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Approximate solution of the whispering gallery modes and estimation of the spontaneous emission coupling factor for microdisk lasers

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Abstract

A simple approximation is developed for solving the "whispering gallery modes" of a microdisk laser structure using conformal transformation and the WKB approximation. Using this method the spontaneous emission coupling factor of microdisk lasers is estimated. The result predicts a β value of the order of 10^{-1}.

Recently, McCall et al. have demonstrated a 2D guided microdisk structure to achieve low-threshold lasing [1]. This structure is an example of photonic confined microcavities, where the photon density of states and the spontaneous emission characteristics are significantly modified [2]. The threshold current of a microcavity laser with low transparency current is dependent mainly on the spontaneous emission coupling factor β [3]. We have previously calculated the decay rate of exciton in a one-dimensional cylindrical dielectric waveguide with high index guiding [4]. In the calculations, the spontaneous decay is modeled as stimulated decay due to stochastic vacuum field fluctuations [2–4]. The spontaneous emission rate (γ) is proportional to the modal density of states and the field intensity of a vacuum mode at the location of the dipoles. More precisely, the contribution to γ from the guided mode n is [4]

$$\gamma_{gn} = \frac{2\pi}{\hbar^2} \frac{\hbar\omega}{2L_z A} \, |\bar{E}_n \cdot \bar{\mu}|^2 \frac{L_z}{2\pi} \frac{\mathrm{d}k_{nz}}{\mathrm{d}\omega} , \tag{1}$$

where A is basically the spatial mode area (i.e. the transverse area of the mode) times the dielectric constant, $\bar{E}_n$ is the mode function for the guided mode, and $\bar{\mu}$ is the usual dipole matrix element. In the case of the one-dimensional waveguide structure, the field is quantized in the longitudinal direction (z-axis) via the traveling-wave modes [4], where the number of states per unit angular frequency is $\rho_F = (L_z/2\pi)\, \mathrm{d}k_{nz}/\mathrm{d}\omega$ for each waveguide mode. To extend our theory to a microdisk laser for estimating the β values, a simple, approximate method for solving the waveguide modes and the density of states is developed here. Knowledge of the mode size and the density of states allows us to obtain the value of γ_{gn} according to Eq. (1), and thus estimate the β value of the microdisk laser.

A microdisk laser as shown in Fig. 1a is characterized by two size parameters, d and R, where d is the thickness and R the radius of the disk. The indices in

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the figure are taken to be that given by a semiconductor microdisk with $n_1=3.4$ and $n_2=1.0$. The waveguide modes travel around the edge of the disk by repeated total internal reflections, in a way similar to the "whispering gallery modes" [5]. In this paper the whispering gallery modes are solved approximately using conformal transformation and Wentzel-Kramers-Brillouin (WKB) approximation. The modal density of states and the mode field sizes are then evaluated as a function of d and R. Finally, an estimation for the β values of microdisk lasers is given.

The solutions for the conventional cylindrical waveguide are Bessel functions, which are the solutions of the wave equation in cylindrical coordinates. For the case of microdisk, we adopt an alternative treatment based on the method of conformal transformation developed in Ref. [6]. Our approach differs from Ref. [6] in that we treat a disk waveguide with finite thickness. In this approach, the wave equation for the cylindrical waveguide is first transformed into the Cartesian coordinate system (u, v) which is related to (r, ϕ) by a conformal transformation as follows [6]:

$$u=R\ln(r/R), \quad v=R\phi . \tag{2}$$

The "circular" waveguide is thus equivalent to a "straight" waveguide in the (u, v) plane, and the index distribution has become a function of u.

Under the transformation, the forms of ψ is transformed to $\psi=F(u)\exp(ik_v v)\cos(k_z z)$, where $k_v v=m\phi$. The cosinusoidal variation of ψ in the z direction accounts for the waveguiding in the plane of the disk. Let us define an effective index n_v to be $k_v=2\pi n_v/\lambda$ (λ is the free-space wavelength). Since $k_v v=m\phi$, using $v=R\phi$ from Eq. (2) to eliminate ϕ we have $\lambda=2\pi n_v R/m$.

Note that in the disk waveguide, for each k vector there are two modes corresponding to two polarizations. We shall define them as transverse electric (TE) and transverse magnetic (TM) modes with the electric field and magnetic field, respectively, parallel to the r direction as shown in Fig. 1a (in the figure the direction of the electric field is indicated). This corresponds to having the electric field and magnetic field, respectively, parallel to the u direction in the transformed coordinate. Under our definition of TE and TM modes, ψ is the z-component of the magnetic field for the case of TE mode (i.e. the electric field

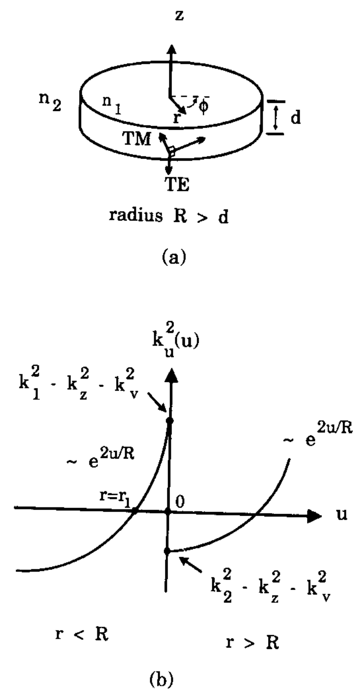

Fig. 1. (a) Schematic structure of a disk waveguide with radius R and thickness d. n_1 and n_2 are the refractive indices inside and outside the disk. (b) Illustration of the form for $k_u^2(u)$ as a function of u.

has no z-component) [7]. Likewise, ψ is the z-component of the electric field for the case of TM mode. The radial function $F(u)$ is then the solution of

$$d^2F/du^2=-k_u^2(u)F, \tag{3}$$

where $k_u^2(u)=(k_1^2-k_z^2)\exp(2u/R)-k_v^2$ for $r<R$, $k_u^2(u)=(k_2^2-k_z^2)\exp(2u/R)-k_v^2$ for $r>R$, and k_v must satisfy the condition for guided mode: $(k_2^2-k_z^2)<k_v^2<(k_1^2-k_z^2)$. The variables k_1 and k_2 are defined as $2\pi n_1/\lambda$ and $2\pi n_2/\lambda$ respectively. The form of $k_u^2(u)=[2\pi n_u(u)/\lambda]^2$ is illustrated in Fig. 1b, where $n_u(u)$ is an effective index distribution in the u direction. The form of $n_u(u)$ inside the waveguide entails that the radial distribution of the mode is skewed toward the edge (i.e. $r=R$ or $u=0$) since the index has a maximum there.

As in a planar waveguide, k_z is determined by matching boundary conditions at the interface. For the lowest-order TE modes, $\tan(k_z d/2)=\gamma/k_z$, where d is the thickness of the disk, and $k_z^2+\gamma^2=k_0^2(n_1^2-n_2^2)$ with $k_0=2\pi/\lambda$. Let $n_z=k_z/k_0$, from these equations we can obtain n_z as a function of d/λ ($\lambda=\omega/c$).

The most interesting region of d/λ is below $0.5/n_1$ where only the lowest TE mode is guide [4].

Next, k_v (n_v or the mode number m) is determined using the WKB approximation, by requiring the quantization condition [8]:

$$\int k_u(u)\,\mathrm{d}u = l\pi + \phi_1 + \phi_2, \quad l=0, 1, 2\ldots, \tag{4}$$

where the integral is taken over the classical region in Fig. 1b, i.e. from the lower limit defined by $k_u(u)=0$ to the upper limit $u=0$ (or $r=R$). In Eq. (4), l is a radial mode number, and ϕ_1, ϕ_2 are phases determined by the shape of the potential function at the turning points. For $l=0$, the calculated n_v as a function of R/λ and d/λ are shown by the discrete data points in Fig. 2. Note that (i) n_v is relatively insensitive to R/λ for $R/\lambda>2$, and (ii) n_v decreases with d/λ. This is because n_z increases with decreasing d/λ. To facilitate later calculations, we fit the calculated n_v with simple curves shown by the solid lines in Fig. 2, which have the following simple expression:

$$n_v = \alpha\sqrt{n_1^2 - n_z^2}, \quad \alpha = s_1 - \frac{s_2}{R/\lambda}, \tag{5}$$

where $s_1=0.984$ and $s_2=0.163$. Note that n_v still depends on d/λ via n_z. Using Eq. (5) and the definition of m, it follows that for a fixed d/λ, and hence fixed $\sqrt{n_1^2-n_z^2}$, there is a *linear* relationship between R/λ and m. Since m is an integer, this determines, for a given R, the resonance wavelength λ_m corresponding to each value of m. Specifically, $\lambda_m = 2\pi n_v R/m$.

The modal density of states, $\mathrm{d}k_v/\mathrm{d}\omega$, for any guided mode can be derived by making use of Eq. (4), and noting that $k_v=\omega n_v/c$. Hence if n_v is a function of only R/λ and d/λ, then a general and complete expression for $\mathrm{d}k_v/\mathrm{d}\omega$ is

$$c\frac{\mathrm{d}k_v}{\mathrm{d}\omega} = n_v + \frac{R}{\lambda}\frac{\mathrm{d}n_v}{\mathrm{d}(R/\lambda)} + \frac{d}{\lambda}\frac{\mathrm{d}n_v}{\mathrm{d}(d/\lambda)}. \tag{6}$$

In Fig. 3, $c\,\mathrm{d}k_v/\mathrm{d}\omega$ for the lowest-order mode is plotted as a function of d/λ for various values of R/λ. It is noteworthy that $\mathrm{d}k_v/\mathrm{d}\omega$ is not sensitive to R/λ, and that the dependence on d/λ is significant only if d/λ is smaller than 0.2.

In order to evaluate the spontaneous emission rate, we also need to determine the mode area A besides the modal density. For the lowest-order modes, we obtained A by approximating the two-dimensional mode profiles with a product of cosine functions of the form $\cos(\pi x/w_r)\cos(\pi y/d_z)$, where w_r is the effective width of the radial mode function, and d_z is the effective mode thickness in the direction perpendicular to the disk. The effective mode thickness d_z is approximated by the width of the lowest order planar waveguide mode. The value of w_r is determined by the radial functions for the (m, l) modes, which are Bessel functions of order m, $J_m(qr)$, where $q=k_0\sqrt{n_1^2-n_z^2}$. The outer limit of w_r is simply given by $r_{\mathrm{out}}=R$ (or $u=0$) in Fig. 1b. The inner limit of w_r, denoted by r_{in}, is approximately given by the classical

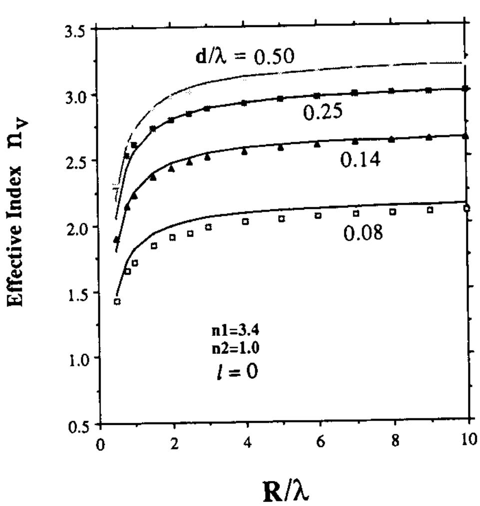

Fig. 2. The calculated effective index n_v for the $l=0$ radial mode as a function of R/λ for various values of d/λ. The solid lines are curve fitting for the calculated points.

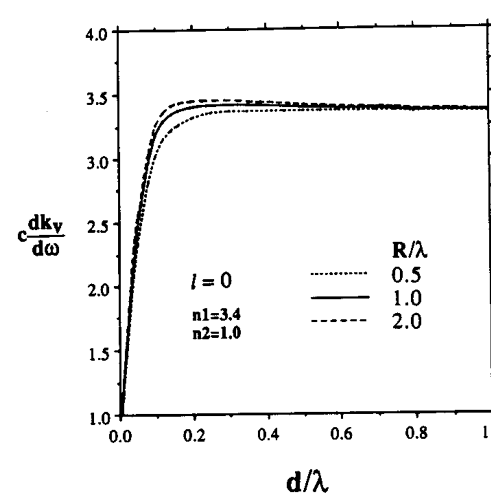

Fig. 3. The dependence of the mode density $\mathrm{d}k_v/\mathrm{d}\omega$ for the lowest-order TE mode on d/λ and R/λ.

turning point at $r=r_1$ shown in Fig. 1b at which $k_u(u)=0$. Setting $k_u(u)=0$ gives $\sqrt{n_1^2-n_z^2}\exp(u/R)=n_v$. Using Eq. (5) we then obtain $r_1=R\alpha=Rn_v/\sqrt{n_1^2-n_z^2}$. Using r_1 as an estimate for $r_{\rm in}$ tends to underestimate w_r because of the weak mode confinement and a better estimate is given by $r_{\rm in}\simeq Rn_v/n_1$ which is slightly smaller than r_1. Using this estimation for r_1 we obtain the following equation for w_r/λ: $w_r/\lambda=(R/\lambda)(1-n_v/n_1)$. In this form, w_r/λ is not only a function of R/λ, but also of d/λ. Note that the radial mode widths are actually larger for the disks with smaller d/λ since they have larger n_z and hence smaller n_v.

The modal density of states and the effective mode size allow us to determine the spontaneous emission coupling factor β for a microdisk laser. It is given by $\beta=R_{\rm L}/R_{\rm T}$, where $R_{\rm L}$ is the emission rate into the lasing mode, and $R_{\rm T}$ is the total emission rate. We shall take $R_{\rm L}$ and $R_{\rm T}$ as normalized rates normalized by the spontaneous emission rate in a bulk medium of uniform index n_1. We assume that the spontaneous emission into the lasing mode only emits into one guided mode spatially and into one cavity resonance spectrally. The condition for the spontaneous emission to go spectrally into one single resonance mode is $\Delta\nu_{\rm c}>\Delta\nu_{\rm sp}$ where $\Delta\nu_{\rm sp}$ is the spontaneous emission width, and $\Delta\nu_{\rm c}$ is the intermode frequency spacing, given by $\Delta\nu_{\rm c}\simeq c/(2\pi Rn_v)$ [3]. If $\Delta\nu<\Delta\nu_{\rm sp}$, then the spontaneous emission will emit into other non-lasing modes spectrally and the β value will decrease. This condition, therefore, determines the maximum size of the disk before the β value decreases. For example, the spontaneous emission width of a quantum well is typically 1% of the optical frequency, so at $\lambda=1.5$ µm, the largest diameter of the disk that satisfies the condition $\Delta\nu>\Delta\nu_{\rm sp}$ is 9.5 µm. If several radial modes are allowed, the dominant mode is one with the largest cavity Q. The evaluation of Q is complicated as it depends on the particular dominant loss mechanism. In the case of radiation loss due to tunneling from the disk, it can be shown that the Q is larger for the modes with the larger values of m. We note that the values of m for the $l=0$ modes are significantly larger than those for the $l=1$ modes. Hence, the $l=0$ modes will have larger Q values than the $l>0$ modes and will be the most likely to lase. The cavity resonance width $\Delta\nu_{\rm cav}$ is also related to Q, according to $\Delta\nu_{\rm cav}=\Delta\nu_{\rm c}/Q$. If $\Delta\nu_{\rm sp}\ll\Delta\nu_{\rm cav}$, then the spontaneous emission rate into the guided modes will be enhanced by a factor of Q. On the other hand, if $\Delta\nu_{\rm sp}>\Delta\nu_{\rm cav}$, then the spontaneous emission rate into the guided mode will not be strongly affected by the cavity. The cavity enhancement factor will be averaged to around unity if the spontaneous emission width approaches the intermode spacing. We will assume for our estimation of β that this is the case [4].

It turns out that for the microdisk case $R_{\rm T}$ is close to the bulk emission rate because of the high probability of spontaneous emission out through the center and then out from the side of the disk. Thus we can take $R_{\rm T}\simeq 1$. In fact even for the strongly confined waveguide of a microring cavity, $R_{\rm T}$ does not vary from 1 by more than 20% (see Ref. [3]). By knowing $dk_v/d\omega$ and A as a function of R/λ and d/λ, one can determine $R_{\rm L}$ for a specific combinations of R/λ and d/λ. We will restrict d/λ to $0<d/\lambda<0.5/n_1$, which is the region of interest for microdisk lasers where we want all the emissions into a single lasing mode. The calculated β is plotted in Fig. 4 with respect to R/λ and d/λ. Note that the β value for particular R/λ is around the highest at $d/\lambda\simeq 0.14$, which is right at the disk thickness that cutoff the second order planar guided mode. The β value reduces with reduced disk thickness d because of the broadening of the guided mode at small d, leading to a larger mode width w_r. On the other hand the β value increases with decreasing disk radius R due to the reduction of the

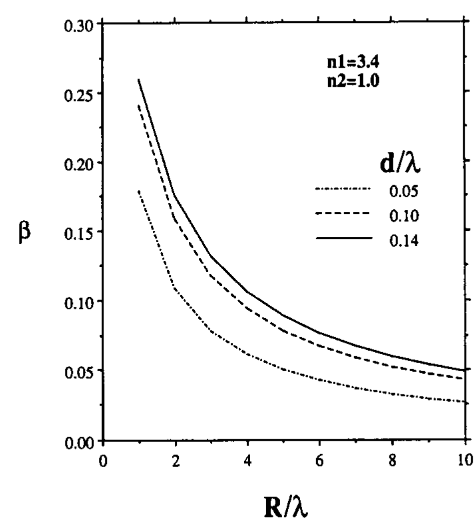

Fig. 4. The calculated spontaneous emission coupling factor, β, for microdisk lasers with varying size parameters $(R/\lambda, d/\lambda)$, assuming only one guided (lasing) mode is supported.

mode width at smaller disk radius. Physically, a smaller mode width corresponding to a larger mode guiding angle, which allows the lasing mode to capture more spontaneous emission.

The calculation of the β value is done with the assumption that only the radial dipole is excited by the pump (i.e. the non-isotropic pumping). In the case of isotropic excitation (i.e. all three dipoles are equally excited) which is normally the case in practice, the β value will vary. However, we note that it is shown in Ref. [4] that the radiation is non-isotropic even when the excitation is isotropic. In particular, the z-dipole emission is suppressed by the thin disk due to the reduction of the z component of the vacuum field. In fact the z dipole emission rate will be further reduced if quantum wells are used inside the disk as the active medium [9]. As a result there will only be r and ϕ dipole emissions even under isotropic excitation. Since only r dipole emits into the lasing mode, the value will be approximately half of that considered here.

The β value obtained here can be compared to the case of microring laser considered in Ref. [3]. The β value of a microdisk laser is generally smaller than that of a micro-ring laser, this is primarily because of the weaker mode confinement in a microdisk structure, except when the radius is of the order of a wavelength. For the microdisk case, if the laser is multimode with several radial modes ($l=0, 1, ...$) then the β value will be even smaller.

In conclusion, we have developed an approximate method for solving the whispering gallery modes in a microdisk laser. Conformal transformation of the wave equation for the circular disk is used to show the effective radial index distribution which provides guiding action to confine the mode near the edge of the disk. The effect of the disk thickness is also evident in this approach. We showed that the spontaneous emission coupling factor of a microdisk laser is smaller than a laser with an ideal cylindrical waveguide structure with strong index guiding. Nevertheless, a considerably high value of β, 0.1–0.2, can still be achieved in a microdisk laser with a cavity Q value of unity. In practice, the microring laser geometry is not easy to achieve.

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