Sampling-Based Methods for Stochastic Optimization

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The model

Objective is to solve

$$\min_{x \in X} \{g(x) := \mathbb{E} G(x, \boldsymbol{\xi})\}$$

where $\boldsymbol{\xi}$ is a random vector representing the "uncertainty" in the model (e.g. future demand). Denote by F the (joint) distribution of $\boldsymbol{\xi}$.

We are interested in the case where $\boldsymbol{\xi}$ either has a continuous distribution or a huge number of scenarios, so *sampling* is required.

Two approaches

1. *External Sampling:*

   Let $\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^N$ be an iid random sample drawn from F.

   Solve

$$\min_{x \in X} \left\{ \widehat{g}_N(x) := N^{-1} \sum_{j=1}^{N} G(x, \boldsymbol{\xi}^j) \right\}.$$

   Let

$$\begin{aligned} x^* &= \text{optimal solution of original problem} \\ \nu^* &= \text{optimal value of original problem} \\ \widehat{x}_N &= \text{optimal solution of approximating problem} \\ \widehat{\nu}_N &= \text{optimal value of approximating problem} \end{aligned}$$

   Many convergence properties exist: under proper assumptions,

   - $\widehat{x}_N \to x^*$ w.p.1, $\widehat{\nu}_N \to \nu^*$ w.p.1
   - For any given $\varepsilon > 0$, $P(\|\widehat{x}_N - x^*\| \geq \varepsilon)$ approaches zero exponentially fast as $N \to \infty$.
   - etc.

   ISSUES: How to choose N, quality of solution for given N.

Two approaches (cont.)

2. *Internal Sampling:*

   Idea is to *incorporate* sampling into an optimization algorithm.

   In our context, this means the following:

   (i) Let $\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^N$ be an iid random sample drawn from F.

   (ii) Compute $\widehat{g}_N(x) := N^{-1} \sum_{j=1}^{N} G(x, \boldsymbol{\xi}^j)$.

   (iii) Do some optimization steps with $\widehat{g}_N(x)$.

   (iv) Repeat (i)-(iii) above.

   ISSUES:

   - Convergence
   - Choice of sample sizes
   - New samples vs. cumulative samples
   - Testing optimality conditions
   - etc.

   Statistical tests are often used as auxiliary tools.

My interests

- Development of efficient sampling-based methods, especially for *non-smooth* stochastic optimization
  - A stochastic bundle method?
- Use of variance-reduction techniques from simulation literature
  - Guarantees of efficiency, practical performance
- Applications
  - Manufacturing, logistics, *revenue management*