

# An iterative method for the analysis of rectangular waveguides with external effective index correction

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## ABSTRACT

We discuss a simple, self-consistent iterative method for the analysis of rectangular waveguides with strong guidance. We show that this method removes the ambiguity of the conventional effective index method and gives a convergent value of the propagation constant both near and far from cutoff. Near the cut-off region, the accuracy can be improved by including an external effective index correction which takes into account the finite mode effect outside the waveguide.

**Keywords:** Iterative method, Rectangular waveguide, Effective index method, Index correction

## 1. INTRODUCTION

The effective index method (EIM) is the most widely used approximate method for solving rectangular waveguides.<sup>1,2</sup> In this method, a two-dimensional waveguide is converted into two one-dimensional (slab) waveguides of which one has an effective index in the guiding layer obtained from the other slab waveguide. The refractive index outside the waveguide is usually assumed to be the bulk index of the surrounding medium. The slab waveguide can be solved exactly, and the approximation in the effective index method originates from the effective indices assumed for the guiding layer and for the surrounding medium. The error due to the approximation is distributed in the various components of the propagation vector of the waveguide mode. Moreover, the results obtained depend on the order in which the two possible slab waveguides are taken.<sup>3</sup> The two possible results are not very different in the case of large waveguides far above cutoff, but for waveguides near cutoff, and also for small strongly guided waveguides, the ambiguity between the two results become significant.

In this paper, we propose a new iterative method that is able to produce a unique value for all the components of the propagation vector, thereby removing the ambiguity of the EIM. The method is simple and computationally efficient. Furthermore, we take into account the finite mode field effect which gives rise to an effective index for the external medium which is smaller than the bulk index. This external effective index correction is essential to improve the accuracy of the iterative method near the cut-off region. However, the correction is valid only up to a limit. For very small waveguides, especially when the index contrast between the waveguide and the surrounding medium is large, even this correction is not sufficient to give accurate results. In section 2, we first discuss the limitations of the conventional EIM. Section 3 presents the iterative method. Section 4 discusses the external effective index correction.

## 2. THE AMBIGUITY OF THE EIM

For simplicity, we consider a rectangular waveguide with width  $a$  and height  $b$ , and core index  $n_1$  surrounded by a medium of lower index  $n_2$ . Without loss of generality, we assume  $a \geq b$ , and define the  $x$  direction to be along the larger (width) dimension. In a rectangular waveguide, the modes are not strictly TE or TM. We follow the mode field convention of Marcatili's paper to define TE-like modes and TM-like modes.<sup>4</sup> The  $E$  field of TE-like modes is mainly in  $x$  direction and  $y$  direction for that of TM-like modes. For illustration we consider the lowest order TE-like modes as the method for TM-like modes is similar.

The basic concept of the EIM is to approximate a rectangular waveguide by a slab waveguide with an effective index equal to the mode index of another slab waveguide oriented in the perpendicular direction. To distinguish between the two slab waveguides, we shall call the slab waveguide in which the plane is parallel to the  $x$  direction the TE-waveguide, and that in which the plane is parallel to the  $y$  (vertical) direction the TM-waveguide. Fig. 1 illustrates the two valid sequences in which the EIM can be carried out. In the *TE-TM* method, the effective index  $n_{eff}$  is first obtained from the TE-waveguide, then incorporated into the TM-waveguide. In the *TM-TE* method, the sequence is reversed. The ambiguity of the EIM is that the results of these two methods are in general different. This ambiguity can be traced to the different orders of approximation involved in the two methods, as shown below.

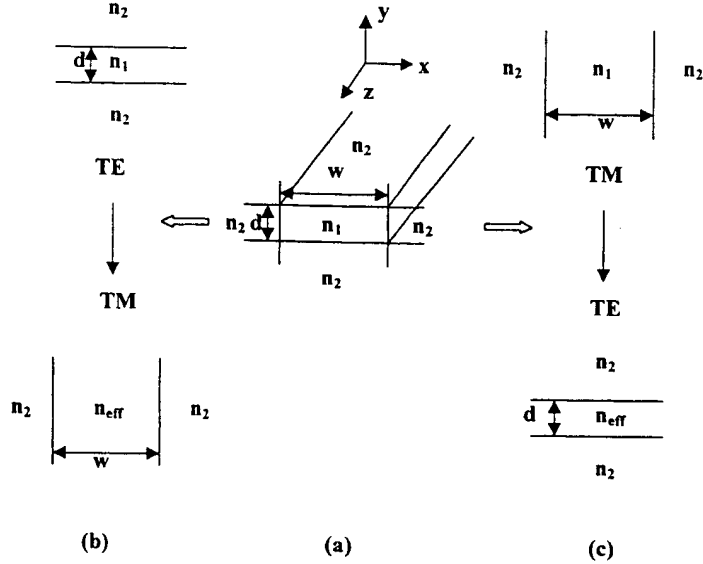


Fig. 1 (a) Configuration of a rectangular waveguide (b) Schematic description of the effective index method by the *TE-TM* approach. (c) Schematic description of the effective index method by the *TM-TE* approach

To anticipate the iterative method to be discussed later, we shall explicitly label the order of the wavevector components to reflect the sequence in which they are derived. In the *TE-TM* approach, solving the TE mode in the TE-waveguide gives the zeroth-order wavenumber component  $k_{1y}^{(0)}$ . From  $k_{1y}^{(0)}$  we can obtain the zeroth-order wavenumber component on the  $x-z$  plane in medium 1  $k_{1xz}^{(0)}$  given by :

$$k_{1xz}^{(0)} \equiv \sqrt{k_1^2 - k_{1y}^{(0)2}}, \quad (1)$$

where  $k_1 = k_0 n_1$ , and  $k_0$  is the propagation constant in free space. Next, we go to the TM waveguide, assume that the center refractive index for the TM-waveguide is given by the effective index defined as  $n_{1eff, TM}^{(0)} = k_{1xz}^{(0)} / k_0$ , and then solve the TM modes in the waveguide. In this step, we obtain the  $x$ -component  $k_{1x}^{(1)}$ , a first-order result because it incorporates the vertical confinement from the previous step. Subsequently, the propagation constant is given by

$$k_{1z, TE-TM}^{(\frac{1}{2})} = \sqrt{k_{1xz}^{(0)2} - k_{1x}^{(1)2}} = \sqrt{k_1^2 - k_{1y}^{(0)2} - k_{1x}^{(1)2}}, \quad (2)$$

Note that at this level,  $k_{1z, TE-TM}^{(\frac{1}{2})}$  contains a mixture of zeroth-order and first-order quantities, and therefore is labeled 1/2.

The TM-TE approach is similar. We start by solving the TM modes of the TM waveguide and then the TE modes of the TE waveguide. In the first step, we obtain the zero-order components  $k_{1x}^{(0)}$ . From  $k_{1x}^{(0)}$  we can obtain the zeroth-order wavenumber component on the  $y-z$  plane in medium 1  $k_{1yz}^{(0)}$  given by :

$$k_{1yz}^{(0)} \equiv \sqrt{k_1^2 - k_{1x}^{(0)2}}, \quad (3)$$

In the second step, we use the effective index  $n_{1\text{eff},TE}^{(0)} = k_{1yz}^{(0)} / k_0$  as the center refractive index, and obtain  $k_{1y}^{(1)}$  and the propagation constant

$$k_{1z,TE-TM}^{(1/2)} = \sqrt{k_{1yz}^{(0)2} - k_{1y}^{(1)2}} = \sqrt{k_1^2 - k_{1x}^{(0)2} - k_{1y}^{(1)2}}, \quad (4)$$

It is clear from Eqs. (2) and (4) that the two approaches give two different values of  $k_{1z}^{(1/2)}$ . In Fig. 2, we show the discrepancy  $\delta k_{1z}$  which is the difference between  $k_{1z,TE-TM}^{(1/2)}$  and  $k_{1z,TM-TE}^{(1/2)}$  divided by the average value  $(k_{1z,TE-TM}^{(1/2)} + k_{1z,TM-TE}^{(1/2)}) / 2$ , as a function of waveguide dimensions, for the case of a large index contrast of  $n_1 : n_2 = 3.4 : 1$ . We note that  $\delta k_{1z}$  is negligible when the waveguide dimensions are large, but increases as the dimensions get smaller. This shows that the ambiguity  $\delta k_{1z}$  is due to the coupling between the  $x$  and  $y$  components, or between the TE and TM polarizations, as this coupling is weak for large waveguides but becomes stronger as the waveguide dimensions decrease. The apparent rollover at  $d = 0.2 \mu m$  is related to the small and uncertain values of  $k_{1z}$  at this point.

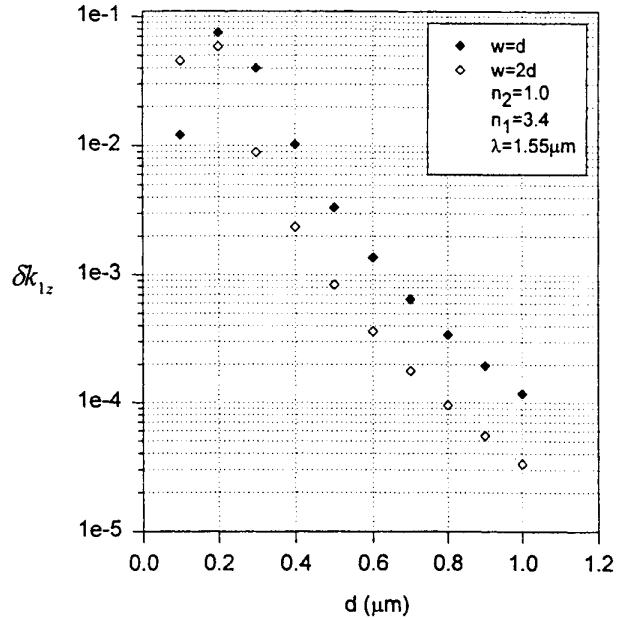


Fig. 2 The discrepancy  $\delta k_{1z}$  between  $k_{1z,TE-TM}^{(1/2)}$  and  $k_{1z,TM-TE}^{(1/2)}$  vs the waveguide thickness  $d$ .

### 3. THE ITERATIVE METHOD

Iterative feedback can be used to give higher orders of approximation to  $k_{1z}$  and eliminate the ambiguity. To illustrate this procedure, we detail the steps involved in the first iteration below. The iteration requires an initial input value of  $k_{1y}^{(1)}$  (or  $k_{1x}^{(1)}$ ) generated from an initial round of TE-TM (or TM-TE) calculations.

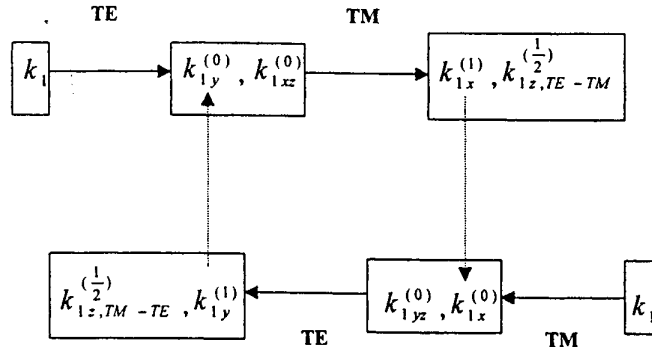


Fig. 3 Illustration of the general procedure of the iterative method. Note that the order of each parameter will change during each cycle.

1. Substitute  $k_{1y}^{(0)}$  in Eq. (1) with  $k_{1y}^{(1)}$  to get the first-order  $k_{1xz}^{(1)} = \sqrt{k_1^2 - k_{1y}^{(1)2}}$  and define the first-order  $n_{1\text{eff},TM}^{(1)} \equiv k_{1xz}^{(1)} / k_0$ .

2. Do the TM step of the TE-TM approach to get the second-order  $k_{1x}^{(2)}$  and the first-and-a-half-order

$$k_{1z,TE-TM}^{(1\frac{1}{2})} = \sqrt{k_{1xz}^{(1)2} - k_{1x}^{(2)2}} = \sqrt{k_1^2 - k_{1y}^{(1)2} - k_{1x}^{(2)2}}.$$

3. Substitute  $k_{1x}^{(0)}$  in Eq. (3) with  $k_{1x}^{(2)}$  to get the second-order  $k_{1yz}^{(2)} = \sqrt{k_1^2 - k_{1x}^{(2)2}}$  and define the second-order  $n_{1yz,TE}^{(2)} \equiv k_{1yz}^{(2)} / k_0$ .

4. Do the TE step of the TM-TE approach and get the third-order  $k_{1y}^{(3)}$  and second-and-a-half-order

$$k_{1z,TM-TE}^{(2\frac{1}{2})} = \sqrt{k_{1yz}^{(2)2} - k_{1y}^{(3)2}} = \sqrt{k_1^2 - k_{1x}^{(2)2} - k_{1y}^{(3)2}}.$$

Again, we use the half-order to exhibit the order mixture property inherited in the iterative method.

Steps 1-4 may be repeated as many times as necessary. Fig. 3 illustrates the basic idea of the iterative procedure. After the  $m$ th iteration, the propagation constant is seen to be given by

$$k_{1z,TE-TM}^{(2m-\frac{1}{2})} = \sqrt{k_1^2 - k_{1y}^{(2m-1)2} - k_{1x}^{(2m)2}} \text{ or}$$

$$k_{1z,TM-TE}^{(2m+\frac{1}{2})} = \sqrt{k_1^2 - k_{1x}^{(2m)2} - k_{1y}^{(2m+1)2}}.$$

The point is that, as the order  $m$  increases, the difference  $k_{1y}^{(2m-1)} - k_{1y}^{(2m+1)}$  would

diminish and  $k_{1z,TE-TM}^{(2m-\frac{1}{2})}, k_{1z,TM-TE}^{(2m+\frac{1}{2})}$  would converge to a unique value. Fig. 4 shows that  $k_{1z}$  indeed converges as the number of iterations increases. Because of the stronger coupling between the  $x$  and  $y$  components,  $k_{1z}$  converges more slowly for the smaller waveguides.

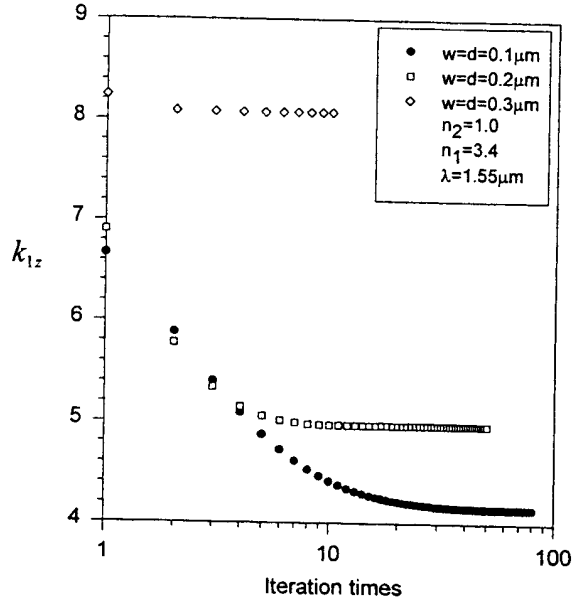


Fig.4  $k_{1z}$  converges as the number of iterations increases. The smaller the waveguide, the larger the number of iterations needed.

#### 4. THE EXTERNAL EFFECTIVE INDEX CORRECTION

One correction is needed to improve the accuracy of the iterative method described above. We recall that in the second step of both the TE-TM and TM-TE approaches, the slab waveguide assumes an effective index in the guiding layer but the refractive index of the surrounding medium remains constant and is equal to the bulk index. This is not correct in view of the finite transverse extent of the waveguide mode. Note that the effective index is defined by the propagation constant. The propagation constant of a plane wave in an infinite medium is related to the bulk index of the medium by

$k = \frac{\omega}{c} n$ , while that of a beam of finite transverse extent is related to some effective propagation constant by

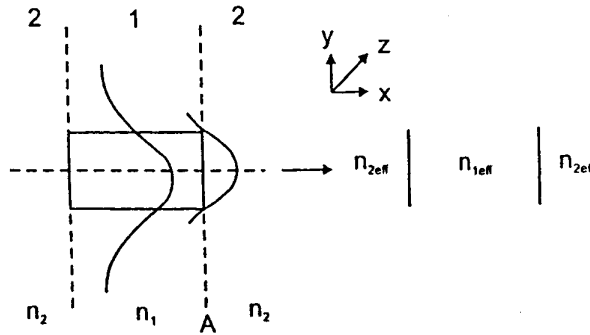
$k_{eff} = \frac{\omega}{c} n_{eff}$ , where  $n_{eff} < n$ . The later is because a beam of finite radius  $\omega_0$ , propagating in the  $z$  direction, is composed of a superposition of plane waves. The transverse components of the propagation vector of these waves are typically  $k_x = k_y \approx \frac{1}{\omega_0}$ . Thus,  $k_z^2 + k_x^2 + k_y^2 = k^2$  gives the effective propagation constant,

$$k_{eff} = k_z = \sqrt{k^2 - k_x^2 - k_y^2} < k.$$

The inconsistency in treatment of the guiding layer and the surrounding layer can be removed if one defines an effective index for the external medium in the same way that one defines an effective index for the guiding layer. For

example, let us consider the *TE-TM* approach (Fig. 1b). In this approach, the rectangular waveguide is first approximated by a TE slab waveguide, which is solved to obtain the effective index for the TM waveguide. In the effective index approximation, the mode field profile inside a rectangular waveguide is given by a function of the form

$$\varphi(x, y, z) \sim \cos(k_{1x}x) \cos(k_{1y}y) e^{-jk_{1z}z}, \quad (5)$$



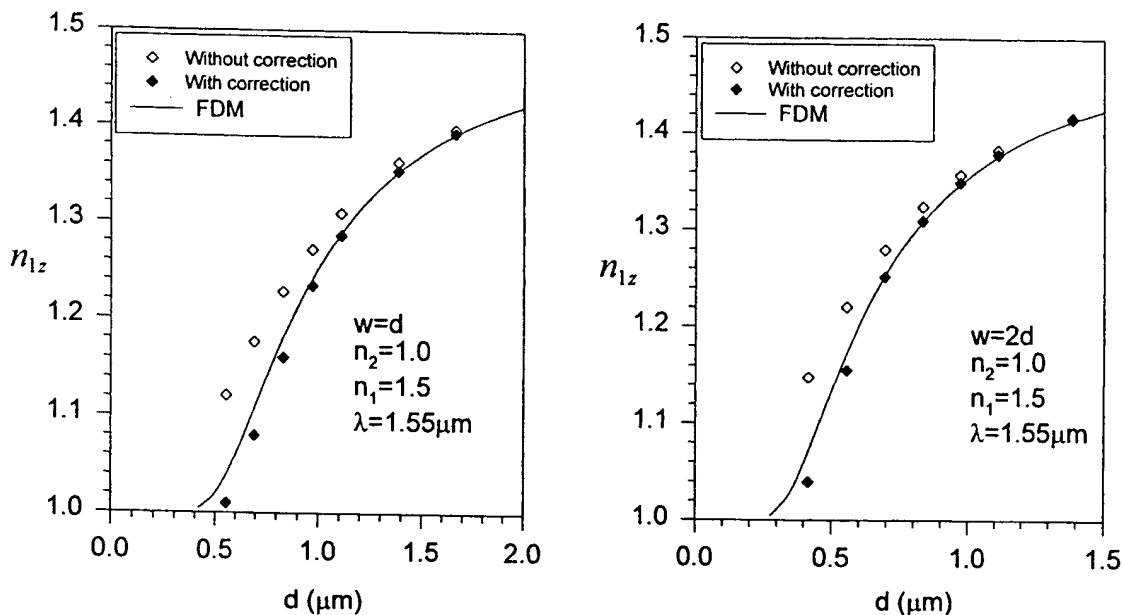
**Fig.5** Schematic diagram of the finite mode which modifies the refractive index outside the waveguide in the case of *TE-TM* approach. For *TM-TE* approach, we can define an effective index in regime 2 for the TE waveguide in a similar way.

where  $k_{1x}^2 + k_{1y}^2 + k_{1z}^2 = k_1^2 = \frac{\omega^2}{c^2} n_1^2$ . In the TE slab waveguide, the transverse mode profile is given by  $\cos(k_{1y}y)$ , which is a superposition of two plane waves with transverse wavevector component  $k_{1y}$  and  $-k_{1y}$ . In this case, the effective propagation constant (in the plane) would be given by  $k_{1eff} = k_{1xz} = \sqrt{k_1^2 - k_{1y}^2}$ , which gives the effective index  $n_{1eff}$  for the TM waveguide as depicted in Fig. 5. By assuming a finite mode in regime 2 immediately outside the waveguide boundary at *A* (see Fig. 5), one is effectively assuming an *effective* index for the region outside the guiding layer of the TM waveguide. Just as the finite mode size (in the *y* direction) modifies the refractive index inside the waveguide, it has the same effect on the external refractive index  $n_2$ . Thus, for the TM waveguide, instead of  $n_2$ , the effective index would be given by  $n_{2eff}^x = k_{2xz} / k_0$ , where  $k_{2xz} = \sqrt{k_2^2 - k_{1y}^2}$ , which we have assumed the continuity of the tangential *k*-vector  $k_y$  at the boundary *A* (i.e.  $k_{1y} = k_{2y}$ ). Similarly, in the *TM-TE* approach, the external effective index for the TE waveguide would be given by  $n_{2eff}^y = k_{2yz} / k_0$ , where  $k_{2yz} = \sqrt{k_2^2 - k_{1x}^2}$ . The effective refractive index in the surrounding medium is thus less than the bulk index ( $n_2$ ) and depends also on the direction. One could in principle use these new effective indices to recalculate  $k_{1x}$  and  $k_{1y}$ , so as to obtain even more accurate values for  $n_{2eff}$  in an iterative fashion. These effective indices, however, are only defined as long as  $k_{1x}, k_{1y} < k_2$ , and therefore may not be valid for smaller waveguides, especially when the waveguides are strongly confined.

Fig. 6 shows the results of  $n_{1z}$  for the fundamental mode as a function of the waveguide thickness *d*, calculated by the iterative method with and without the external effective index correction. For comparison, we also plot the results obtained by the Finite Difference Method (FDM) represented by the solid lines in Fig.6. It is clear that this correction gives a significant improvement in accuracy near the cut-off region with reference to the numerical results. From Fig. 6, we can also see that the iteration method without the correction always overestimates  $n_{1z}$ . The reason has been discussed by Kumar *et al.*<sup>5</sup> On the contrary, the iteration method with the correction tends to underestimate  $n_{1z}$ .

## 5. CONCLUSION

A new iterative method including the external effective index correction has been discussed in this article. This new method gives a unique value of the propagation constant, with significant improvement in accuracy compared with the conventional effective index method. It is more computationally efficient than numerical methods such as FDM. Therefore, this iterative effective index method is expected to be useful for first-cut nanoscale waveguide device design.



**Fig.6**  $n_{1z}$  obtained by the iterative method with and without the external effective index correction for the fundamental TE-like mode. The solid lines are given by the numerical Finite Difference Method (FDM), which is considered more exact. Near cut-off, the index correction significantly improves the accuracy.

## 6. REFERENCES

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