New constraint qualifications in convex optimization

M.A. Goberna

Dep. of Statistics and Oper. Res., Univ. of Alicante (Spain)

(joint work with N. Dinh, V. Jeyakumar and M.A. López)

BASIC REFERENCES

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LINEARIZING CONVEX SYSTEMS

We consider

$$\sigma := \{f_t(x) \leq 0, t \in T;\ x \in C\},$$

where

♦ T is an arbitrary (possibly infinite) index set;

♦ C is a nonempty closed convex subset of a Hausdorff t.v.s. X; and

♦ $f_t : X \to \mathbb{R} \cup \{+\infty\}$ is a proper l.s.c. convex function, $\forall\, t \in T$.

<u>The basic tool:</u> the *conjugate* of $h : X \to \overline{\mathbb{R}}$, i.e.,

$$h^*(u) = \sup\{u(x) - h(x) \mid x \in \operatorname{dom} h\}$$

Given $t \in T$,
$f_t(x) \le 0$

$$\iff f_t^{**}(x) \le 0$$
$$\iff u(x) - f_t^*(u) \le 0,\ \forall u \in \mathrm{dom} f_t^*$$
$$\iff u(x) \le f_t^*(u),\ \forall u \in \mathrm{dom} f_t^*$$
$$\iff u(x) \le f_t^*(u) + \alpha,$$
$$\forall u \in \mathrm{dom} f_t^*,\ \ \forall \alpha \in \mathbb{R}_+$$

Analogously,
$x \in C \Leftrightarrow \delta_C(x) \le 0$

$$\iff u(x) \le \delta_C^*(u),\ \forall u \in \mathrm{dom}\delta_C^*$$
$$\iff u(x) \le \delta_C^*(u) + \beta,$$
$$\forall u \in \mathrm{dom}\delta_C^*\ ,\ \forall \beta \in \mathbb{R}_+$$

Consequently, the following linear systems are equivalent to σ :

$$\left\{ \begin{array}{l} u(x) \leq f_t^*(u),\ u \in \mathrm{dom} f_t^*,\ t \in T \\ u(x) \leq \delta_C^*(u),\ u \in \mathrm{dom}\delta_C^* \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} u(x) \leq f_t^*(u) + \alpha, u \in \mathrm{dom} f_t^*, t \in T, \alpha \in \mathbb{R}_+ \\ u(x) \leq \delta_C^*(u) + \beta, u \in \mathrm{dom}\delta_C^*, \beta \in \mathbb{R}_+ \end{array} \right\}$$

We associate with σ the convex cones

$$N = \text{cone}\left\{\bigcup_{t\in T} \text{gph}\, f_t^* \cup \text{gph}\, \delta_C^*\right\}$$
$$K = \text{cone}\left\{\bigcup_{t\in T} \text{epi}\, f_t^* \cup \text{epi}\delta_C^*\right\}$$
$$P = \text{cone}\left\{\bigcup_{t\in T} \text{epi}\, f_t^* + \text{epi}\delta_C^*\right\}$$

Introduction

N and K: Charnes, Cooper & Kortanek (1965), in LSIP.

K: Chu (1966), in LISs.

P: Jeyakumar, Dinh & Lee (2004), in CP.

From the existence theorems for linear systems (Chu, 1966; Fan, 1968) we get that σ is consistent iff

$$(0,-1) \notin \operatorname{cl} K\left(\operatorname{cl} N, \operatorname{cl} P\right)$$

K is weak*-closed if either

♦ N is weak*-closed and σ is consistent (DGL, 2006)

or

♦ P is weak*-closed (DGL, 2006) or

♦ σ satisfies some interior type regularity condition (JDL, 2004)

FARKAS LEMMA CONVEX-CONVEX

From now on we assume that σ is consistent with solution set $A \neq \emptyset$.

Farkas Lemma linear - linear **(**Chu, 1966)**:** given $v \in X^*$, $v(x) \leq \alpha$ cons. of $\{a_s(x) \leq b_s,\ s \in S\}$ (consistent, with $a_s \in X^* \, \forall s \in S$) $\Leftrightarrow (v, \alpha) \in \operatorname{cl} \operatorname{cone} \{(a_s, b_s),\ s \in S;\ (0,1)\}$

Farkas Lemma linear - convex: given $v \in X^*$ and $\alpha \in \mathbb{R}$, $v(x) \leq \alpha$ cons. of σ

$\Leftrightarrow v(x) \leq \alpha$ cons. of

$$\left\{ \begin{array}{l} u(x) \leq f_t^*(u) + \alpha, u \in \operatorname{dom} f_t^*, t \in T, \alpha \in \mathbb{R}_+ \\ u(x) \leq \delta_C^*(u) + \beta, u \in \operatorname{dom} \delta_C^*,\ \beta \in \mathbb{R}_+ \end{array} \right\}$$

$$\Leftrightarrow (v, \alpha) \in \operatorname{cl} K$$

From now on $f : X \to \mathbb{R} \cup \{+\infty\}$ denotes a proper l.s.c. convex function.

Since $(f-\alpha)^*(u) = f^*(u) + \alpha$ and $\operatorname{dom}(f-\alpha)^* = \operatorname{dom} f^*$, $\{(f-\alpha)(x) \leq 0\}$ is equivalent to $\{u(x) \leq f^*(u) + \alpha + \beta, u \in \operatorname{dom} f^*, \beta \in \mathbb{R}_+\}$

Farkas Lemma convex - convex (DGL, 2006): $f(x) \leq \alpha$ cons. of σ

$\Leftrightarrow$ $\{(f-\alpha)(x) \leq 0\}$ cons. of σ

$\Leftrightarrow$ $u(x) \leq f^*(u) + \alpha + \beta$ cons. of σ $\forall u \in \operatorname{dom} f^*$ $\forall \beta \in \mathbb{R}_+$

$$\Leftrightarrow (0, \alpha) + \operatorname{epi} f^* \subset \operatorname{cl} K$$

FARKAS-MINKOWSKI SYSTEMS

♦ σ is *FM* if K is weak*-closed.

Literature: (ChCK, 1965), (JDL, 2004), (DGL, 2006), and Boţ & Wanka (2006).

Example: $\{\delta_A(x) \leq 0\}$ is a FM representation of A.

If σ is FM, then every continuous linear cons. of σ is also cons. of a finite subsystem of σ (DGL, 2006).

The converse statement fails.

Example: Let $X = C = \mathbb{R}^n$ and $\sigma = \left\{ f_1(x) := \frac{1}{2}\|x\|^2 \leq 0 \right\}$.

Since $f_1^*(v) = \frac{1}{2}\|v\|^2$, $K = (\mathbb{R}^n \times \mathbb{R}_{++}) \cup \{0\}$ is not closed.

Non-asymptotic Farkas Lemma linear - convex (DGL, 2006)**:** let σ be FM and $v \in X^{*} \diagdown \{0\}$. Then:

(i) $v\left(x\right) \geq \alpha$ is cons. of σ

$\Updownarrow$

(ii) $-(v, \alpha) \in K$

$\Updownarrow$

(iii) $\exists \overline{\lambda} \in \mathbb{R}_{+}^{(T)}$ such that

$$v(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq \alpha, \ \forall x \in C$$

Non-asymptotic Farkas Lemma convex - convex (DGL, 2006)**:** if σ is FM, then $f(x) \leq \alpha$ is cons. of σ

$$\Leftrightarrow (0, \alpha) + \mathrm{epi} f^* \subset K$$

Motivating the next result: a is a minimizer of f over A

$\Leftrightarrow f(x) \geq f(a) \; \forall x \in A$

$\Leftrightarrow f(x) \geq f(a)$ is a cons. of σ.

Asymptotic Farkas Lemma reverse-convex - convex (DGLS, 2007)**:** If σ is FM, then $f(x) \geq \alpha$ is cons. of σ

$$\Leftrightarrow (0, -\alpha) \in \mathrm{cl}\left(\mathrm{epi} f^* + K\right)$$

The following *closedness condition* was introduced by Burachik & Jeyakumar (2005):

(CC)

$\operatorname{epi} f^* + \operatorname{cl} K$ is weak*-closed.

Each of the following conditions implies **(CC)**:

♦ $\operatorname{epi} f^* + K$ is weak*-closed.

♦ σ is FM and f is linear.

♦ σ is FM and f is continuous at some point of A.

Non-asymptotic Farkas Lemma reverse-convex - convex (DGLS, 2007)**:** if σ is FM and **(CC)** holds, then

(i) $f(x) \geq \alpha$ is consequence of σ

$\Updownarrow$

(ii) $(0, -\alpha) \in \operatorname{epi} f^* + K$

$\Updownarrow$

(iii) $\exists \overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq \alpha, \; \forall x \in C$$

<u>Precedents:</u> Gwinner (1987) and Dinh, Jeyakumar & Lee (2005) under strong assumptions.

FM SYSTEMS IN CONVEX OPTIMIZATION

Consider the CP problem

$$\begin{aligned} \text{(P)}\ \ \text{Min}\ \ & f(x) \\ \text{s.t.}\ \ & f_t(x) \leq 0,\ t \in T, \\ & x \in C. \end{aligned}$$

Two basic concepts in optimality:

The *normal cone* of $D \subset X$ at $x \in D$ is

$$N_D(x) := \{u \in X^* \mid u(y-x) \leq 0 \,\forall\, y \in D\}$$

The *subdifferential* of h at $x \in \text{dom} h$ is

$$\partial h(x) = \{u \in X^* \mid h(y) \geq h(x) + u(y-x) \ \ \forall y \in X\}$$

KKT optimality theorem (DGLS, 2007)**:** assume that σ is FM, that **(CC)** holds, and let $a \in A \cap \operatorname{dom} f$. Then a is a minimizer of (P) iff $\exists \lambda \in \mathbb{R}_+^{(T)}$ such that

(i) $\partial f_t(a) \neq \emptyset \; \forall t \in \operatorname{supp} \lambda$

(ii) $\lambda_t f_t(a) = 0, \; \forall t \in T$, and

(iii) $0 \in \partial f(a) + \sum_{t \in T} \lambda_t \partial f_t(a) + N_C(a)$

Precedent: without the FM property, the optimality condition is $0 \in \partial f(a) + N_A(a)$ (BJ, 2005).

Following Rockafellar (1970) and Laurent (1972), we define

$$
\begin{array}{lll}
(\mathrm{P}_u) & \mathrm{Min} & f(x) \\
 & \text{s.t.} & f_t(x) \leq u_t,\ t \in T, \\
 & & x \in C,
\end{array}
$$

with feasible set A_u, $u \in \mathbb{R}^T$. Defining $\psi(x,u) := f(x) + \delta_{A_u}(x)$, we reformulate

$$(\mathrm{P}_u)\ \mathrm{Min}\ \psi(x,u), x \in X,$$

so that

$$(\mathrm{P}) \equiv (\mathrm{P}_0)\ \mathrm{Min}\ \psi(x,0), x \in X.$$

Since $\psi(x,0)+\psi^*(0,\lambda)\geq 0$, we have $\mathrm{v(D)}\leq \mathrm{v(P)}$ for

$$\mathrm{(D)}\ \ \mathrm{Max}\ -\psi^*(0,\lambda), \lambda\in\mathbb{R}_+^{(T)}.$$

By the non-asymptotic Farkas Lemma, if $\mathrm{v(P)}\in\mathbb{R}$, $\exists\overline{\lambda}\in\mathbb{R}_+^{(T)}$ such that

$$L(x,\overline{\lambda}) := f(x)+\sum_{t\in T}\overline{\lambda}_t f_t(x)\geq \mathrm{v(P)},\ \forall x\in C.$$

Since

$$\inf_{x\in C} L(x,\overline{\lambda}) = -\psi^*(0,\overline{\lambda})\leq \mathrm{v(D)},$$

we get the

Duality theorem (DGLS, 2007)**:** if $\mathrm{(P)}$ is bounded, σ is FM, and **(CC)** holds, then $\mathrm{v(D)}=\mathrm{v(P)}$ and $\mathrm{(D)}$ is solvable.

Precedents: Rockafellar (1974) and Bonnans & Shapiro (2000).

Lagrange optimality theorem (DGLS, 2007)**:** suppose that σ is FM and that **(CC)** holds. Then a point $a \in A$ is minimizer of (P) iff $\exists\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(a, \bar{\lambda})$ is a *saddle point* of the Lagrangian function L, i.e.,

$$L(a,\lambda) \le L(a,\bar{\lambda}) \le L(x,\bar{\lambda}),$$
$$\forall\lambda \in \mathbb{R}_+^{(T)}\ \forall x \in C.$$

In that case $\bar{\lambda}$ is a maximizer of (D).

Denote by $v\,(u)$ the value of (P_u), so that $\mathrm{v(P)} = v(0)$. The following stability concepts (Laurent, 1972) involve the *value function* $v : \mathbb{R}^T \to \overline{\mathbb{R}}$, whose directional derivative at 0 in the direction u is denoted by $v'(0, u)$.

(P) is called:

♦ *inf-stable* if $v(0) \in \mathbb{R}$ and v is l.s.c. at 0.

♦ *inf-dif-stable* if $v(0) \in \mathbb{R}$ and $\exists \lambda_0 \in \mathbb{R}^{(T)}$ such that

$$v'(0, u) \geq \lambda_0(u),\ \forall u \in \mathbb{R}^T.$$

(P) inf-dif-stable $\Longrightarrow$ (P) inf-stable because:

(P) inf-dif-stable

$\Updownarrow$

$\partial v(0) \neq \emptyset$ (called *calmness* in Clarke, 1976)

$\Updownarrow$

$\mathrm{v}(\mathrm{D}) = \mathrm{v}(\mathrm{P})$ and (D) is solvable, and

(P) inf-stable

$\Updownarrow$

$\mathrm{v}(\mathrm{D}) = \mathrm{v}(\mathrm{P}) \in \mathbb{R}$ (called *normality* in Zălinescu, 2002)

Stability Theorem (DGLS, 2007)**:** if (P) is bounded, σ is FM, and **(CC)** holds, then (P) is inf-dif-stable.

CHARACTERIZING THE FM SYSTEMS

Theorem (GJL, 2007)**:** if each f_t is continuous at least at one point of A, then the following statements are equivalent:

(i) σ is FM.

(ii) For every f such that $A \cap \operatorname{dom} f \neq \emptyset$ and $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is weak*-closed, the *strong duality* holds, i.e.

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\}$$

(iii) For every $v \in X^*$

$$\inf_{x \in A} v(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{v(x) + \sum_{t \in T} \lambda_t f_t(x)\}$$

LOCALLY FARKAS-MINKOWSKI SYSTEMS

♦ σ is *LFM* at $x \in A$ if

$$N_A(x) \subseteq N_C(x) + \text{cone}\left(\bigcup_{t \in T(x)} \partial f_t(x)\right),$$

where $T(x) := \{t \in T \mid f_t(x) = 0\}$.

Literature: Puente & Vera de Serio (1999), Hiriart Urruty & Lemarechal (1993), and Li & Ng (2005).

♦ σ is said to be *LFM* if it is LFM at every feasible point $x \in A$.

As a consequence of the optimality theorem,

$$\sigma \text{ FM} \Rightarrow \sigma \text{ LFM}$$

If σ is LFM at $x \in A$, then every continuous linear consequence of σ which is binding at x is also consequence of a finite subsystem of σ. The converse statement fails.

Theorem (GJL, 2007)**:** the following statements are equivalent:

(i) σ is *LFM*.

(ii) For every f that is continuous at least at a point of $A \cap \operatorname{dom} f$ and that attains $\min_{x \in A} f(x)$, the Lagrangian *min-max duality* holds, i.e.

$$\min_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\}$$

(iii) For every $v \in X^*$ that attains $\min_{x \in A} v(x)$ one has

$$\min_{x \in A} v(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{v(x) + \sum_{t \in T} \lambda_t f_t(x)\}$$

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