l = k + 1$ *we have* $\phi_j \geq B_j^{k+1}$,
- *If* $\sum_{l \in F} y_l = t \notin \{k, k+1\}$ *then by convexity,* $B_j^t \geq RHS$

Variable upper bounds on flow variables

- *Remember that for $i \in F$ and $j \in C$,*

$$x^*_{ij} \; = \; \frac{1}{q_{ij}} \phi^*_j \, y^*_i$$

holds for all optimal solutions and $y^*_i \leq 1$.

Lemma : *For for $i \in F$ and $j \in C$, the* flow lower bound

$$q_{ij} \, x_{ij} \;\; \leq \;\; \phi_j \;\; - \;\; (1 - y_i) LB_{ij}$$

is valid for all **optimal** *solutions (LB_{ij} is a lower bound on ϕ_j when $y_i = 0$)*

Not valid for all feasible solutions. Consider slightly increasing an optimal flow $x^*_{ij} > 0$

$$q_{ij} x^*_{ij} = \phi^*_j = \sum_{i \in F} q_{ij} (x^*_{ij})^2$$

More bounds on flow variables

- *Assume* $\sum y_l^* = k$ *for some* $k \in \mathbb{Z}$ *and let* $j \in C$.
- *As* $q_{ij}\, x_{ij}^* \;=\; \phi_j^*\; y_i^*$ *holds for all optimal solutions, and*

$$\phi_j^* \;\geq\; L_{ij}^k \;=\; \frac{1}{1/q_i^j + \sum_{l \in F_i^{k-1}} 1/q_l^j}$$

when $y_i = 1$ *and* $\sum y_l = k$, *we have*

$$q_{ij}\, x_{ij} \geq L_{ij}^k y_i$$

Lemma : L_{ij}^k *is convex in* k *for all* j *and the* customer-cost lower bound

$$\phi_j \geq L_{ij}^k \left(k + 1 - \sum_{l \in F} y_l \right) + L_{ij}^{k+1} \left(-k + \sum_{l \in F} y_l \right)$$

is valid for all feasible solutions.

More bounds on flow costs

- *Assume* $\sum y_l^* = k$ *for some* $k \in \mathbb{Z}$ *and let* $\bar{C} \subseteq C$*.*
- *As*

$$\frac{1}{\phi_j^*} = \sum_{l \in F} \frac{1}{q_{lj}} y_l^* \qquad \text{we have} \qquad \sum_{j \in \bar{C}} \frac{1}{\phi_j^*} = \sum_{l \in F} \left(\sum_{j \in \bar{C}} \frac{1}{q_{lj}} \right) y_l^* \leq B^k(\bar{C})$$

- *Furthermore,* $1/B^k(\bar{C})$ *is convex in* k *for all* $\bar{C} \subseteq C$*.*

Lemma : *For* $\bar{C} \subseteq C$ *and* $k = 1, 2, \ldots, |F| - 1$, *the* customer-cost lower bound

$$\frac{1}{\sum_{j \in \bar{C}} 1/\phi_j} \geq \frac{1}{B^k(\bar{C})} \left(k + 1 - \sum_{l \in F} y_l \right) + \frac{1}{B^{k+1}(\bar{C})} \left(-k + \sum_{l \in F} y_l \right)$$

is valid for all feasible solutions.

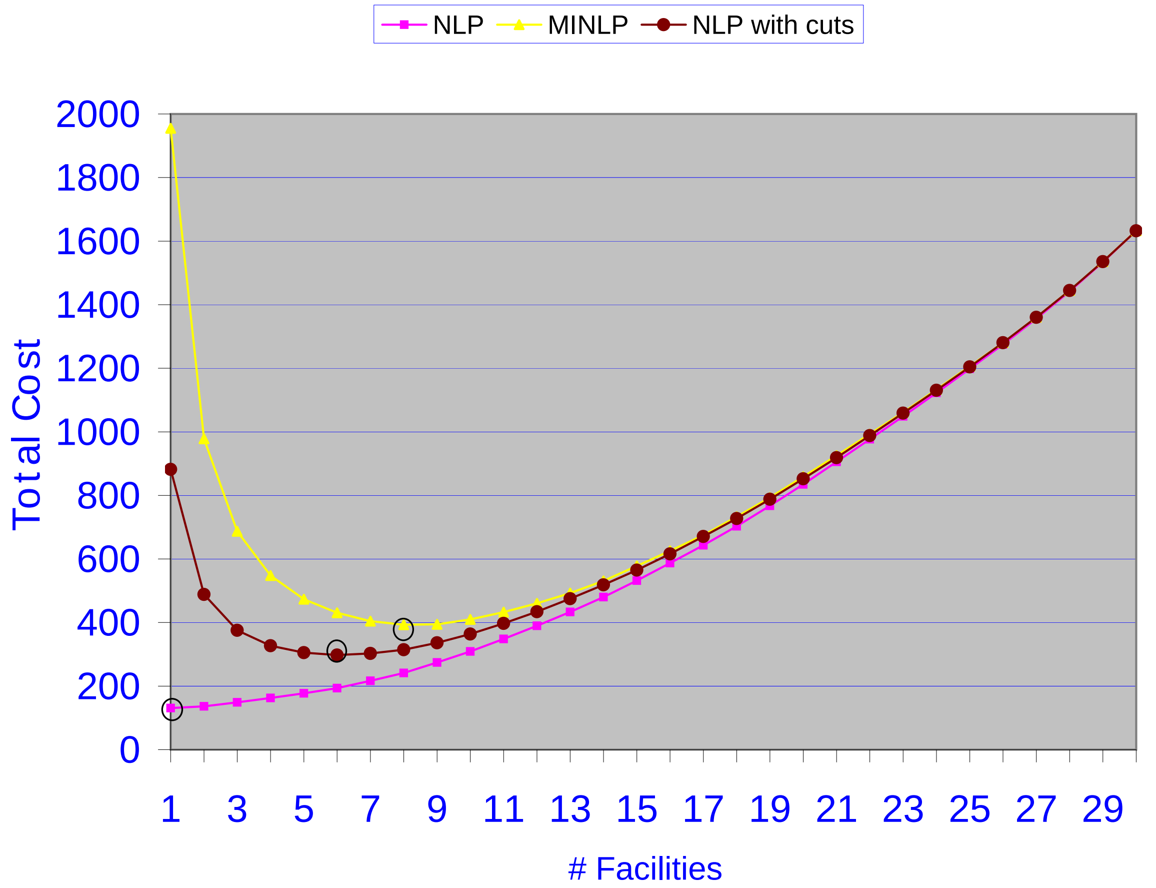
Computational results

- *Test data is generated randomly*
  - *Customers and candidate facilities thrown on the unit square*
  - *Flow cost is the Euclidian distance*
  - *Fixed cost of opening a facility is generated uniformly*

m,n	No cuts	Basic cuts	+ VUB cuts	+ customer-pairs	Opt
10,30	140.5	286.8	296.0	318.8 (5)	348.7
15,50	141.2	261.4	280.1	302.5 (14)	384.0
20,65	122.5	206.9	223.1	238.5 (29)	289.3
25,80	121.3	201.4	220.4	244.6 (36)	315.8
30,100	128.0	248.0	265.1	297.5 (62)	393.1

Summary

- *Linear/non-linear/feasibility/optimality cuts help tighten the continuous relaxation*



Thank you.