

# "Good" vs. "Bad" Expressions

`; interp-expr FAE ... -> FAE-Value`

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- Does **interp-expr** produce a value for all expressions?

# "Good" vs. "Bad" Expressions

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- Does `interp-expr` produce a value for all expressions?
- Of course not!

# "Good" vs. "Bad" Expressions

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- Does `interp-expr` produce a value for all expressions?
- Of course not!
- `(interp-expr (parse '{5 5}))` etc ...

# "Good" vs. "Bad" Expressions

`; interp-expr FAE ... -> FAE-Value`

- Does `interp-expr` produce a value for all expressions?
- Of course not!
- `(interp-expr (parse '{5 5}))` etc ...
- But do we know enough about expressions to tell before actually calling `interp-expr`?

# Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

# Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

- **Wrong answer: 0**

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{ + 1 2 }

- **Wrong answer: 0**
- **Wrong answer: 42**



# Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

- **Wrong answer: 0**
- **Wrong answer: 42**
- **Answer: 3**

# Quiz

- **Question #2:** What is the value of the following expression?

```
{+ fun 17 8}
```

# Quiz

- **Question #2:** What is the value of the following expression?

```
{+ fun 17 8}
```

- **Wrong answer: error**

# Quiz

- **Question #2:** What is the value of the following expression?

`{+ fun 17 8}`

- **Wrong answer: error**
- **Answer:** Trick question! `{+ fun 17 8}` is not an expression

# Language Grammar for Quiz

```
<MFAE> ::= <num>
| true
| false
| {+ <MFAE> <MFAE>}
| {- <MFAE> <MFAE>}
| {= <MFAE> <MFAE>}
| <id>
| {fun {<id>*} <MFAE>}
| {<MFAE> <MFAE>*}
| {if <MFAE> <MFAE> <MFAE>}
```

# Quiz

- Question #3: Is the following an expression?

```
{{fun {x y} 1} 7}
```

# Quiz

- Question #3: Is the following an expression?

```
{{fun {x y} 1} 7}
```

- Wrong answer: **No**

# Quiz

- **Question #3:** Is the following an expression?

`{{fun {x y} 1} 7}`

- **Wrong answer: No**
- **Answer: Yes** (according to our grammar)



# Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

# Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer:** `{fun {y} 1}` (according to some interpreters)

# Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer:** `{fun {y} 1}` (according to some interpreters)
- But no *real* language would accept `{{fun {x y} 1} 7}`

# Quiz

- **Question #4:** What is the value of the following expression?

`{{fun {x y} 1} 7}`

- **Answer:** `{fun {y} 1}` (according to some interpreters)
- But no *real* language would accept `{{fun {x y} 1} 7}`
- Let's agree to call `{{fun {x y} 1} 7}` an ***ill-formed expression*** because `{fun {x y} 1}` should be used only with two arguments
- Let's agree to never evaluate ill-formed expressions

# Quiz

- **Question #5:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

# Quiz

- **Question #5:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer: None** - the expression is ill-formed

# Quiz

- **Question #6:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

# Quiz

- **Question #6:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

- **Answer: Yes**



# Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

# Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

- **Answer: None** - it produces an error:

*numeric operation expected number*

# Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

- **Answer: None** - it produces an error:

*numeric operation expected number*

- Let's agree that a **fun** expression cannot be inside a **+** form

# Quiz

- **Question #8:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

# Quiz

- **Question #8:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

- **Answer: No**

# Quiz

- **Question #9:** Is the following a well-formed expression?

`{+ {{fun {x} x} 7} 5}`

# Quiz

- **Question #9:** Is the following a well-formed expression?

{+ {{fun {x} x} 7} 5}

- **Answer:** Depends on what we meant by *inside* in our most recent agreement
  - *Anywhere inside* - **No**
  - *Immediately inside* - **Yes**

# Quiz

- **Question #9:** Is the following a well-formed expression?

`{+ {{fun {x} x} 7} 5}`

- **Answer:** Depends on what we meant by *inside* in our most recent agreement
  - *Anywhere inside* - **No**
  - *Immediately inside* - **Yes**
- Since our interpreter produces **12**, and since that result makes sense, let's agree on *immediately inside*



# Quiz

- **Question #10:** Is the following a well-formed expression?

```
{+ {{fun {x} x} {fun {y} y}} 5}
```

# Quiz

- **Question #10:** Is the following a well-formed expression?

```
{+ {{fun {x} x} {fun {y} y}} 5}
```

- **Answer: Yes**, but we don't want it to be!

# Quiz

- **Question #11:** Is it possible to define **well-formed** (as a decidable property) so that we reject all expressions that produce errors?

# Quiz

- **Question #11:** Is it possible to define **well-formed** (as a decidable property) so that we reject all expressions that produce errors?
- **Answer: Yes:** reject *all* expressions!

# Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?

# Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?
- **Answer: No**

# Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?

- **Answer: No**

```
{+ 1 {if ... 1 {fun {x} x}}}
```

- If we always knew whether ... produces true or false, we could solve the halting problem

# Types

- Solution to our dilemma
  - In the process of rejecting expressions that are certainly bad, also reject some expressions that are good

```
{+ 1 {if {prime? 131101} 1 {fun {x} x}}}
```



# Types

- Solution to our dilemma
  - In the process of rejecting expressions that are certainly bad, also reject some expressions that are good

```
{+ 1 {if {prime? 131101} 1 {fun {x} x}}}
```

- Overall strategy:
  - Assign a **type** to each expression *without evaluating*
  - Compute the type of a complex expression based on the types of its subexpressions

# Types

`1 : num`

`true : bool`

# Types

`1 : num`

`true : bool`

`{+ 1 2}`

# Types

`1 : num`

`true : bool`

`{+ 1 2}`

`num`

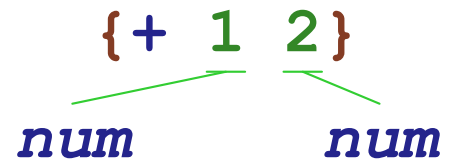


# Types

`1 : num`

`true : bool`

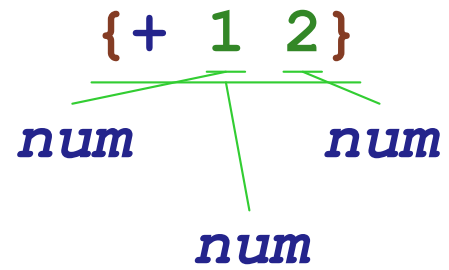
`{+ 1 2}`  
*num*                      *num*



# Types

`1 : num`

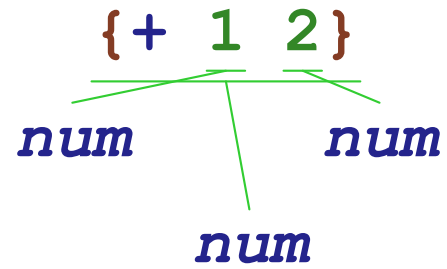
`true : bool`



# Types

`1 : num`

`true : bool`



`{+ 1 false}`

# Types

`1 : num`

`true : bool`

`{+ 1 2}`  
  
`num`      `num`  
`num`

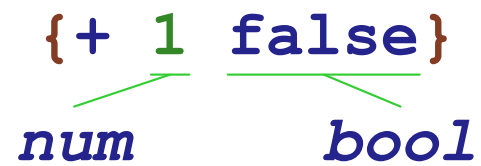
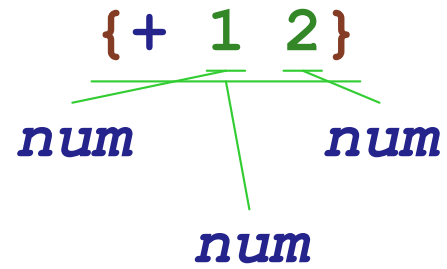
`{+ 1 false}`  
  
`num`



# Types

`1 : num`

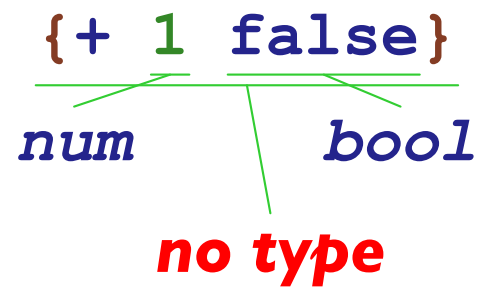
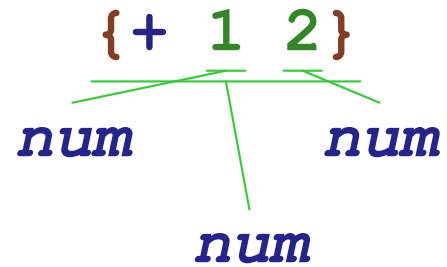
`true : bool`



# Types

`1 : num`

`true : bool`



# Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

---

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

# Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

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$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

# Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

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$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

---

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num}$

$2 : \text{num}$

---

$\{ + 1 2 \} : \text{num}$

# Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

---

$\{+ \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2\} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num}$

$2 : \text{num}$

---

$\{+ 1 2\} : \text{num}$

$1 : \text{num}$

$\text{false} : \text{bool}$

---

$\{+ 1 \text{false}\} : \text{no type}$

# Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

---

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$2 : \text{num}$

---

$\{ + 1 2 \} : \text{num}$

$3 : \text{num}$

---

$\{ + \{ + 1 2 \} 3 \} : \text{num}$

# Types: Conditionals

```
{if true 1 2}
```



# Types: Conditionals

```
{if true 1 2}
```

*bool*

# Types: Conditionals

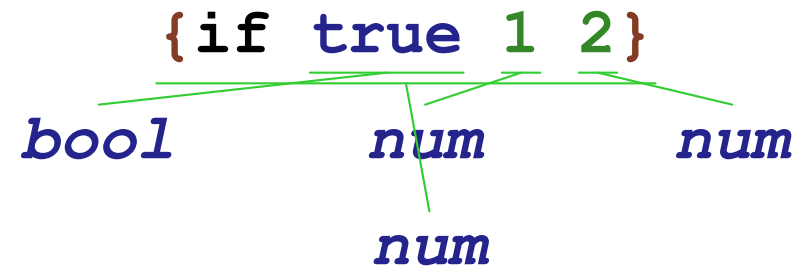
```
{if true 1 2}
```

*bool*                      *num*

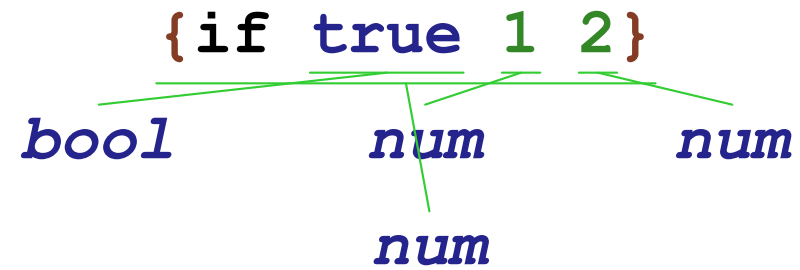
# Types: Conditionals



# Types: Conditionals

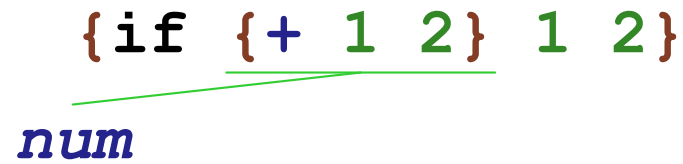
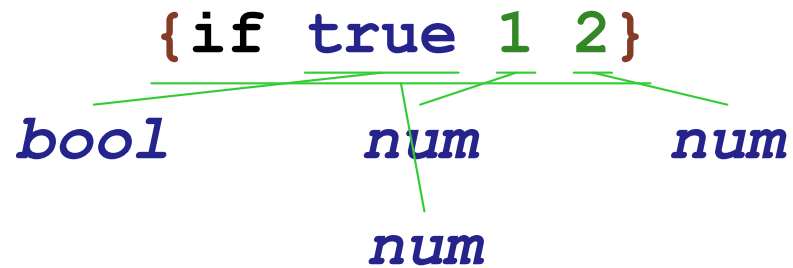


# Types: Conditionals

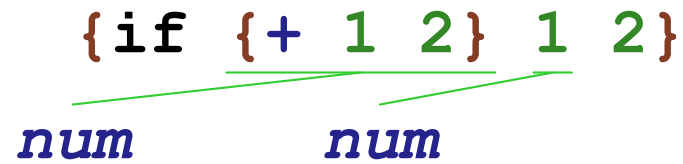
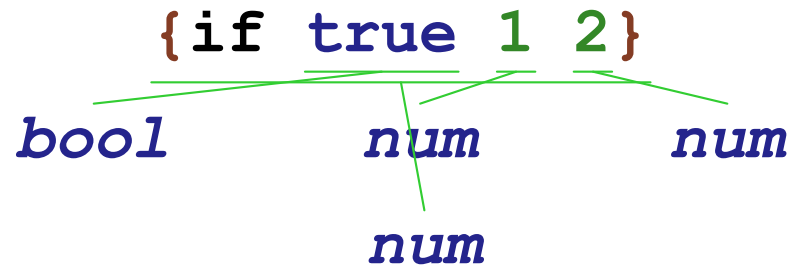


`{if {+ 1 2} 1 2}`

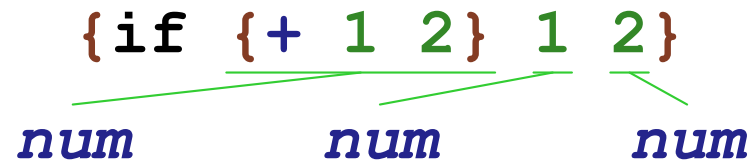
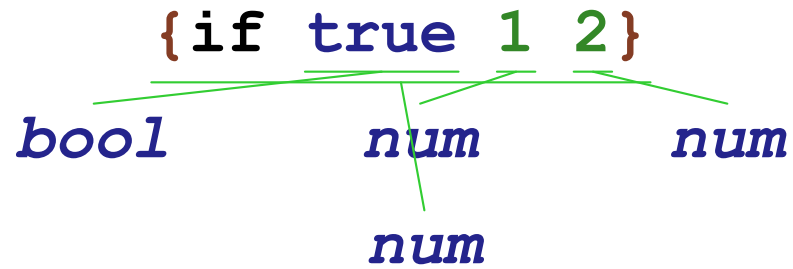
# Types: Conditionals



# Types: Conditionals

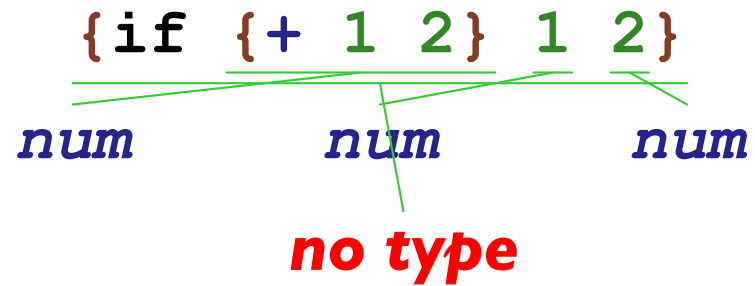
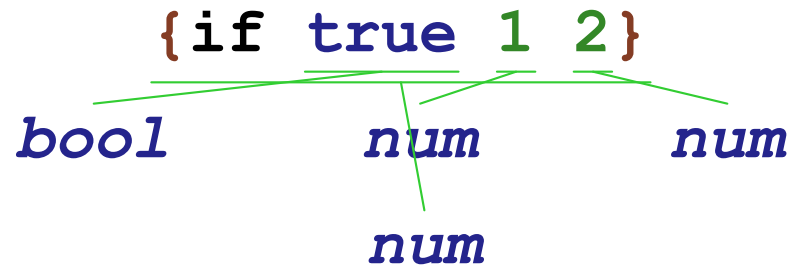


# Types: Conditionals

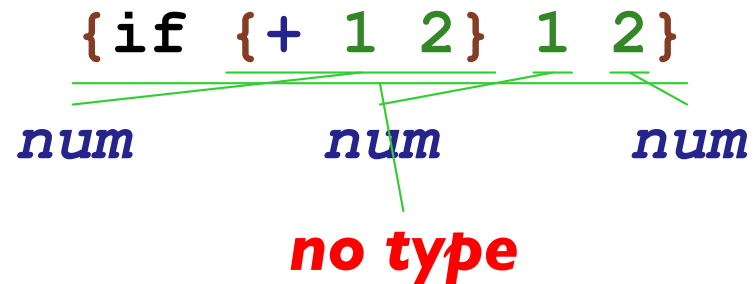
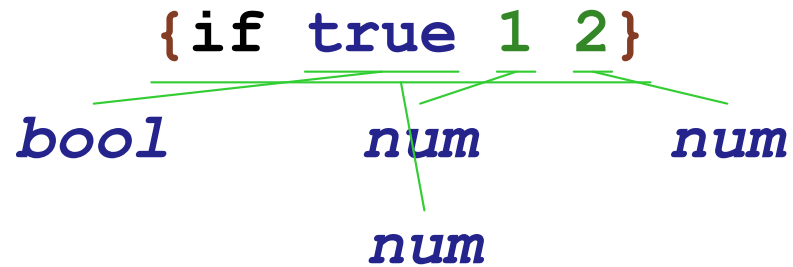




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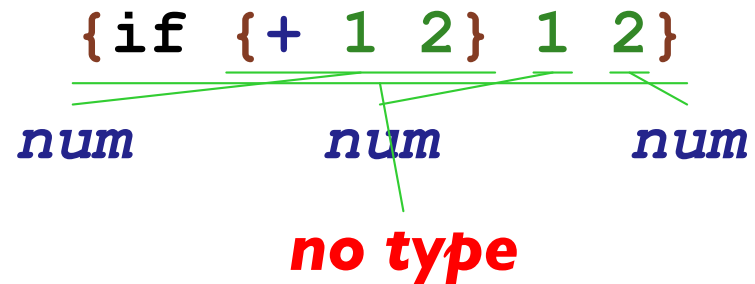
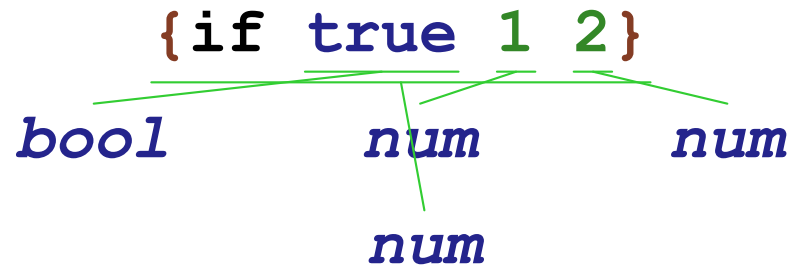


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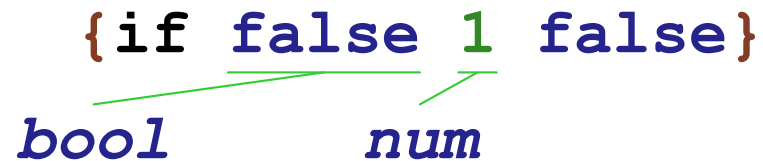
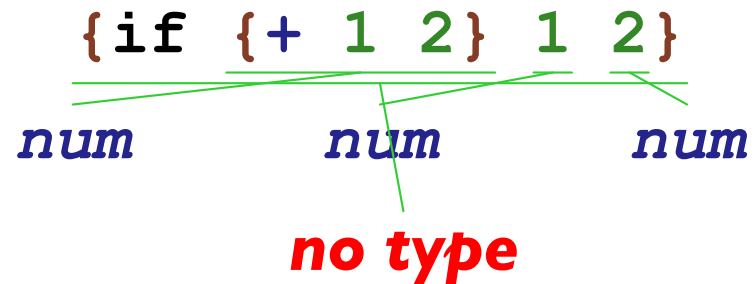
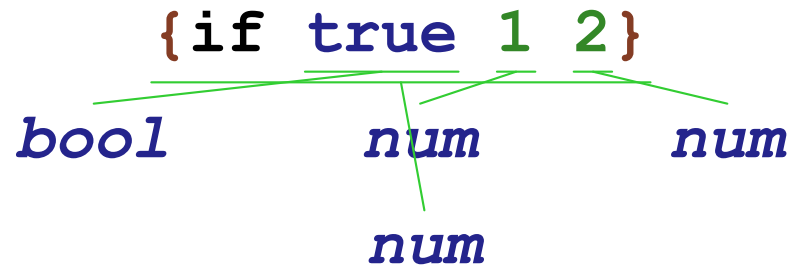


`{if false 1 false}`

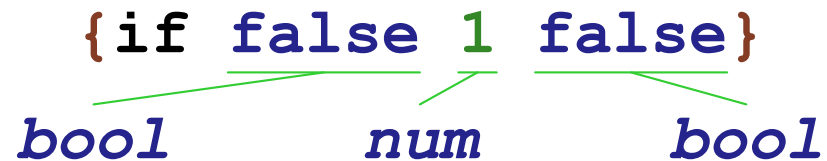
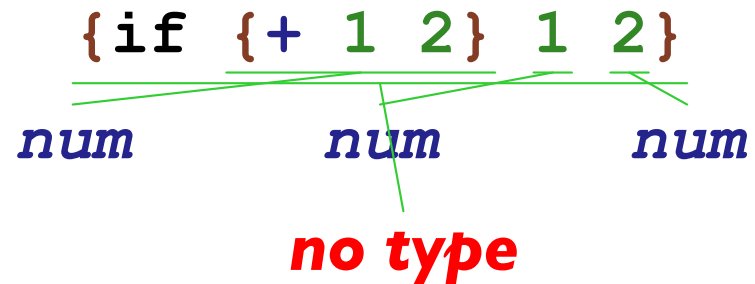
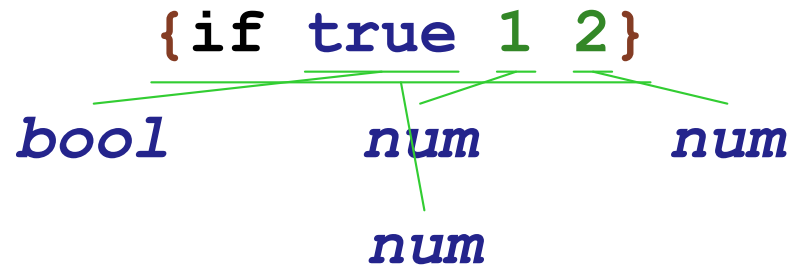
# Types: Conditionals



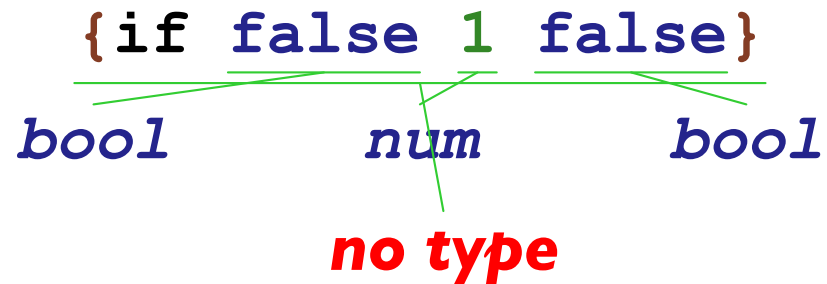
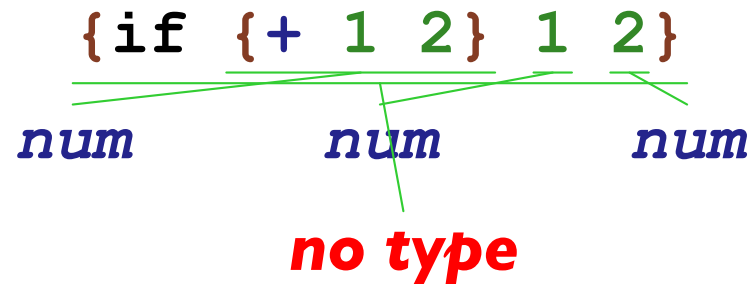
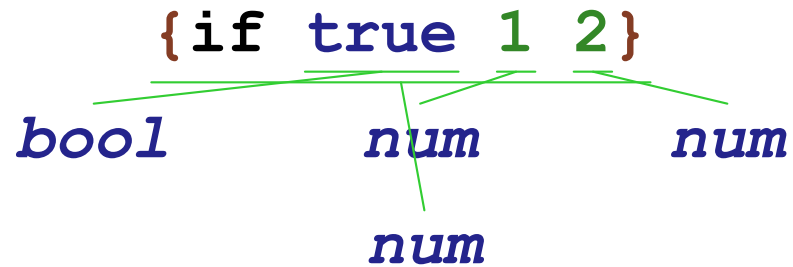
# Types: Conditionals



# Types: Conditionals



# Types: Conditionals



# Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \textit{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \textit{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \textit{type} \rangle_0}{\{\textit{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \textit{type} \rangle_0}$$

# Conditional Type Rules

$\langle \text{MFAE} \rangle_1 : \text{bool}$        $\langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0$        $\langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0$

---

$\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0$

$\text{true} : \text{bool}$        $1 : \text{num}$        $2 : \text{num}$

---

$\{\text{if true } 1 \ 2\} : \text{num}$



# Conditional Type Rules

$\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0$

---

$\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0$

$\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}$

---

$\{\text{if true } 1 \ 2\} : \text{num}$

$\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}$

---

$\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}$

# Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$
$$\frac{\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if true } 1 \ 2\} : \text{num}}$$
$$\frac{\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}}$$
$$\frac{\text{false} : \text{bool} \quad 1 : \text{num} \quad \text{false} : \text{bool}}{\{\text{if false } 1 \ \text{false}\} : \text{no type}}$$

# Types: Variables and Functions

**x** : **no type**

# Types: Variables and Functions

`x : no type`

`{fun {x : bool} x}`

# Types: Variables and Functions

**x** : **no type**

```
{fun {x : bool} x}
      bool
```

# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

---

`bool`

`(bool → bool)`

# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

---

`bool`

`(bool → bool)`

`{fun {x : bool} {if x 1 2}}`

# Types: Variables and Functions

**x : no type**

```
{fun {x : bool} x}
```

---

*bool*

*(bool → bool)*

```
{fun {x : bool} {if x 1 2}}
```

---

*bool*



# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

*bool*

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

*bool*

*num*

# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

*bool*

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

*bool*

*num*

*num*

# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

*bool*

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

*bool*

*num*

*num*

*num*

# Types: Variables and Functions

**x : no type**

`{fun {x : bool} x}`

*bool*

*(bool → bool)*

`{fun {x : bool} {if x 1 2}}`

*bool*

*num*

*num*

*num*

*(bool → num)*

# Variable and Function Type Rules

$$[ \dots \langle \text{id} \rangle \leftarrow \tau \dots ] \vdash \langle \text{id} \rangle : \tau$$

$$\Gamma [ \langle \text{id} \rangle \leftarrow \tau_1 ] \vdash \mathbf{e} : \tau_2$$

---

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$

Abbreviations:  $\tau = \langle \text{type} \rangle$     $\mathbf{e} = \langle \text{MFAE} \rangle$     $\Gamma = \langle \text{env} \rangle$

# Variable and Function Type Rules

$$[ \dots \langle \text{id} \rangle \leftarrow \tau \dots ] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [ \langle \text{id} \rangle \leftarrow \tau_1 ] \vdash \mathbf{e} : \tau_2$$

---

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$

# Variable and Function Type Rules

$$[ \dots \langle \text{id} \rangle \leftarrow \tau \dots ] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [ \langle \text{id} \rangle \leftarrow \tau_1 ] \vdash \mathbf{e} : \tau_2$$

---

$$\Gamma \vdash \{ \mathbf{fun} \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \mathbf{no\ type}$$
$$[ \mathbf{x} \leftarrow \mathbf{bool} ] \vdash \mathbf{x} : \mathbf{bool}$$

---

$$\emptyset \vdash \{ \mathbf{fun} \{ \mathbf{x} : \mathbf{bool} \} \mathbf{x} \} : (\mathbf{bool} \rightarrow \mathbf{bool})$$

# Variable and Function Type Rules

$$[ \dots \langle \text{id} \rangle \leftarrow \tau \dots ] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [ \langle \text{id} \rangle \leftarrow \tau_1 ] \vdash \mathbf{e} : \tau_2$$

---

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[ \mathbf{x} \leftarrow \text{bool} ] \vdash \mathbf{x} : \text{bool}$$

---

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \mathbf{x} \} : (\text{bool} \rightarrow \text{bool})$$
$$[ \mathbf{x} \leftarrow \text{bool} ] \vdash \mathbf{x} : \text{bool} \quad [ \mathbf{x} \leftarrow \text{bool} ] \vdash \mathbf{1} : \text{num} \quad [ \mathbf{x} \leftarrow \text{bool} ] \vdash \mathbf{2} : \text{num}$$

---

$$[ \mathbf{x} \leftarrow \text{bool} ] \vdash \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} : \text{num}$$

---

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} \} : (\text{bool} \rightarrow \text{num})$$



# Revised Rules

$$\Gamma \vdash \langle \text{num} \rangle : \text{num}$$
$$\Gamma \vdash \text{true} : \text{bool}$$
$$\Gamma \vdash \text{false} : \text{bool}$$
$$\Gamma \vdash \mathbf{e}_1 : \text{num} \quad \Gamma \vdash \mathbf{e}_2 : \text{num}$$

---

$$\Gamma \vdash \{+ \mathbf{e}_1 \ \mathbf{e}_2\} : \text{num}$$
$$\Gamma \vdash \mathbf{e}_1 : \text{bool} \quad \Gamma \vdash \mathbf{e}_2 : \tau_0 \quad \Gamma \vdash \mathbf{e}_3 : \tau_0$$

---

$$\Gamma \vdash \{\mathbf{if} \ \mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3\} : \tau_0$$

# Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}
```

# Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}  
  (bool → num)
```

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{bool} \rightarrow \text{num} \quad \text{bool}}$

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$        $\text{bool}$

$\text{num}$

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$        $\text{bool}$

$\text{num}$

$\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}$

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$        $\text{bool}$

$\text{num}$

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num})}$

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$   
 $\text{num}$

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$



# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

*(bool → num)*      *bool*

*num*

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{\text{no type}}$

*(bool → num)*      *num*

**no type**

# Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$   
*num*

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$   
**no type**

{7 5}

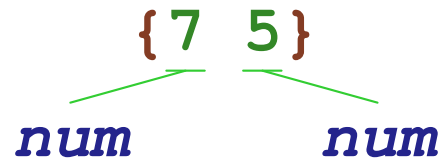
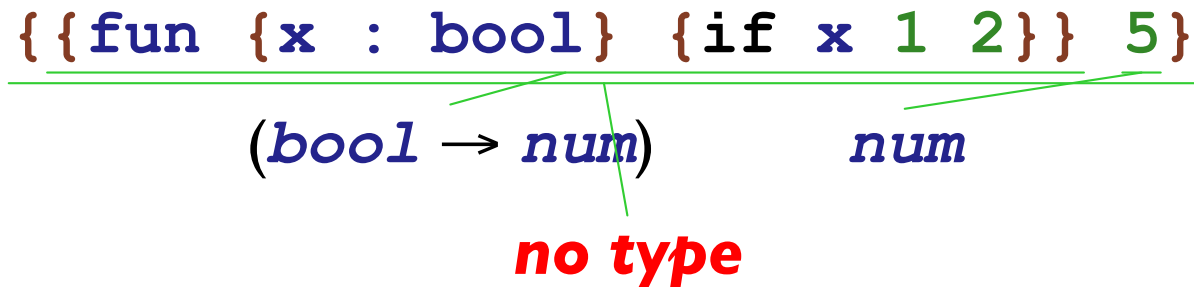
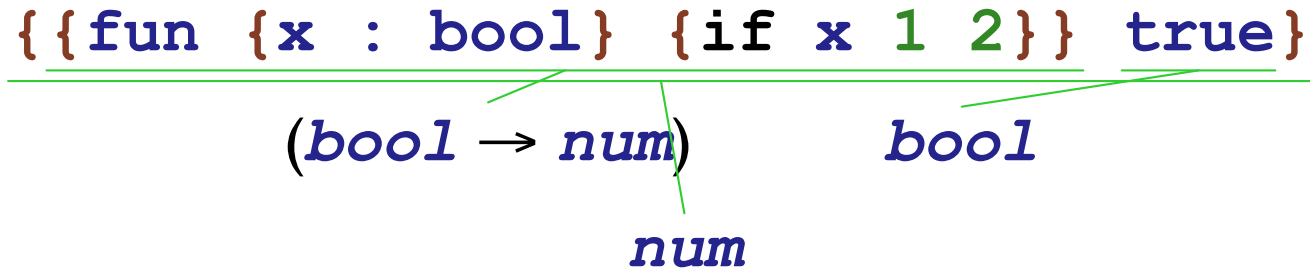
# Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}  
*(bool → num)*      *bool*  
*num*

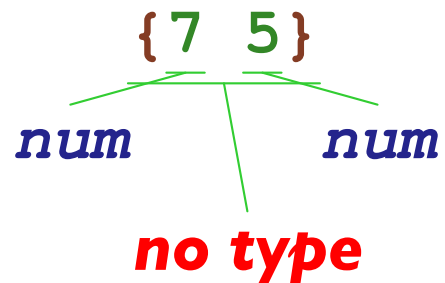
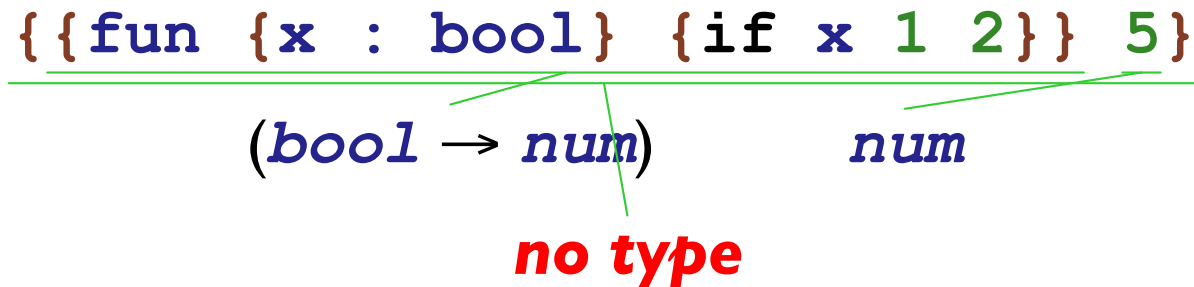
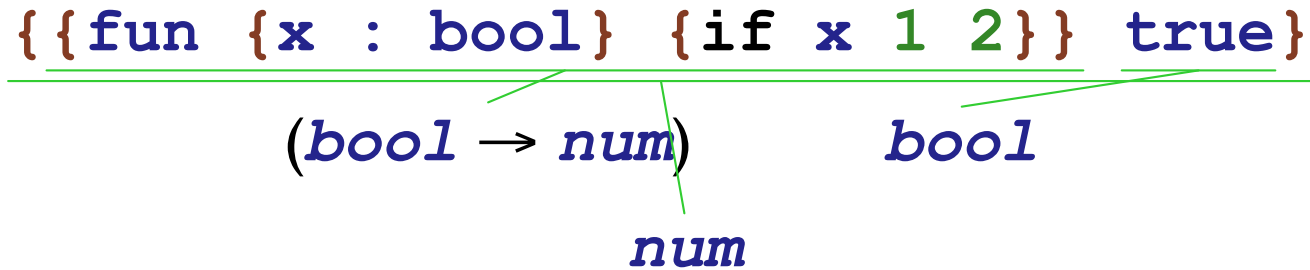
{{fun {x : bool} {if x 1 2}} 5}  
*(bool → num)*      *num*  
**no type**

*{7 5}*  
*num*

# Types: Function Calls



# Types: Function Calls



# Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

# Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

# Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathit{if} \ \mathbf{x} \ 1 \ 2\}\} : (\mathit{bool} \rightarrow \mathit{num}) \quad \emptyset \vdash \mathit{true} : \mathit{bool}}{\emptyset \vdash \{\{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathit{if} \ \mathbf{x} \ 1 \ 2\}\} \ \mathit{true}\} : \mathit{num}}$$

$$\frac{\emptyset \vdash \{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathit{if} \ \mathbf{x} \ 1 \ 2\}\} : (\mathit{bool} \rightarrow \mathit{num}) \quad \emptyset \vdash 5 : \mathit{num}}{\emptyset \vdash \{\{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathit{if} \ \mathbf{x} \ 1 \ 2\}\} \ 5\} : \mathbf{no \ type}}$$



# Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{num} \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{7 \ 5\} : \text{no type}}$$

# Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y} }
```

# Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y}}
```

*num*



# Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y}}
```

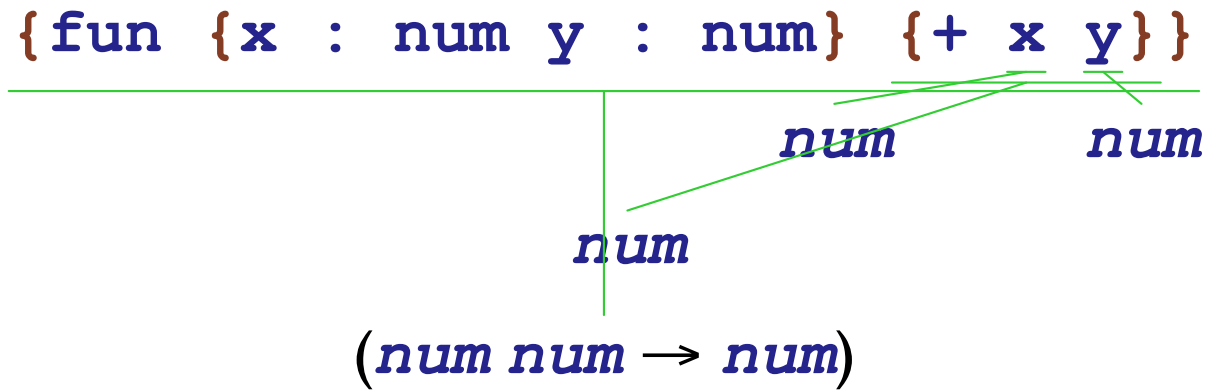
*num*                      *num*

# Types: Multiple Arguments

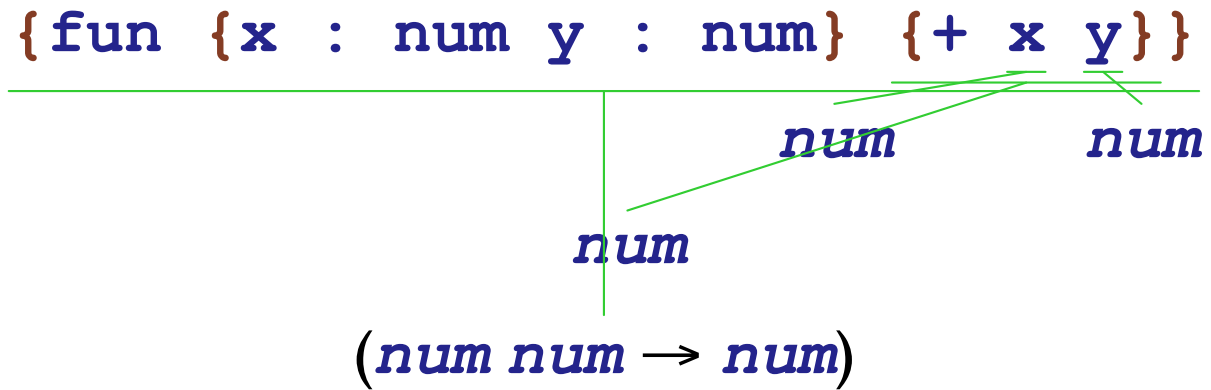
```
{ fun {x : num y : num} {+ x y}}
```

*num*                      *num*                      *num*

# Types: Multiple Arguments



# Types: Multiple Arguments



`{{fun {x : num y : num} {+ x y}} 5 6}`

# Types: Multiple Arguments

$\frac{\{ \text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num} \quad \text{num}}$

$(\text{num num} \rightarrow \text{num})$

$\frac{\{ \{ \text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} \ 5 \ 6 \}}{(\text{num num} \rightarrow \text{num})}$



# Types: Multiple Arguments

$\frac{\text{\{fun \{x : num y : num\} \{+ x y\}\}}{\text{\{num num \to num\}}$

$\frac{\text{\{\{fun \{x : num y : num\} \{+ x y\}\} 5 6\}}{\text{\{num num \to num\} num}}$

# Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

---

*num*                      *num*

*num*

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

---

$(num\ num \rightarrow num)$                       *num*                      *num*

# Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

---

$\text{num}$        $\text{num}$        $\text{num}$

$\text{num}$

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

---

$(\text{num } \text{num} \rightarrow \text{num})$        $\text{num}$        $\text{num}$

$\text{num}$

# Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

---

*num*      *num*

*num*

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

---

$(num\ num \rightarrow num)$       *num*      *num*

*num*

`{{fun {x : num y : num} {+ x y}} 5}`

# Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$   
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{\text{num} \quad \text{num}} \text{num}$   
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{\text{num}} \text{num}$   
 $(\text{num num} \rightarrow \text{num})$

# Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$   
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$

# Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$   
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$   
**no type**

# Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_1 \leftarrow \tau_1 \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{\text{fun } \{\langle \text{id} \rangle_1 : \tau_1 \dots \langle \text{id} \rangle_n : \tau_n\} \mathbf{e}\} : (\tau_1 \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \mathbf{e}_1 : \tau_1 \quad \dots \quad \Gamma \vdash \mathbf{e}_n : \tau_n}{\Gamma \vdash \{\mathbf{e}_0 \ \mathbf{e}_1 \ \dots \ \mathbf{e}_n\} : \tau_0}$$