

# **Recent research**

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## Interests

### In the past

- Hard constraint approaches for fixed time-step simulation of nonsmooth dynamics.
- Theory and algorithm analysis for degenerate nonlinear programming, in particular, mathematical programming with equilibrium constraints.

## Interests

### In the future

- Robust algorithms for degenerate nonlinear programming.
- Iterative, parallel methods for QP, NLP.
- Coupling the hard constraint approach with other physics, such as fluid dynamics (Lattice Boltzmann?).
- Applications of HCA for nonsmooth dynamics to robotics (some application involving MPEC with Srinivas Akella and Jufeng Peng), haptics (Ed Colgate ?), granular flow, ...
- Uncertainty quantification, stochastic optimization ...
- ...

## Degenerate Nonlinear Programming

- Extended Robinson's result about having a locally isolated KKT solution near a maximum using much weaker second-order conditions (the weakest known: quadratic growth conditions).
- MPECs do not satisfy a constraint qualification. **Linearizations may be infeasible arbitrarily close to the solution.** However, the elastic mode ( $L_1$  relaxation/penalization) converges locally for a **finite** value of the penalty parameter for the generic case.
- Fletcher, Leyffer, Sholtes and Ralph have shown that exact second derivative SQP algorithms converge superlinearly if either (1) they start from a complementarity point or (2) all subproblems are feasible. However an adaptive elastic mode approach wrapped around FLSR converges superlinearly **without** under the same assumptions, except the preceding alternative.

## Nonsmooth multi-rigid-body dynamics

Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces.

- non-interpenetration.
- collision.
- joint constraints
- adhesion
- Dry friction – Coulomb model.
- global forces: electrostatic, gravitational.

■ These we cover in our approach.

## Matrix Form of the Integration Step

At each time step solve the problem

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \quad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Can progress with fixed time step and no backtrack!

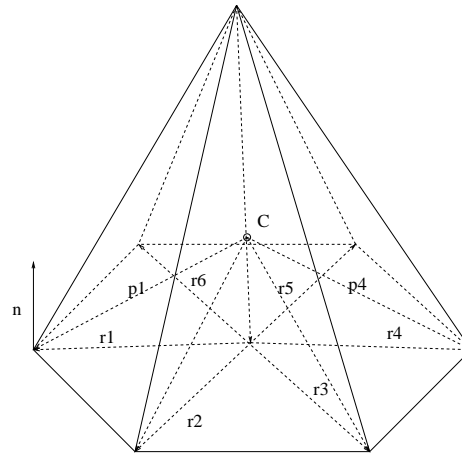
Method implemented in KARMA, a commercial physical (nr.2) content provider for gaming (who cite our work).

## Solving the LCP

Is it possible to obtain an algorithm that has modest conceptual complexity?

- **PATH** after reduction to proper LCP works, but for larger scale problems alternatives to it are desirable. **Works well for tens of bodies, most of the time.**
- **Interior Point methods** work for the frictionless problem ( **since matrices are PSD**), but their applicability to the problem with friction depends on the convexity of the solution set.
- Is the solution set of the complementarity problem at least convex?  
**NO, for the tiniest of friction!**

## Nonconvex solution set



Force Balance:

$$\sum_{j=1}^6 c_n^{(j)} n^{(j)} - hmg \begin{pmatrix} n \\ \mathbf{0}_3 \end{pmatrix} = 0.$$

$$\mu c_n^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \quad j = 1, 2, \dots, 6.$$

## Weaker formulation for NRMD

Measure differential inclusions, since there is no classical solution. Find  $q(\cdot), v(\cdot)$  such that

1.  $v(0)$  is a function of bounded variation (but may be discontinuous).
2.  $q(\cdot)$  is a continuous, locally Lipschitz function that satisfies

$$q(t) = q(0) + \int_0^t v(\tau) d\tau$$

3. The measure  $dv(t)$ , which exists due to  $v$  being a bounded variation function, must satisfy, (where  $f_c(q, v)$  is the Coriolis and Centripetal Force)

$$\frac{d(Mv)}{dt} - k(t, v) - f_c(q, v) \in FC(q(t))$$

4.  $\Phi^{(j)}(q) \geq 0, \forall j = 1, 2, \dots, p.$

$\in$  means in the sense of nonnegative test functions.

## The convex relaxation

Define  $\Theta^{(l)} = -Mv^{(l)} - hk^{(l)}$ . We solve the following LCP

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & -\tilde{\mu} \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} \Theta^{(l)} \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \quad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

The LCP is actually equivalent to a strongly convex QP.

## The new convergence result with convex subproblems

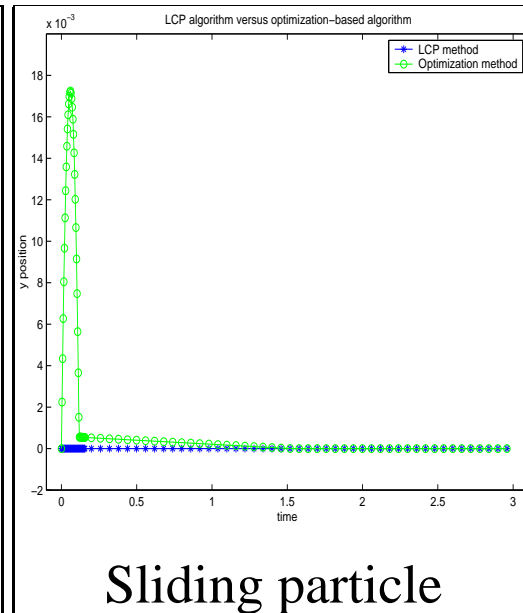
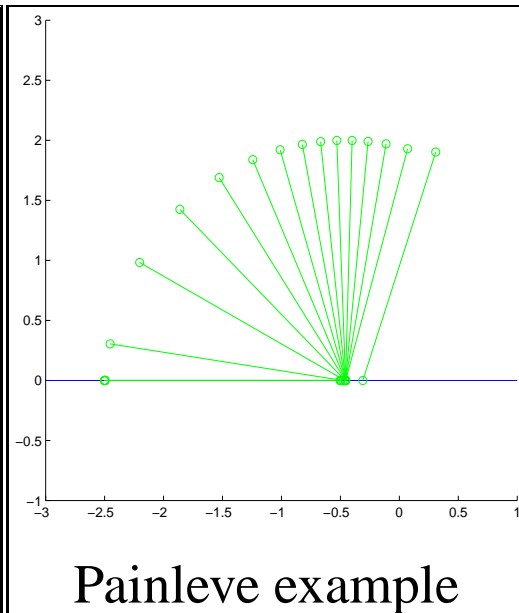
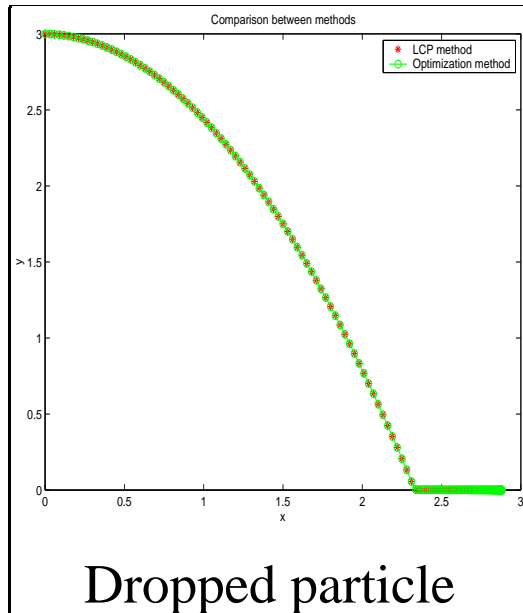
- H1 The functions  $n^{(j)}(q), t_1^{(j)}(q), t_2^{(j)}(q)$  are smooth and globally Lipschitz, and they are bounded in the 2-norm.
- H2 The mass matrix  $M$  is positive definite.
- H3 The external force increases at most linearly with the velocity and position.
- H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence  $h_k \rightarrow 0$  where

- $q^{h_k}(\cdot) \rightarrow q(\cdot)$  uniformly.
- $v^{h_k}(\cdot) \rightarrow v(\cdot)$  pointwise a.e.
- $dv^{h_k}(\cdot) \rightarrow dv(\cdot)$  weak \* as Borel measures. in  $[0, T]$ , and every such subsequence converges to a solution  $(q(\cdot), v(\cdot))$  of **MDI**. Here  $q^{h_k}$  and  $v^{h_k}$  is produced by the relaxed algorithm.

## Significance

There is now hope of catching up with penalty methods, that have 30 years ahead of us, since convex QPs are much easier to solve than general LCP, especially with  $10^3 - 10^6$  bodies.



$$h_k = \frac{0.1}{2^k}, \mu = 0.3$$

$$h_k = \frac{0.1}{2^k}, \mu = 0.75$$

k	$h_k$	$\ y_{QP} - y_{LCP}\ _2$
0	5.6314784e-02	
1	1.7416198e-02	
2	6.7389905e-03	
3	2.1011170e-03	
4	7.6112319e-04	
5	2.6647317e-04	
6	9.2498029e-05	
7	3.2649217e-05	

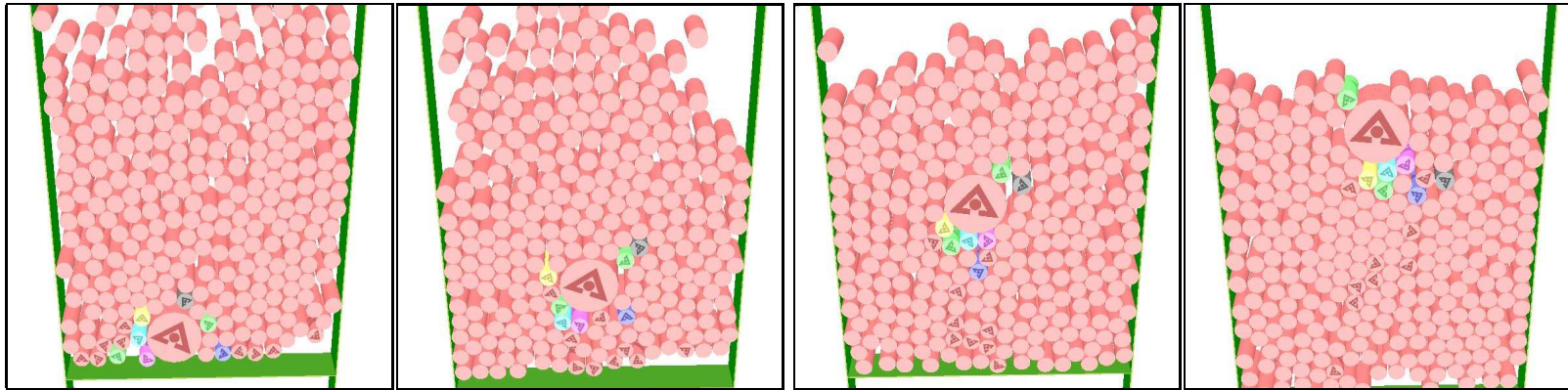
k	$h_k$	$\ y_{QP} - y_{LCP}\ _2$
0	1.5736018e+000	
1	7.2176724e-001	
2	1.4580267e-001	
3	9.2969637e-002	
4	5.5543025e-003	
5	4.3982975e-003	
6	3.7537593e-003	
7	3.7007014e-004	

No convergence, but  
small absolute error.

## Granular matter

- Sand, Powders, Rocks, Pills are examples of granular matter.
- The range of phenomena exhibited by granular matter is tremendous. Size-based segregation, jamming in grain hoppers, but also flow-like behavior.
- There is still no accepted continuum model of granular matter.
- Direct simulation methods (discrete element method) are still the most general analysis tool, but they are also computationally costly.
- The favored approach: the penalty method which works with time-steps of microseconds for moderate size configurations.

## Brazil nut effect simulation



- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. **One QP/step. No collision backtrack.**
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).