Problem Definitions, Reductions & Expressing Them as Programs

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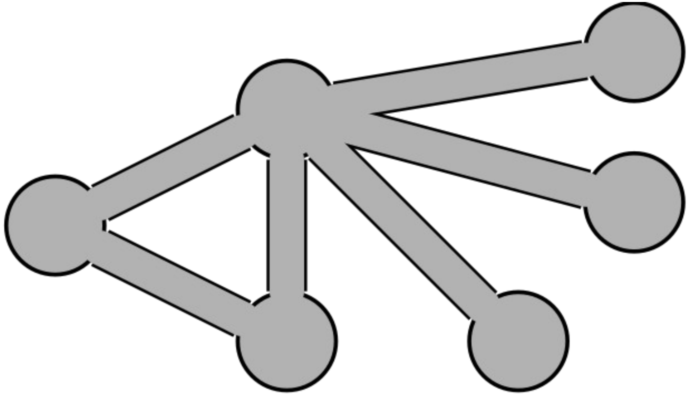
CS396 Fall 2023
Northwestern

Plan of the week

- NP Problem & Reduction -- Monday
- **Examples, Reduction in Karp (Today)**
- Lab, Assignment 4 -- Friday

Review – NP problem

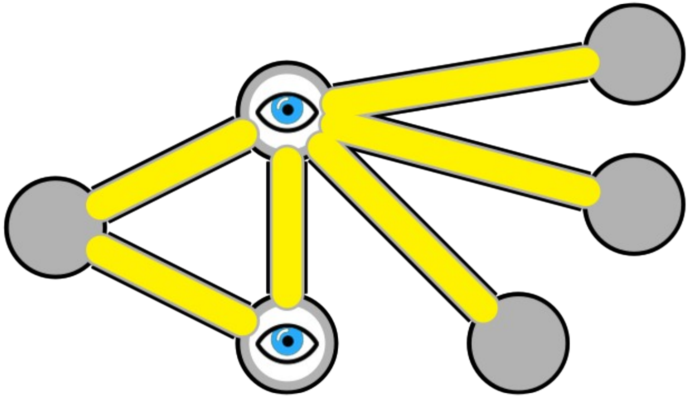
Can we cover all edges by selecting only **2** vertices?



VERTEX-COVER

Review – NP problem

Can we cover all edges by selecting only **2** vertices?



VERTEX-COVER

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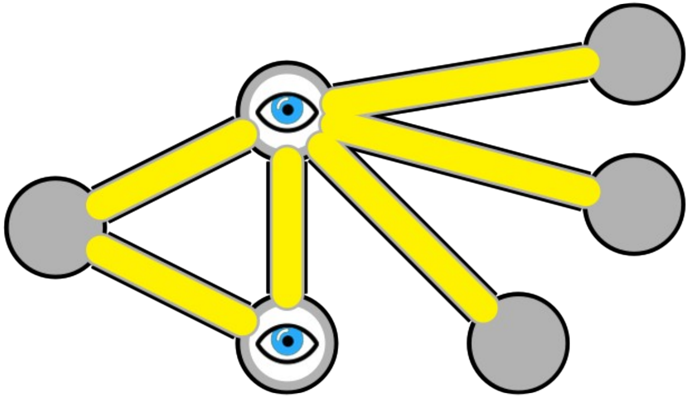
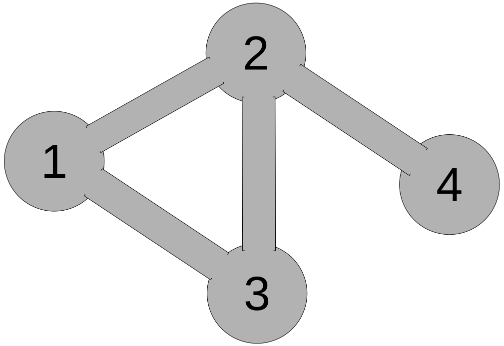


VERTEX-COVER

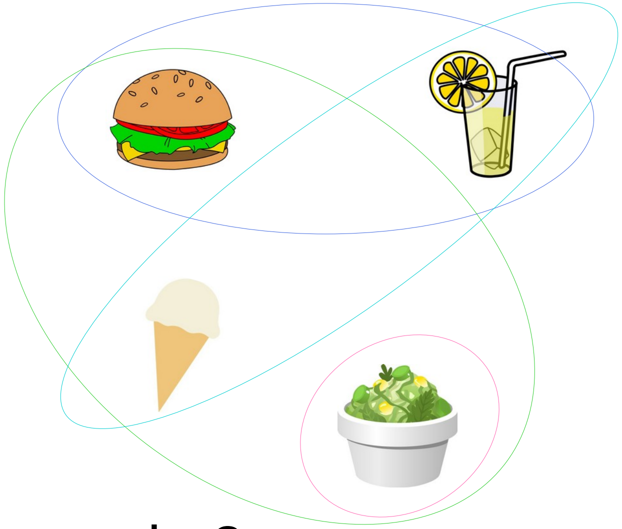
Yes-instance has easy to check certificates

Review – Reduction and Justification



k=2

Instance Construction

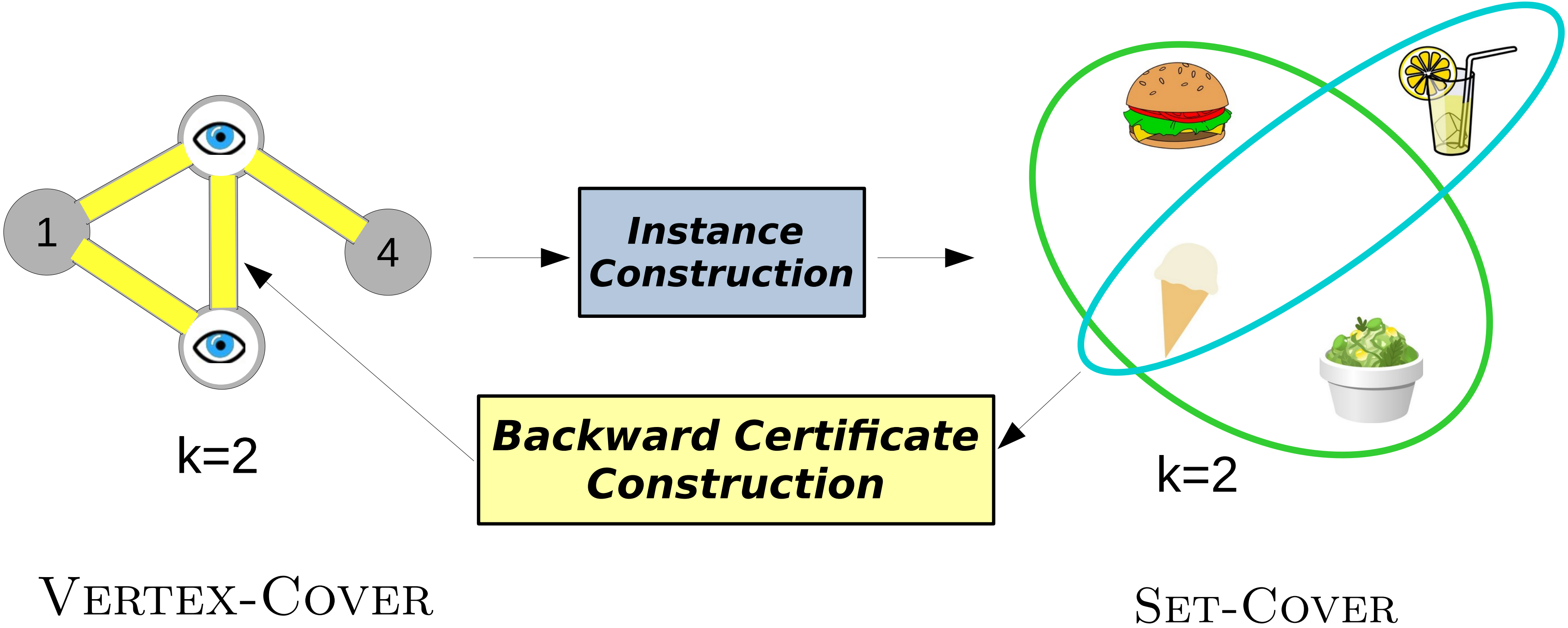


k=2

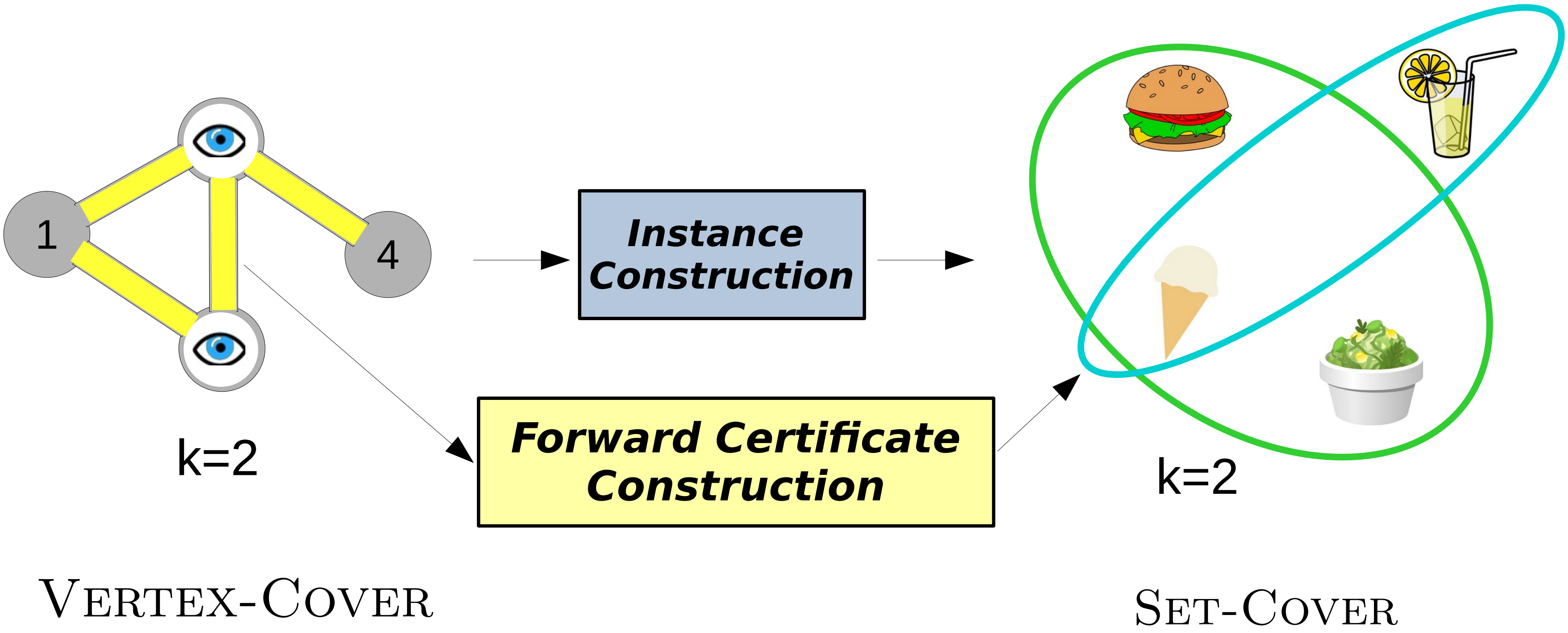
VERTEX-COVER

SET-COVER

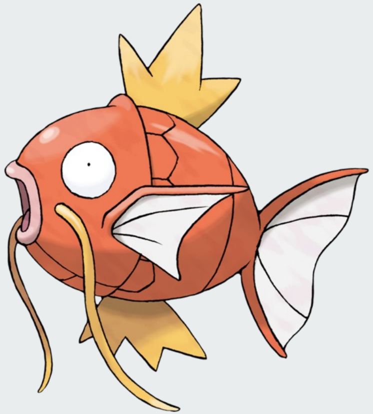
Backward Certificate Construction



Forward Certificate Construction



#lang karp



#lang karp

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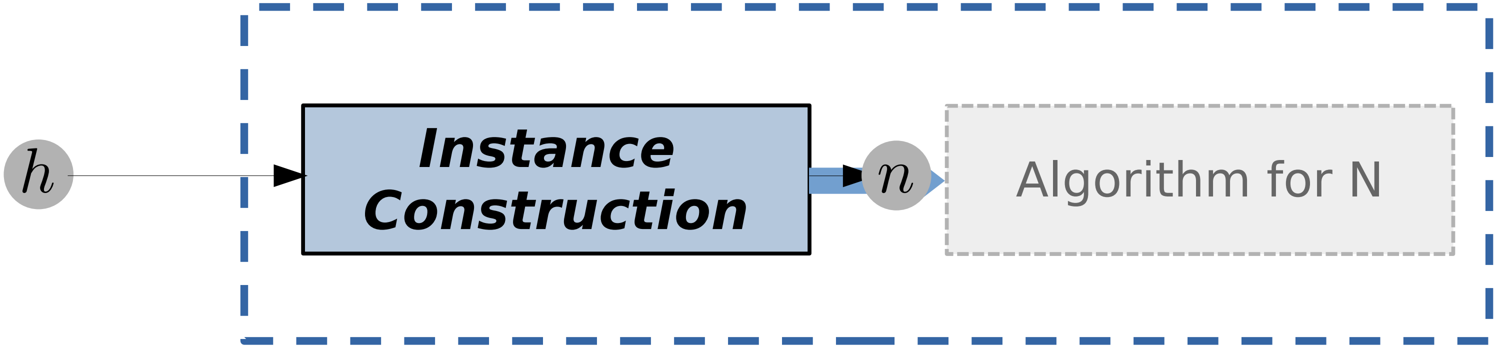
(named after Richard M. Karp)

Reduction Example

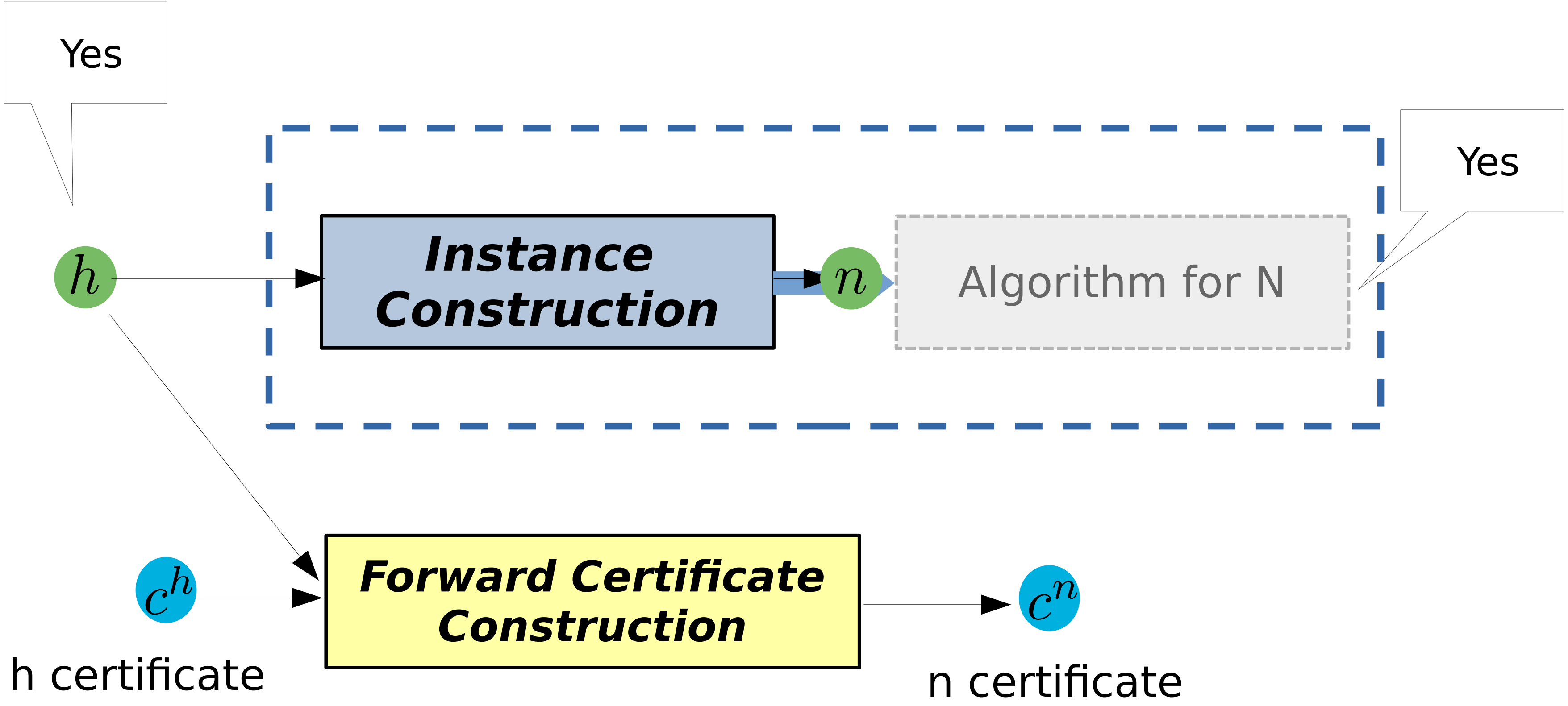
H = $\textsc{3-Sat}$ **N** = $\textsc{Independent-Set}$

Problem Definitions

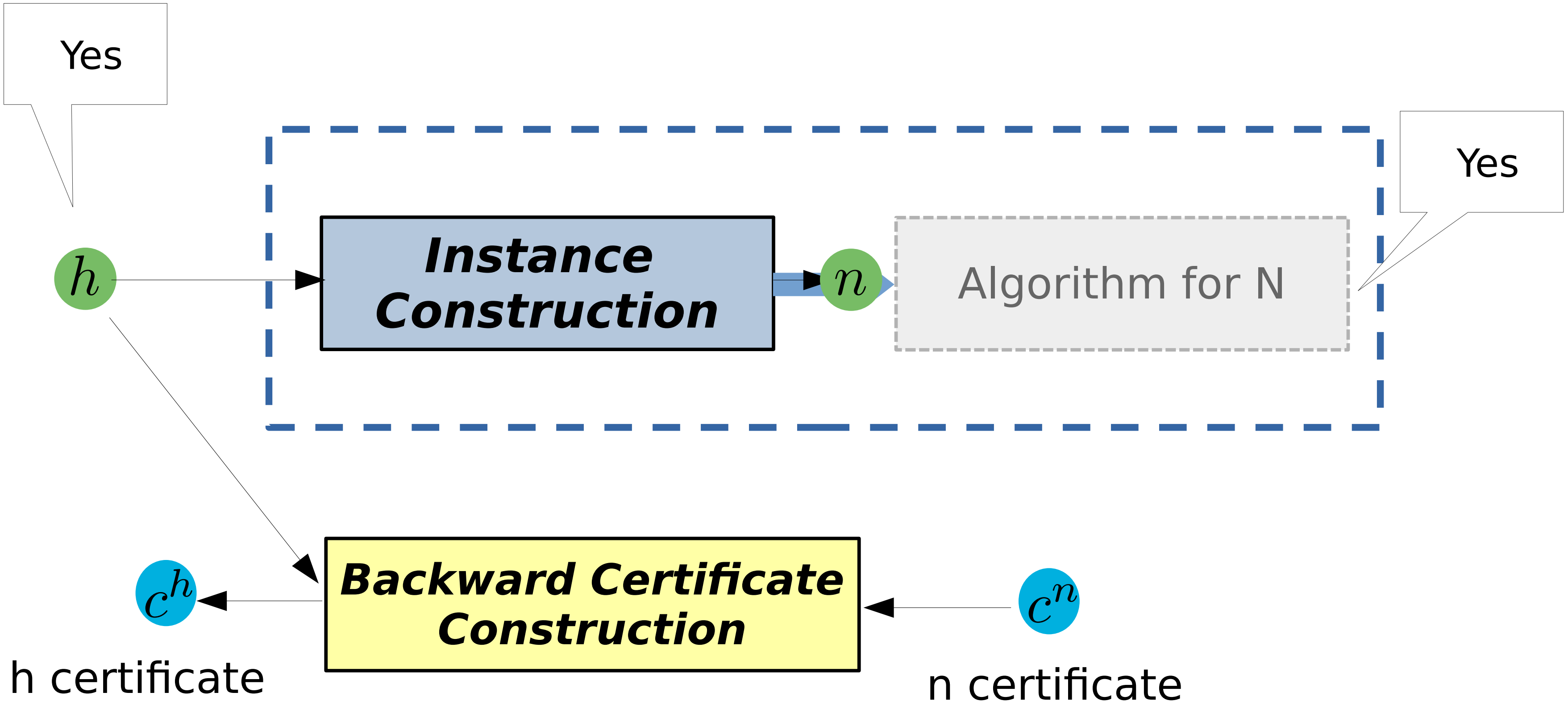
Instance construction



Justifying N No => H No

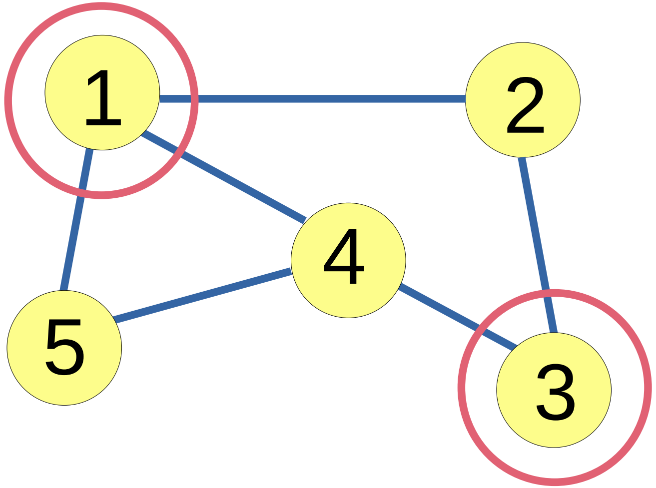


Justifying N Yes => H Yes



Independent-Set

Exists a set of k vertices s.t. no two are neighbors of each other?



k=2

Instance: a graph G and a threshold number k

Certificate: a subset of the vertices of G

INDEPENDENT-SET Instance and Certificate

Instance: graph G and natural k

INDEPENDENT-SET Instance and Certificate

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Assertion for valid certificate C of (G,k):

Forall e in edges of G:

INDEPENDENT-SET Instance and Certificate

Instance: graph G and natural k

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Assertion for valid certificate C of (G,k):
Forall e in edges of G:
Not (And <u>one vertex of</u> e in C
<u>the other vertex of</u> e in C)

INDEPENDENT-SET Instance and Certificate

Instance: graph G and natural k

Certificate: subset of vertices of G

Assertion for valid certificate C of (G,k):

Forall e in edges of G:

Not (And <u>one vertex of</u> e in C

<u>the other vertex of</u> e in C)

and

Size of C >= k

$3\text{-}\textsc{Sat}$ – Mother of All NP-Problems

Exists true/false assignment of the variable satisfying all clauses?

$$(\neg x_1 \vee x_2 \vee x_3)$$

$$(x_1 \vee \neg x_2 \vee x_4)$$

$$(x_2 \vee \neg x_3 \vee \neg x_4)$$

Valid or Not?

Instance: A Boolean formula in 3-conjunctive normal form (CNF)

$$x_1 \rightsquigarrow \mathrm{F} \qquad x_2 \rightsquigarrow \mathrm{T} \qquad x_3 \rightsquigarrow \mathrm{F} \qquad x_4 \rightsquigarrow \mathrm{F}$$

Certificate: Assignment from variables of the CNF to Boolean

3-SAT Instance and Certificate

Instance: 3CNF formula Phi

3-SAT Instance and Certificate

Instance: 3CNF formula Phi

Certificate: mapping from variables of Phi to Booleans

3-SAT Instance and Certificate

Instance: 3CNF formula Phi

Certificate: mapping from variables of Phi to Booleans

Assertion for valid certificate C of Phi:
Forall c in clauses of Phi:
Exists (literal l in c s.t
l is satisfied under c)

Reductions

Instance Construction

$$(x_1 \vee x_2 \vee x_3)$$

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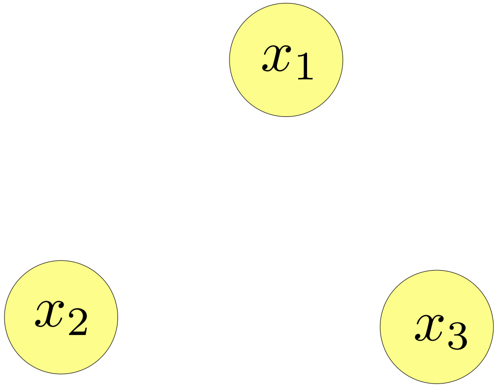

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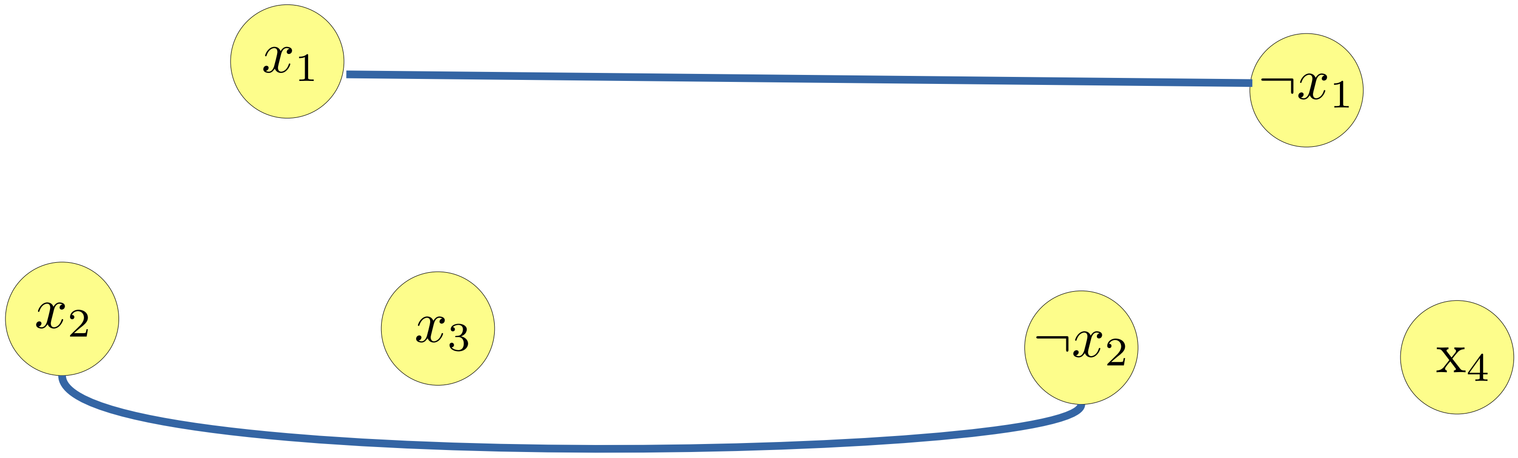

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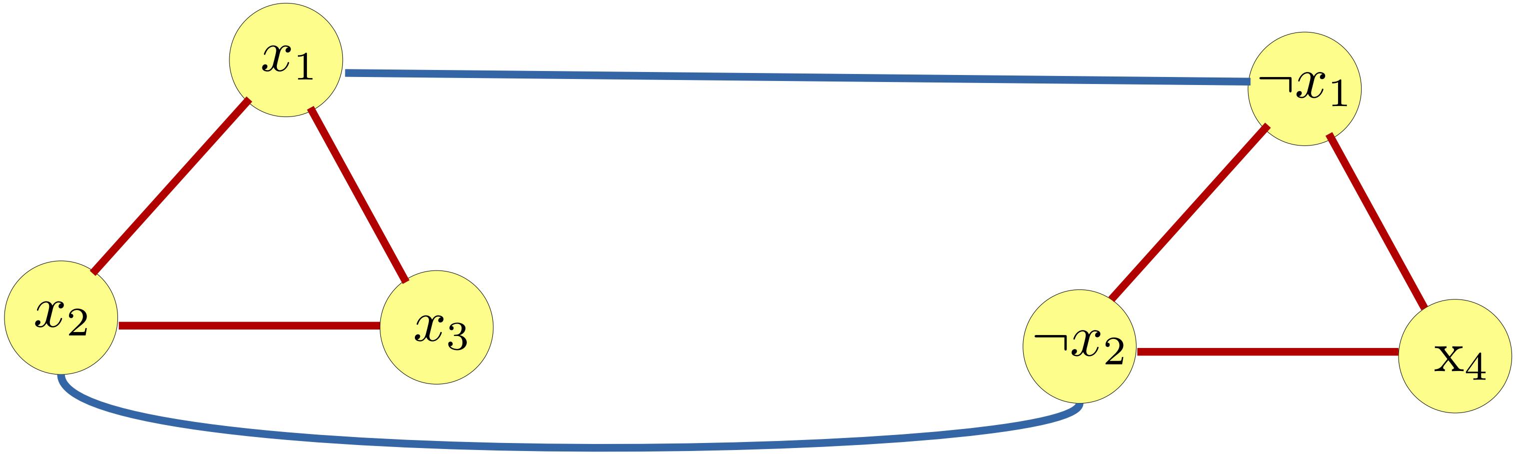
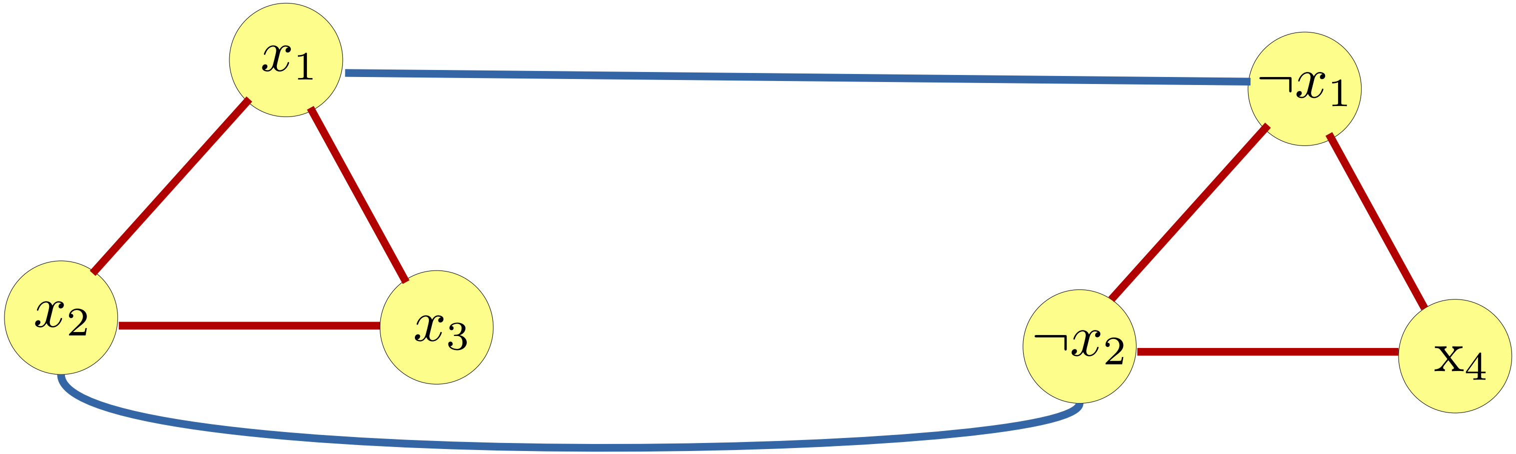


Forward Certificate Construction

$$(x_1 \vee x_2 \vee x_3) \qquad x_1 = T$$

$$(\neg x_1 \vee \neg x_2 \vee x_4) \qquad x_4 = T$$



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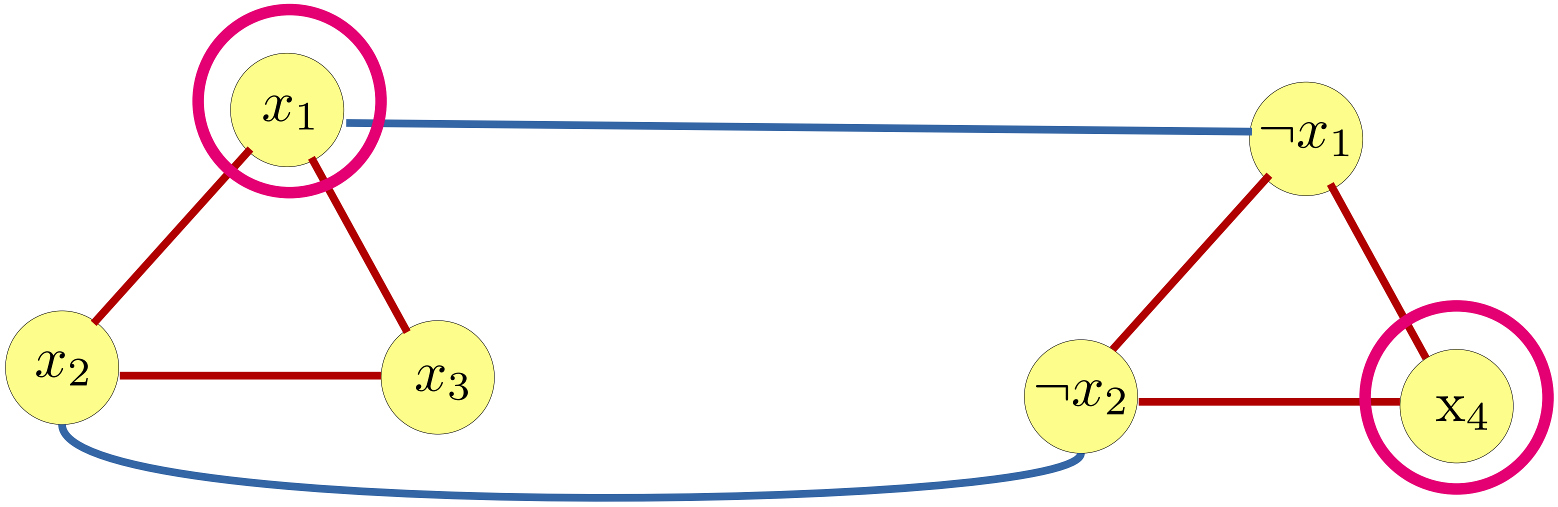

Backward Certificate Construction

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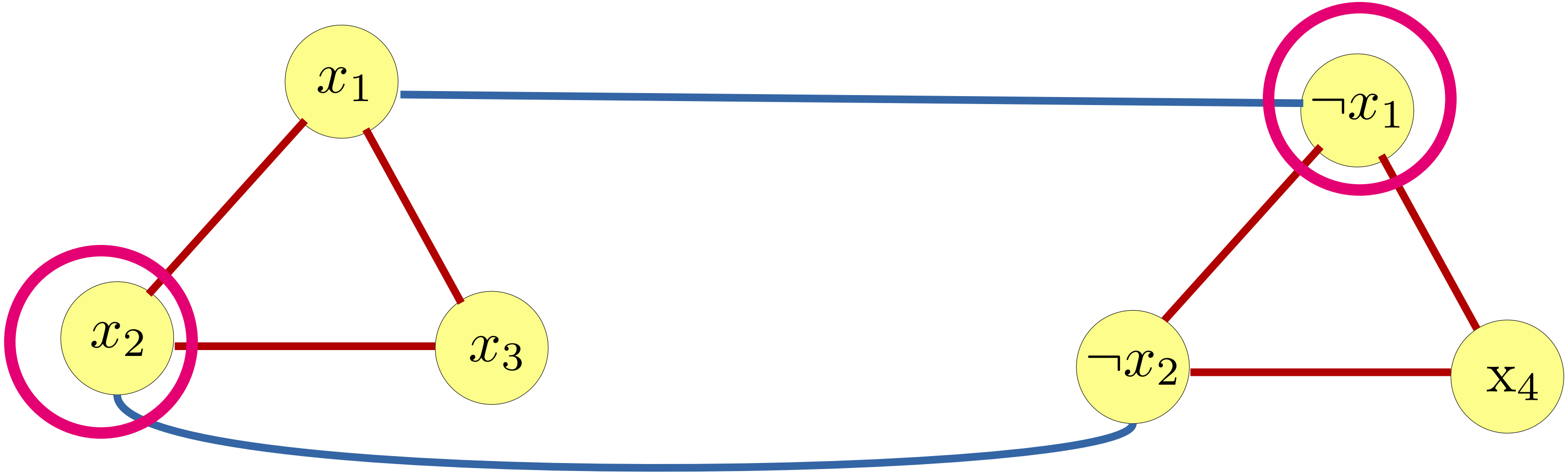


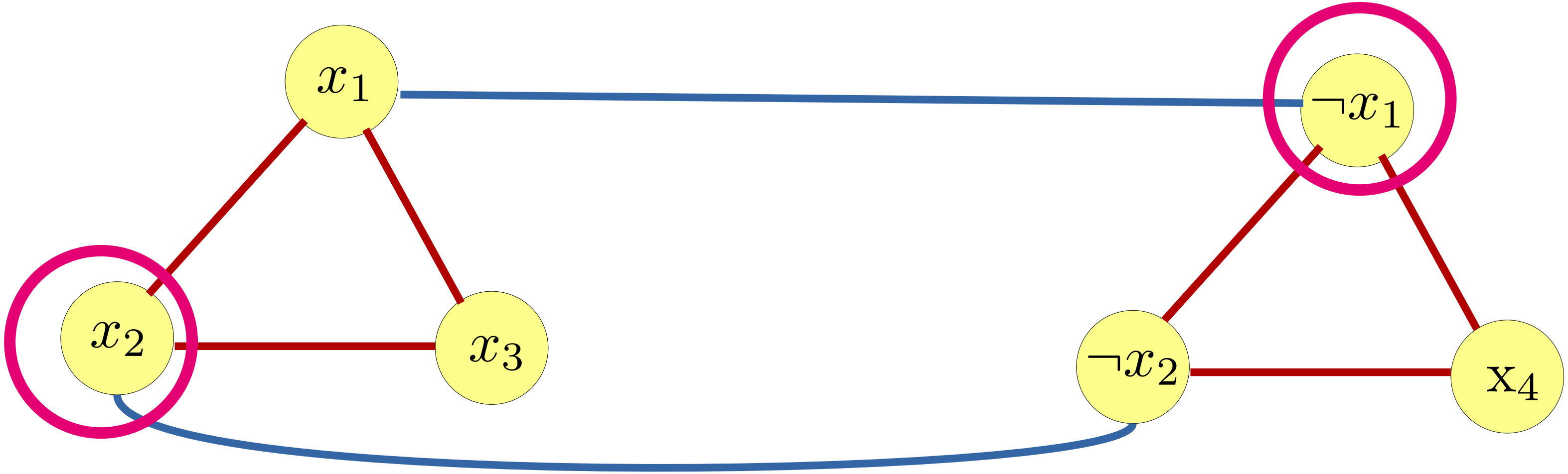
Backward Certificate Construction

$$(x_1 \vee x_2 \vee x_3)$$

$$x_2 = T$$

$$(\neg x_1 \vee \neg x_2 \vee x_4)$$

$$x_1 = F$$



Does this also works?

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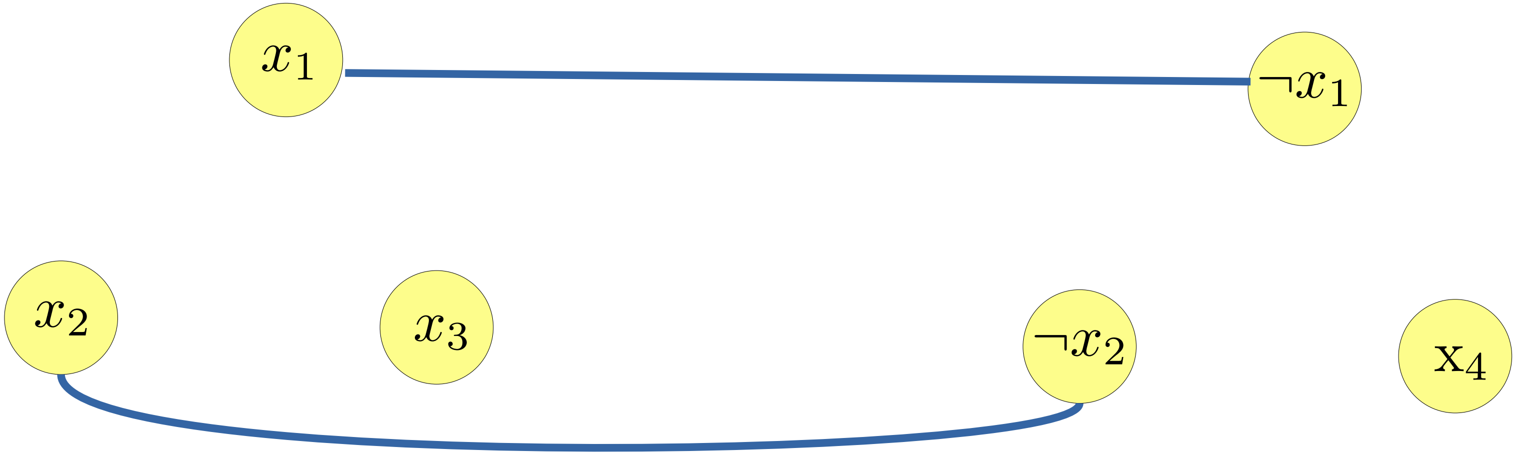

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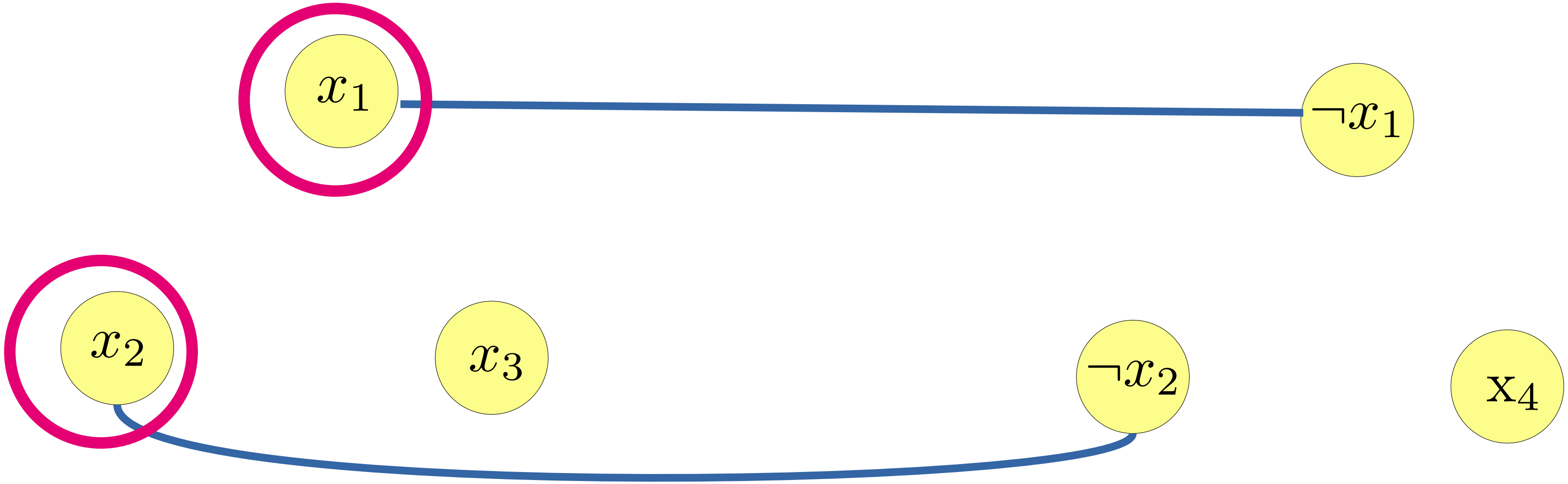
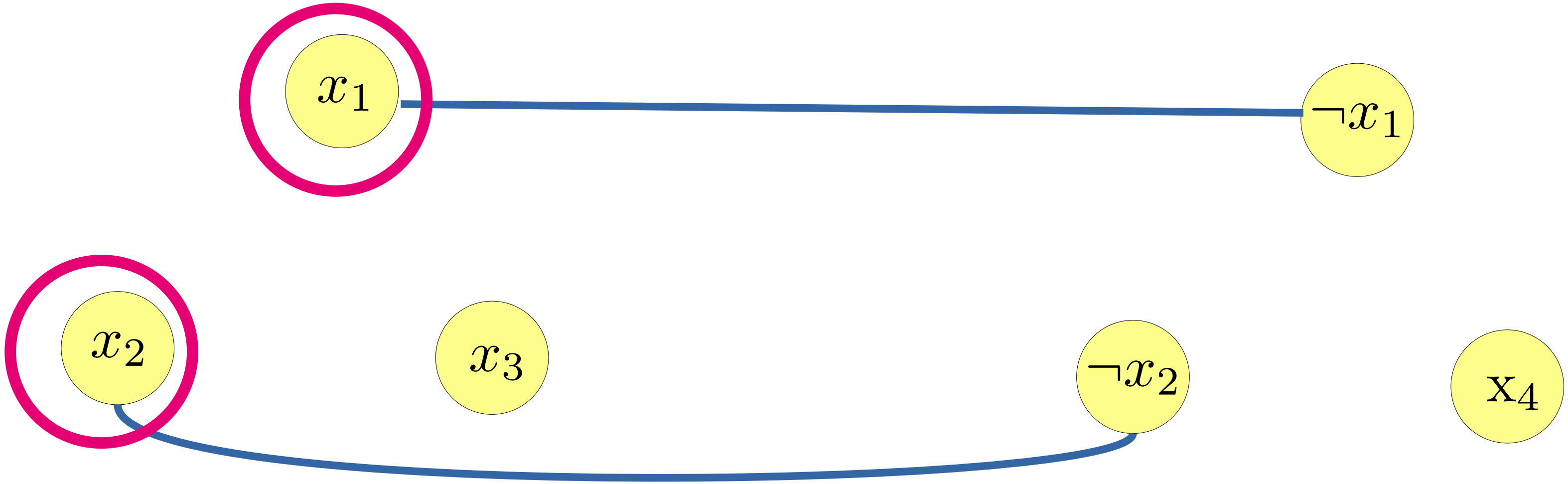


Does this also works?

$$(x_1 \vee x_2 \vee x_3) \qquad x_1 = T \quad x_2 = T$$

$$(\neg x_1 \vee \neg x_2 \vee x_4) \qquad x_3 = x_4 = F$$



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$x_1 = T \quad x_2 = T$

$x_3 = x_4 = F$