

Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

- **Wrong answer: 0**

Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

- **Wrong answer: 0**
- **Wrong answer: 42**

Quiz

- **Question #1:** What is the value of the following expression?

{ + 1 2 }

- **Wrong answer: 0**
- **Wrong answer: 42**
- **Answer: 3**

Quiz

- **Question #2:** What is the value of the following expression?

```
{+ fun 17 8}
```

Quiz

- **Question #2:** What is the value of the following expression?

```
{+ fun 17 8}
```

- **Wrong answer: error**

Quiz

- **Question #2:** What is the value of the following expression?

`{+ fun 17 8}`

- **Wrong answer: error**
- **Answer:** Trick question! `{+ fun 17 8}` is not an expression

Language Grammar for Quiz

```
<MFAE> ::= <num>
| true
| false
| {+ <MFAE> <MFAE>}
| {- <MFAE> <MFAE>}
| {= <MFAE> <MFAE>}
| <id>
| {fun {<id>*} <MFAE>}
| {<MFAE> <MFAE>*}
| {if <MFAE> <MFAE> <MFAE>}
```


Quiz

- Question #3: Is the following an expression?

```
{{fun {x y} 1} 7}
```

Quiz

- Question #3: Is the following an expression?

```
{{fun {x y} 1} 7}
```

- Wrong answer: **No**

Quiz

- **Question #3:** Is the following an expression?

`{{fun {x y} 1} 7}`

- **Wrong answer: No**
- **Answer: Yes** (according to our grammar)

Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer:** `{fun {y} 1}` (according to some interpreters)

Quiz

- **Question #4:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer:** `{fun {y} 1}` (according to some interpreters)
- But no *real* language would accept `{{fun {x y} 1} 7}`

Quiz

- **Question #4:** What is the value of the following expression?

`{{fun {x y} 1} 7}`

- **Answer:** `{fun {y} 1}` (according to some interpreters)
- But no *real* language would accept `{{fun {x y} 1} 7}`
- Let's agree to call `{{fun {x y} 1} 7}` an ***ill-formed expression*** because `{fun {x y} 1}` should be used only with two arguments
- Let's agree to never evaluate ill-formed expressions

Quiz

- **Question #5:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```


Quiz

- **Question #5:** What is the value of the following expression?

```
{{fun {x y} 1} 7}
```

- **Answer: None** - the expression is ill-formed

Quiz

- **Question #6:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

Quiz

- **Question #6:** Is the following a well-formed expression?

`{+ {fun {} 1} 8}`

- **Answer: Yes**

Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

- **Answer: None** - it produces an error:

numeric operation expected number

Quiz

- **Question #7:** What is the value of the following expression?

```
{+ {fun {} 1} 8}
```

- **Answer: None** - it produces an error:

numeric operation expected number

- Let's agree that a **fun** expression cannot be inside a **+** form

Quiz

- **Question #8:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

Quiz

- **Question #8:** Is the following a well-formed expression?

```
{+ {fun {} 1} 8}
```

- **Answer: No**

Quiz

- **Question #9:** Is the following a well-formed expression?

`{+ {{fun {x} x} 7} 5}`

Quiz

- **Question #9:** Is the following a well-formed expression?

{+ {{fun {x} x} 7} 5}

- **Answer:** Depends on what we meant by *inside* in our most recent agreement
 - *Anywhere inside* - **No**
 - *Immediately inside* - **Yes**

Quiz

- **Question #9:** Is the following a well-formed expression?

{+ {{fun {x} x} 7} 5}

- **Answer:** Depends on what we meant by *inside* in our most recent agreement
 - *Anywhere inside* - **No**
 - *Immediately inside* - **Yes**
- Since our interpreter produces **12**, and since that result makes sense, let's agree on *immediately inside*

Quiz

- **Question #10:** Is the following a well-formed expression?

```
{+ {{fun {x} x} {fun {y} y}} 5}
```

Quiz

- **Question #10:** Is the following a well-formed expression?

```
{+ {{fun {x} x} {fun {y} y}} 5}
```

- **Answer: Yes**, but we don't want it to be!

Quiz

- **Question #11:** Is it possible to define **well-formed** (as a decidable property) so that we reject all expressions that produce errors?

Quiz

- **Question #11:** Is it possible to define **well-formed** (as a decidable property) so that we reject all expressions that produce errors?
- **Answer: Yes:** reject *all* expressions!

Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?

Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?
- **Answer: No**

Quiz

- **Question #12:** Is it possible to define **well-formed** (as a decidable property) so that we reject *only* expressions that produce errors?
- **Answer: No**

```
{+ 1 {if ... 1 {fun {x} x}}}
```

- If we always knew whether ... produces true or false, we could solve the halting problem

Types

- Solution to our dilemma
 - In the process of rejecting expressions that are certainly bad, also reject some expressions that are good

```
{+ 1 {if {prime? 131101} 1 {fun {x} x}}}
```

Types

- Solution to our dilemma
 - In the process of rejecting expressions that are certainly bad, also reject some expressions that are good

```
{+ 1 {if {prime? 131101} 1 {fun {x} x}}}
```

- Overall strategy:
 - Assign a **type** to each expression *without evaluating*
 - Compute the type of a complex expression based on the types of its subexpressions

Types

`1 : num`

`true : bool`

Types

`1 : num`

`true : bool`

`{+ 1 2}`

Types

`1 : num`

`true : bool`

`{+ 1 2}`


`num`



Types

`1 : num`

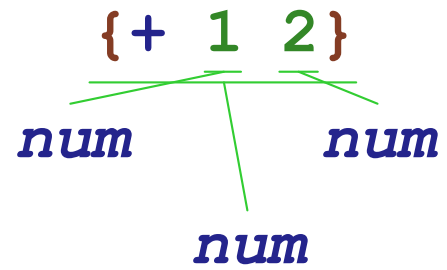
`true : bool`

`{+ 1 2}`

`num num`

Types

`1 : num`

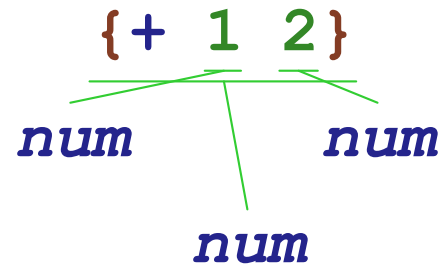
`true : bool`



Types

`1 : num`

`true : bool`



`{+ 1 false}`

Types

`1 : num`

`true : bool`

`{+ 1 2}`

`num` `num`
`num`

`{+ 1 false}`

`num`

Types

`1 : num`

`true : bool`

`{+ 1 2}`

`num` `num`
`num`

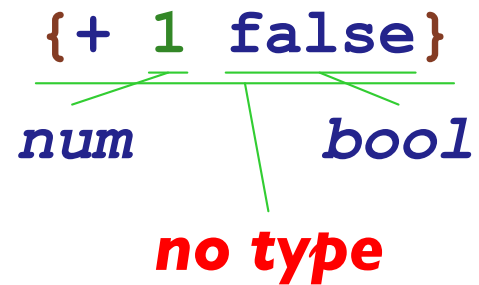
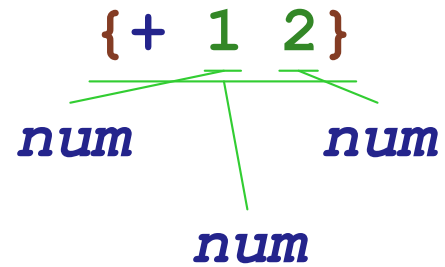
`{+ 1 false}`

`num` `bool`

Types

`1 : num`

`true : bool`



Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num}$

$2 : \text{num}$

$\{ + 1 2 \} : \text{num}$

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{+ \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2\} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num}$

$2 : \text{num}$

$\{+ 1 2\} : \text{num}$

$1 : \text{num}$

$\text{false} : \text{bool}$

$\{+ 1 \text{false}\} : \text{no type}$

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{ + \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \} : \text{num}$

$1 : \text{num}$

$2 : \text{num}$

$\{ + 1 2 \} : \text{num}$

$3 : \text{num}$

$\{ + \{ + 1 2 \} 3 \} : \text{num}$

Types: Conditionals

```
{if true 1 2}
```

Types: Conditionals

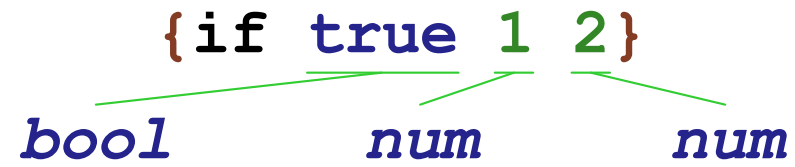
```
{if true 1 2}
```

bool

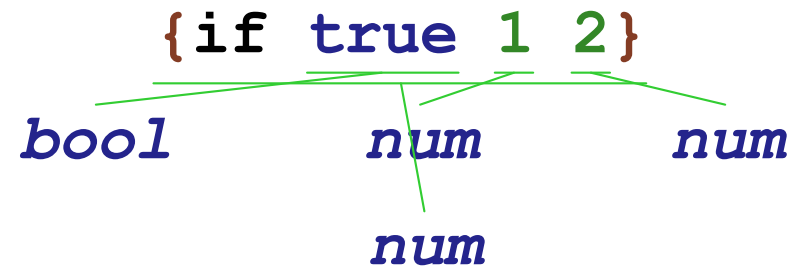
Types: Conditionals

```
      {if true 1 2}  
      ^      ^  
      |      |  
bool      num
```

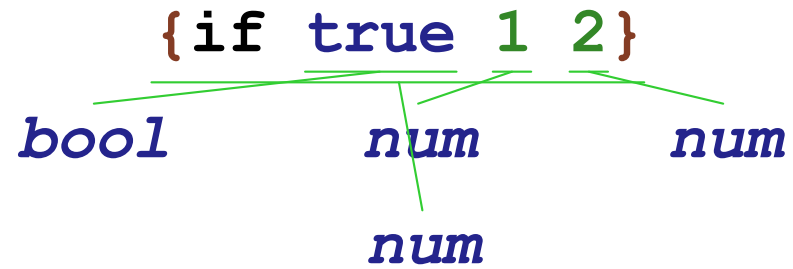
Types: Conditionals



Types: Conditionals



Types: Conditionals



`{if {+ 1 2} 1 2}`

Types: Conditionals

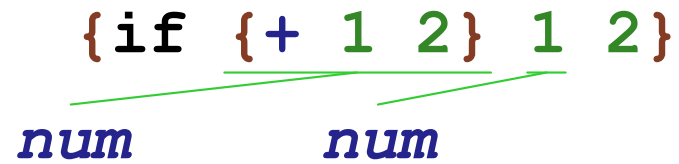
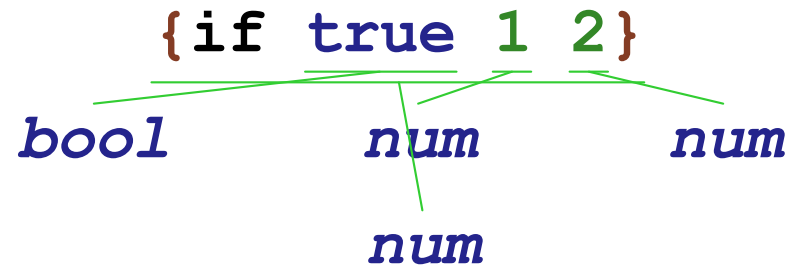
```
{if true 1 2}
```

bool *num* *num*
num

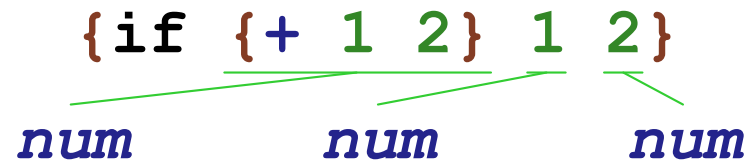
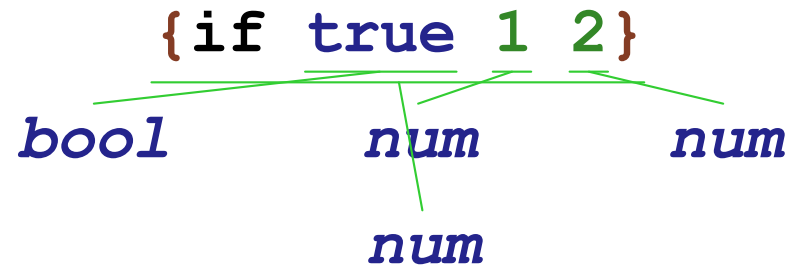
```
{if {+ 1 2} 1 2}
```

num

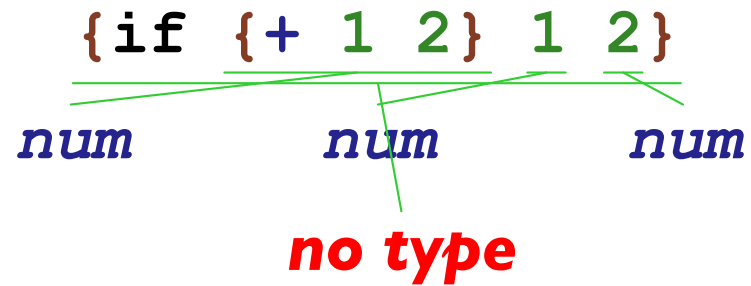
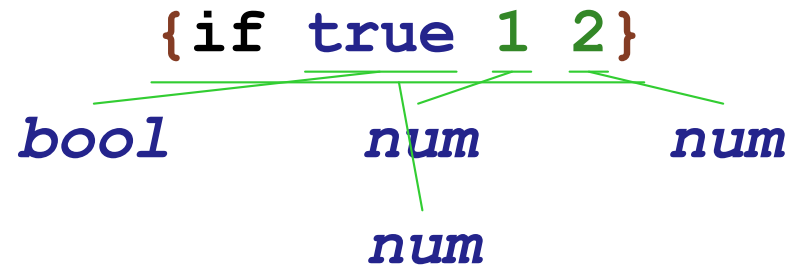
Types: Conditionals



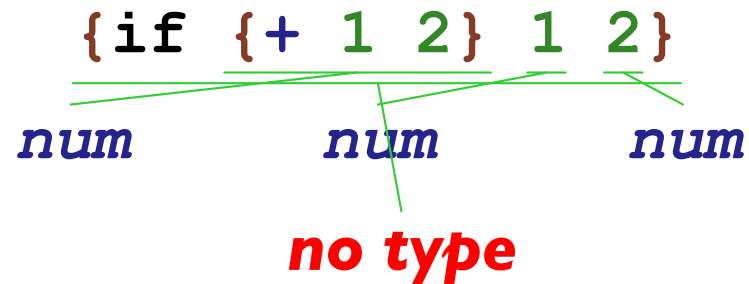
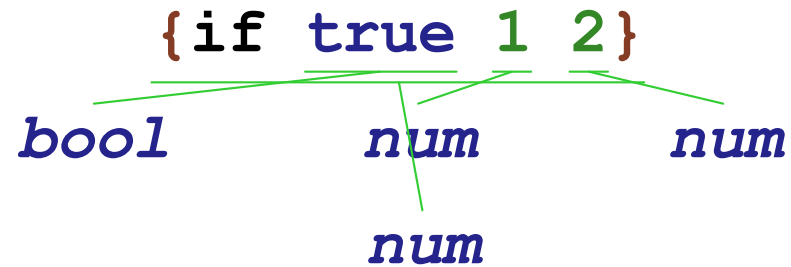
Types: Conditionals



Types: Conditionals

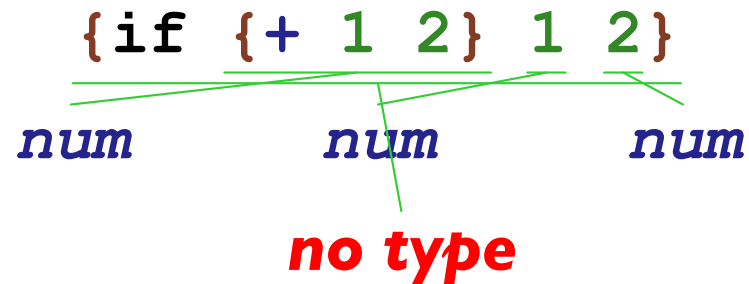
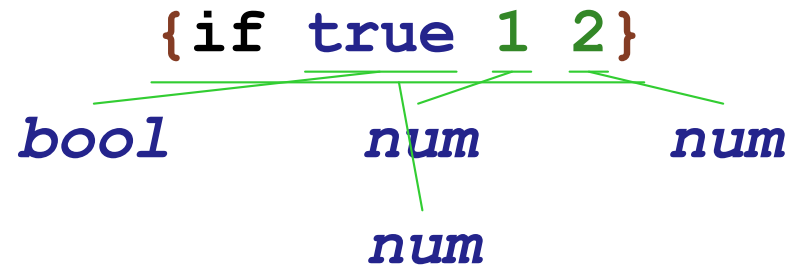


Types: Conditionals

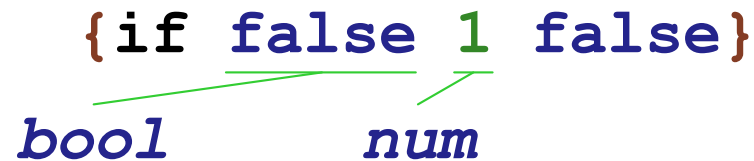
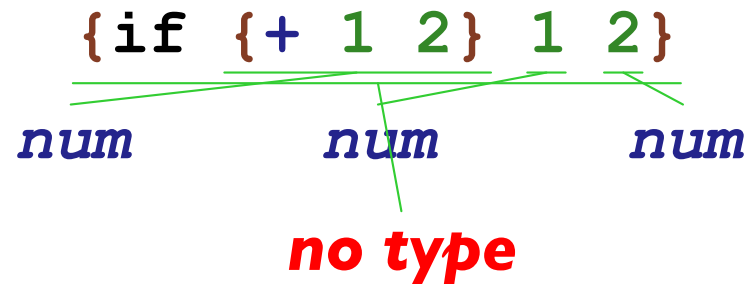
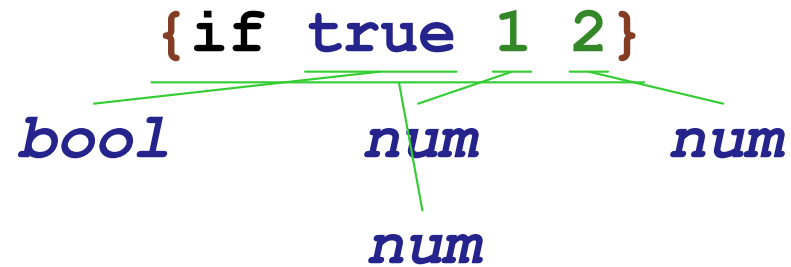


`{if false 1 false}`

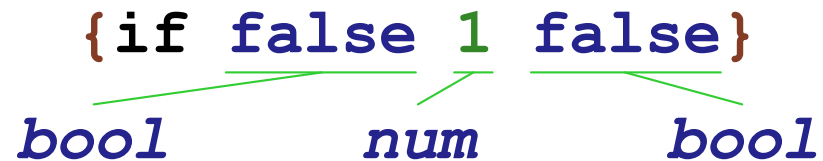
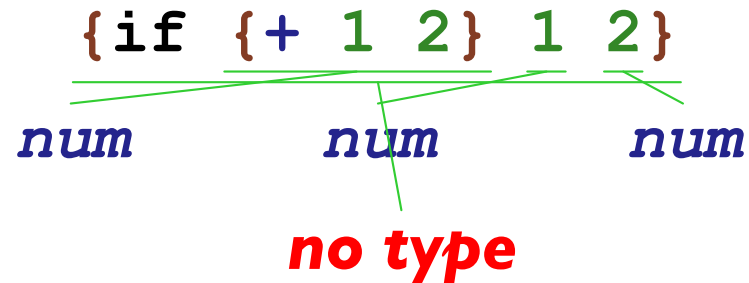
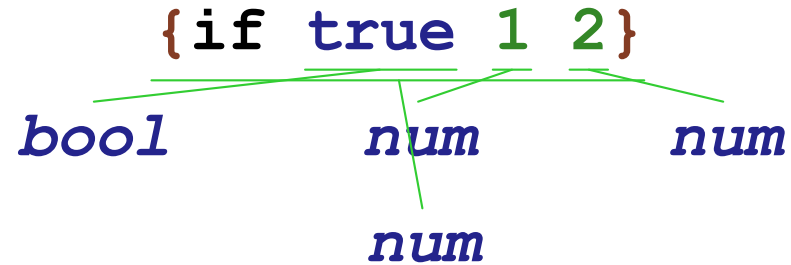
Types: Conditionals



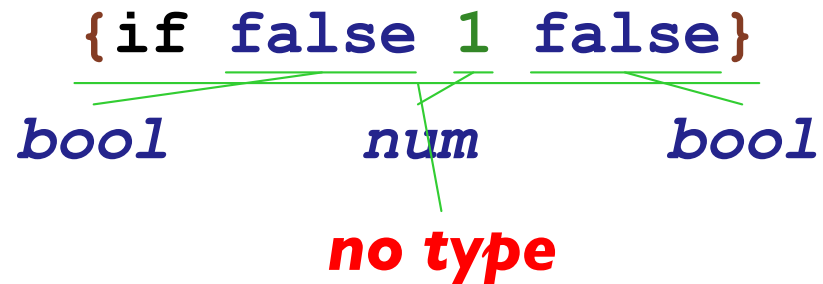
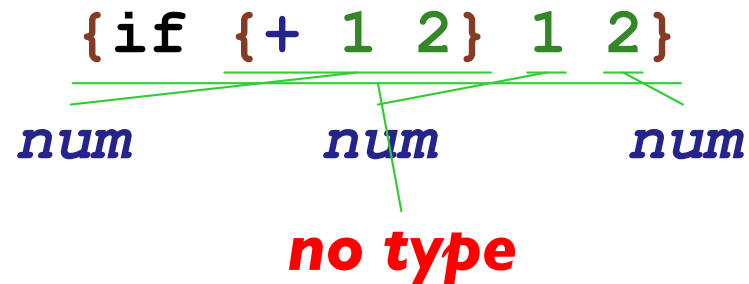
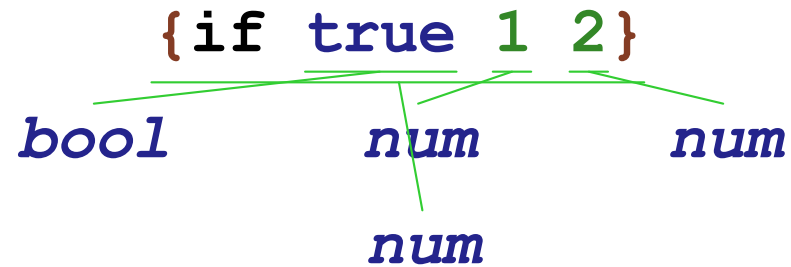
Types: Conditionals



Types: Conditionals



Types: Conditionals



Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \ \langle \text{MFAE} \rangle_2 \ \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$

Conditional Type Rules

$\langle \text{MFAE} \rangle_1 : \text{bool}$ $\langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0$ $\langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0$

$\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0$

$\text{true} : \text{bool}$ $1 : \text{num}$ $2 : \text{num}$

$\{\text{if true } 1 \ 2\} : \text{num}$

Conditional Type Rules

$\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0$

$\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0$

$\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}$

$\{\text{if true } 1 \ 2\} : \text{num}$

$\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}$

$\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}$

Conditional Type Rules

$\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0$

$\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0$

$\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}$

$\{\text{if true } 1 \ 2\} : \text{num}$

$\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}$

$\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}$

$\text{false} : \text{bool} \quad 1 : \text{num} \quad \text{false} : \text{bool}$

$\{\text{if false } 1 \ \text{false}\} : \text{no type}$

Types: Variables and Functions

x : **no type**

Types: Variables and Functions

`x : no type`

`{fun {x : bool} x}`

Types: Variables and Functions

x : **no type**

```
{fun {x : bool} x}  
      bool
```


Types: Variables and Functions

x : **no type**

{fun {x : bool} x}

bool

(bool → bool)

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

`bool`
 $(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool
 $(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

bool

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool
 $(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

bool *num*

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool
 $(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`
bool *num* *num*

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool
 $(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`
bool *num* *num*
num

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

(bool → bool)

`{fun {x : bool} {if x 1 2}}`

bool

num

num

num

(bool → num)

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$

Abbreviations: $\tau = \langle \text{type} \rangle$ $\mathbf{e} = \langle \text{MFAE} \rangle$ $\Gamma = \langle \text{env} \rangle$

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \mathbf{x} \} : (\text{bool} \rightarrow \text{bool})$$

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \mathbf{x} \} : (\text{bool} \rightarrow \text{bool})$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool} \quad [\mathbf{x} \leftarrow \text{bool}] \vdash 1 : \text{num} \quad [\mathbf{x} \leftarrow \text{bool}] \vdash 2 : \text{num}$$

$$[\mathbf{x} \leftarrow \text{bool}] \vdash \{ \text{if } \mathbf{x} \ 1 \ 2 \} : \text{num}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \{ \text{if } \mathbf{x} \ 1 \ 2 \} \} : (\text{bool} \rightarrow \text{num})$$

Revised Rules

$$\Gamma \vdash \langle \text{num} \rangle : \text{num}$$
$$\Gamma \vdash \text{true} : \text{bool}$$
$$\Gamma \vdash \text{false} : \text{bool}$$
$$\Gamma \vdash \mathbf{e}_1 : \text{num} \quad \Gamma \vdash \mathbf{e}_2 : \text{num}$$

$$\Gamma \vdash \{+ \mathbf{e}_1 \ \mathbf{e}_2\} : \text{num}$$
$$\Gamma \vdash \mathbf{e}_1 : \text{bool} \quad \Gamma \vdash \mathbf{e}_2 : \tau_0 \quad \Gamma \vdash \mathbf{e}_3 : \tau_0$$

$$\Gamma \vdash \{\mathbf{if} \ \mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3\} : \tau_0$$

Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}
```

Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}  
  (bool → num)
```

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}
(bool → num) *bool*

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$
 num

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$ bool

num

$\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$ bool

num

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num})}$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$
 num

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$
num

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$
no type

Types: Function Calls

`{{fun {x : bool} {if x 1 2}} true}`

$(bool \rightarrow num)$ $bool$

num

`{{fun {x : bool} {if x 1 2}} 5}`

$(bool \rightarrow num)$ num

no type

`{7 5}`

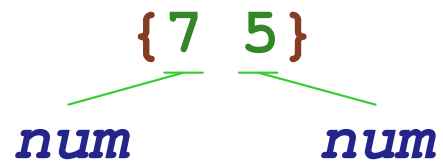
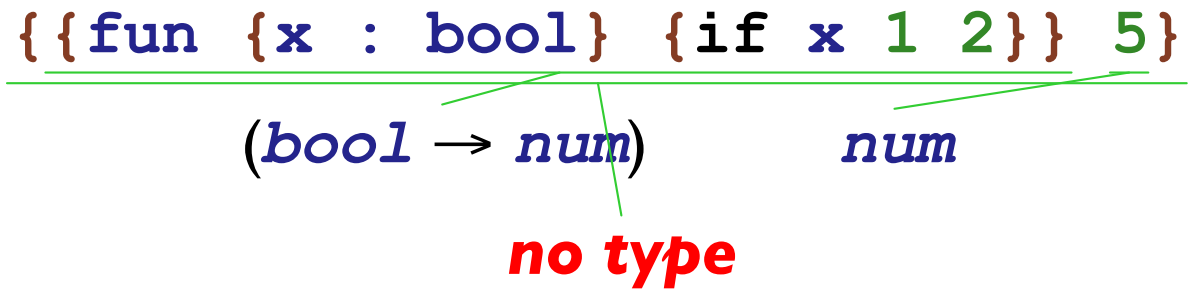
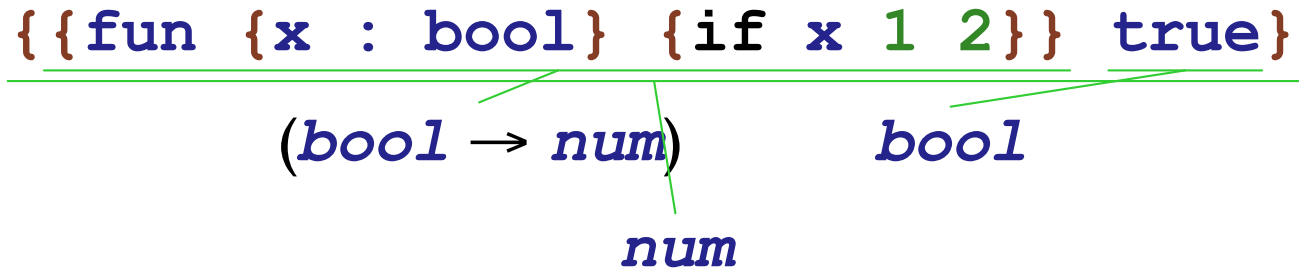
Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$
num

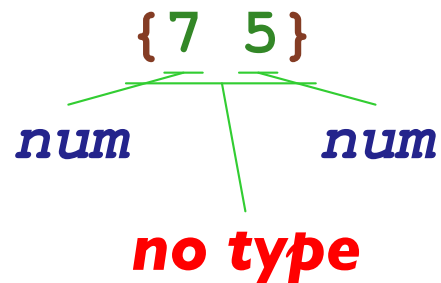
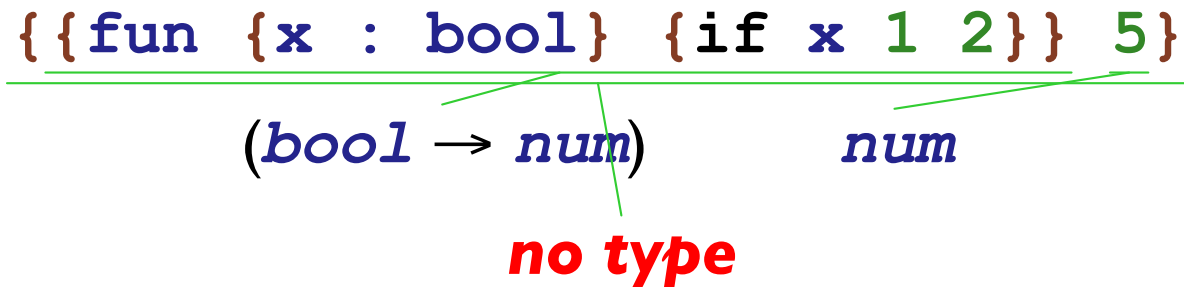
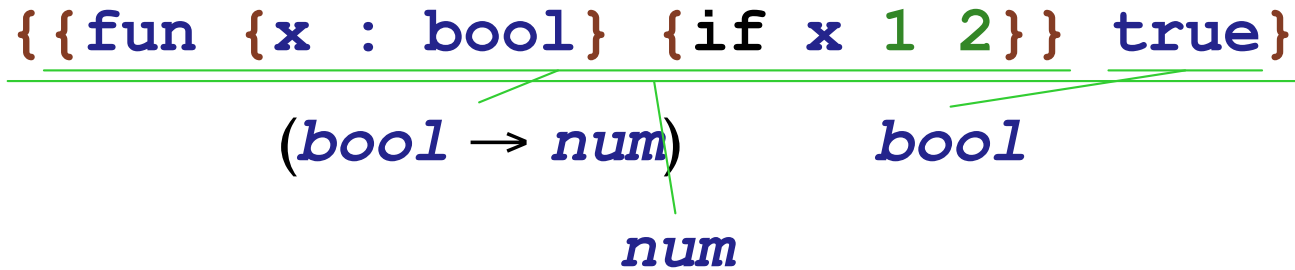
$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$
no type

$\{\text{7 } \text{5}\}$
num

Types: Function Calls



Types: Function Calls



Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{num} \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{7 \ 5\} : \text{no type}}$$


Types: Multiple Arguments

```
{fun {x : num y : num} {+ x y}}
```

Types: Multiple Arguments

```
{fun {x : num y : num} {+ x y}}
```

num



Types: Multiple Arguments

```
{fun {x : num y : num} {+ x y}}
```

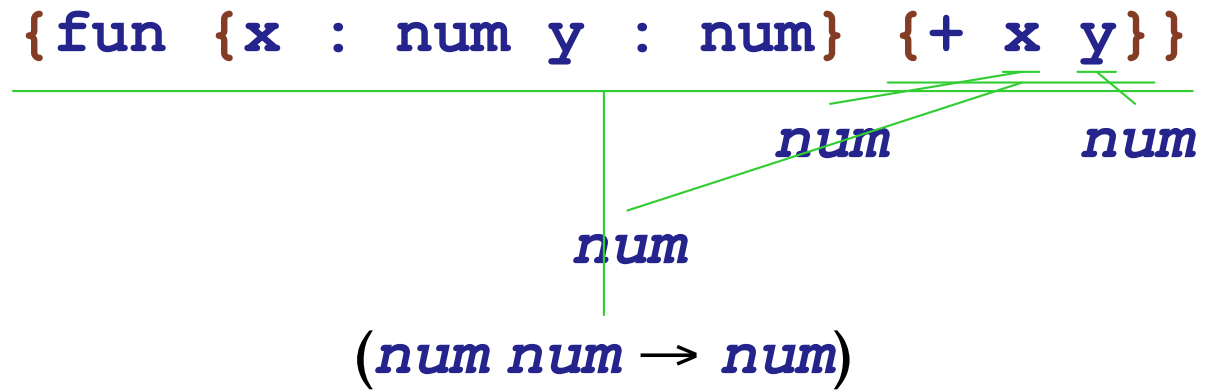
num *num*

Types: Multiple Arguments

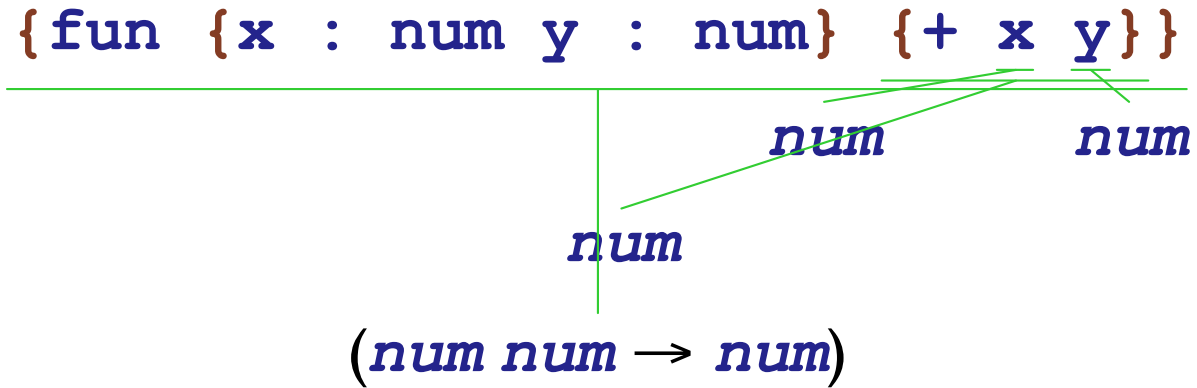
```
{fun {x : num y : num} {+ x y}}
```

num *num* *num*

Types: Multiple Arguments



Types: Multiple Arguments



`{{fun {x : num y : num} {+ x y}} 5 6}`

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num} \quad \text{num}}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num})}$

Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

num *num*

num

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

$(num\ num \rightarrow num)$ *num*

Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

num *num*

num

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

$(num\ num \rightarrow num)$ *num* *num*

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}}$
 num
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}}$
 num

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num})}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num} \quad \text{num}}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}}$
 num

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$
no type

Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_l \leftarrow \tau_l \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle_l : \tau_l \dots \langle \text{id} \rangle_n : \tau_n \} \mathbf{e} \} : (\tau_l \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_l \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \mathbf{e}_l : \tau_l \quad \dots \quad \Gamma \vdash \mathbf{e}_n : \tau_n}{\Gamma \vdash \{ \mathbf{e}_0 \ \mathbf{e}_l \ \dots \ \mathbf{e}_n \} : \tau_0}$$