On Proportional Power Sharing Mechanisms for Secondary Spectrum Markets

Randall A. Berry
Dept. of EECS
Northwestern University
Evanston, IL USA
e-mail: rberry@eecs.northwestern.edu

***Abstract*—We consider a proportional sharing mechanism for allocating spectrum to secondary users. Each user bids for a portion of the received power at a measurement point and receives a share that is proportional to its bid while paying a charge equal to the bid. The users then transmit over a common band treating all interference as noise. Under this mechanism, we model the secondary users as players is a bidding game. The players' interaction in this game is complicated due to the interference among them. We characterize the existence of a Nash equilibrium for both price taking and price anticipating users.**

I. Introduction

There has been a growing interest in exploring new approaches for more efficiently allocating wireless spectrum. One general class of such approaches is based on using markets to enable a spectrum owner (i.e., a primary license holder) to temporarily lease spectrum access to secondary users during periods when the owner does not need to fully utilize the spectrum, e.g. [1]–[7]. Such leasing could also be facilitated by a spectrum broker as in [8], [9].[1] When there are many "small" secondary users, it may be desirable to allow multiple secondary users to access the spectrum simultaneously.[2] Here, we consider one such model introduced in [2], where this is accomplished by allocating transmission power to each secondary user and having each secondary user spread their power over the spectrum band as in a CDMA-based network. The allocated power is assumed to satisfy an "interference constraint," which specifies that the total received power at a fixed measurement point is no greater than a given value. For example, this measurement point could correspond to an access point of the spectrum owner in which case the constraint limits the "interference power" received at the access point.[3]

The problem facing the spectrum owner is allocating a divisible resource (the total power at the measurement point) among multiple secondary users. We assume that the spectrum owner desires to allocate all of the available spectrum. Additionally, the spectrum owner would also like to allocate more spectrum usage to secondary users that value it greater, but is not assumed to have any knowledge of these valuations. Finally, assuming that this allocation is to be done frequently, it is desirable for the allocation procedure to have low communication overhead and low complexity.[4]

In [2] a "power auction" was studied for meeting these goals. In this auction, each secondary user submits a one dimensional bid to the spectrum owner, who also submits a reserve bid β. Each secondary user then receives an allocation of the interference power that is proportional to its bid. Additionally, each secondary user pays an amount for its allocation that is equal to its allocated power multiplied by a fixed "power price" announced by the spectrum owner. In [2], this auction was analyzed via game theory, and it was shown that there exists power prices for which the resulting game has a Nash equilibrium. The efficiency of the equilibrium allocation was characterized in several asymptotic regimes.

There are several shortcomings of this mechanism: (i) achieving a good allocation requires the spectrum owner to determine the correct interference price before the auction is started, (ii) due to the reserve bid, it may not be possible to achieve an allocation that uses all of the interference power; and (iii) a user's bid is not directly related to the price that she pays.[5]

Here, we consider an alternative mechanism, which seeks to overcome some of these shortcoming. This mechanism is based on the proportional allocation mechanism studied for allocating capacity in a wired network in [13], [14]. It differs from the power auction in that there is no reserve bid and the payment of a user is equal to its bid. In [13] this mechanism was studied for "price taking" users in a wire-line network, i.e. users that did not anticipate the effect of their bid on the resulting resource price. In [14], this mechanism was studied for "price anticipating" users. In both of these cases it was

This research was supported in part by NSF grant CNS-0519935.

[1] Alternatives to market-based approaches for secondary spectrum usage have also attracted much interest, such as approaches based on spectrum sensing, e.g. [10], [11].

[2] The problem of allocating the entire spectrum band to one large user is much easier and could be done for example by a simple second-price auction.

[3] Of course, this is not the only way to allocate spectrum usage. For example, an alternative would be to divide the band up into smaller sub-bands and allocate each sub-band to a single secondary user (e.g., see [3], [4], [7]). For a single location this leads to a simpler allocation; however when done over multiple locations it can lead to combinatorial problems similar to those studied for dynamic channel assignment [12].

[4] In particular, this may preclude the use of certain "optimal" auction mechanisms such as the Vickery-Clarke-Groves mechanism.

[5] An alternative "SINR auction" was also studied in [2] in which the users paid a price per received signal-to-interference-plus-noise ratio (SINR). This had better properties with regard to point (ii), but also had the other shortcomings and additionally required the users to truthfully report their SINR to the spectrum owner.

shown that a Nash equilibrium exists and the efficiency of this equilibrium was characterized. However, the setting here differs from the wire-line setting in [13], [14] in that the users interfere with each other. In such a setting it is not clear that a Nash equilibrium will exist and what properties it will have. In this paper, we provide an answer to the first of these questions and give conditions under which this mechanism has a Nash equilibrium for both price taking and price anticipating users.

II. Model

We consider the following simple model based on [2]. There are K secondary users who wish to access a common spectrum band. To do this, each user i may purchase a share of the received power q_i at a given measurement point, where the total received power at the measurement point is constrained to satisfy $\sum_{i=1}^{K} q_i = P$, for a given constant $P > 0$. Each user then transmits over the spectrum band by spreading their share of the power evenly across the band. We assume that each user treats all interference from the other users as noise. Additionally, for simplicity, we make the (more restrictive) assumption that all users are communicating to receivers that are co-located with the measurement point. In this case, each user's performance can be viewed as a function of their SINR at the measurement point, given by[6]

$$\gamma_i = \frac{q_i}{n_o + \sum_{j \neq i} q_i}, \tag{1}$$

where n_0 denotes the total noise power. Specifically, we model user i's performance via a utility function $U_i(\gamma_i)$ which satisfies the following standard assumptions: (*i*) $U_i(\gamma_i)$ is increasing, continuously differentiable, and strictly concave in γ_i and (*ii*) as $\gamma_i \to \infty$, $U_i'(\gamma_i) \to 0$.[7]

We consider the following proportional sharing mechanism. Each secondary user submits a bid $w_i \geq 0$, which can be viewed as the total amount the user is willing to pay. The spectrum owner then allocates to each user an amount of received power proportional to that user's bid, so that the total power budget is utilized (unless all of the agents bid zero in which case no power is allocated). This results in agent i receiving a power allocation of

$$q_i = \frac{w_i}{\sum_j w_j} P, \tag{2}$$

assuming that at least one $w_i > 0$. Equivalently, we can view agent i as receiving an allocation of $q_i = \frac{w_i}{\beta}$, where

$$\beta = \frac{\sum_j w_j}{P} \tag{3}$$

is the effective price per unit power charged to each agent. As pointed out in [14], this can be viewed as a type of market clearing mechanism, i.e. it sets a price so that the total supply meets demand.

[6]Note, here we are assuming that the spectrum owner is not transmitting or equivalently that any power used by the spectrum owner uses is included in the background noise power.

[7]We use the standard notation U_i' to denote the derivate of U_i with respect to its argument.

III. Competitive Equilibrium

We now consider a model with price taking users who do not consider the effect of their bid on the price as in (3). In such a model, for a fixed price β each secondary user i would seek to choose a bid $w_i \geq 0$ to maximize their surplus Π_i given by their utility $(U_i(\gamma_i))$ minus their cost (w_i). Using (1), (2) and (3), this can be written as:

$$\Pi_i(w_i, \mathbf{w}_{-i}, \beta) = U_i\left(\frac{w_i}{\beta n_o + \sum_{j \neq i} w_j}\right) - w_i, \tag{4}$$

where $\mathbf{w}_{-i}$ denotes the vector of bids of all secondary users except user i. In other words, the secondary users are playing a game $\mathcal{G}_\beta$, with payoffs $\Pi_i(w_i, \mathbf{w}_i)$ and strategies w_i.

Note that even though a user does not anticipate its effect on the price, the users are still strategically coupled due to the interference. This is different from the capacity allocation game studied in [13], [14], in which price taking users are faced with only a single-user decision problem.

Lemma 1: For any price $\beta > 0$, $\mathcal{G}_\beta$ has a pure strategy Nash equilibrium.

Proof: From the assumed properties of U_i, it follows that there exists a $\bar{w}_i \geq 0$ so that for all $w_i > \bar{w}_i$

$$U_i\left(\frac{w_i}{\beta n_0}\right) < w_i. \tag{5}$$

A player will never choose a bid $w_i > \bar{w}_i$, since this would result in a negative pay-off, regardless of the other players' actions. Hence, even though each player can choose any bid $w_i \geq 0$, we can think of each player as choosing a bid from the convex strategy set $[0, \bar{w}_i]$. Furthermore, it can be seen that Π_i is concave and continuous in w_i. These properties show that $\mathcal{G}_\beta$ is a concave game and so a pure strategy Nash equilibrium exists. [15] ■

We define a pair $(\mathbf{w}, \mu)$ with $\mathbf{w} \geq 0$ and $\mu > 0$ to be a *competitive equilibrium* if (*i*) $\mathbf{w}$ is a Nash equilibrium of $\mathcal{G}_\beta$ and (*ii*) the price β clears the market, i.e.,

$$\beta = \frac{\sum_i w_i}{P}. \tag{6}$$

Since for any β the game has a Nash equilibrium, it is clear that there exist games for which a competitive equilibrium exists, i.e., we can always choose P so that (6) is tight. A more general question is whether such an equilibrium exists for any P. A sufficient condition for this is given next

Lemma 2: If there exists a continuous function $\mathbf{w}(\beta) : \mathbb{R}_+ \mapsto \mathbb{R}_+^K$ so that for every β, $\mathbf{w}(\beta)$ is a Nash Equilibrium of $\mathcal{G}_\beta$, then a competitive equilibrium always exists.

Proof: Assume that the continuous function $\mathbf{w}(\beta)$ exists and define the corresponding continuous function $s(\beta) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by $s(\beta) = \frac{\sum_i w_i(\beta)}{\beta}$. To prove the existence of a competitive equilibrium, we show that $s(\beta)$ can take any positive value for a suitable choice of β (and thus can be made equal to any given value of P).

First note that as $\beta \to \infty$, for each user i, the $\bar{w}_i$ which satisfies (5) can not be increasing and so it must be that $\frac{w_i(\beta)}{\beta} \to 0$ for each user i. Hence, $s(\beta) \to 0$ as $\beta \to \infty$.

Next we show that given any $M > 0$ we can choose β small enough so that $s(\beta) > M$. First note that for a small enough β, at a Nash equilibrium at least one user must be bidding a positive amount. Let this be user i (where i may vary with β). Then from the first order optimality conditions, this user's bid must satisfy

$$w_i = g_i\left(\beta n_0 + \sum_{j\neq i} w_j\right)\left(\beta n_0 + \sum_{j\neq i} w_j\right),$$

where $g_i(x) = (U')^{-1}(x)$. And so,

$$s(\beta) = g_i\left(\beta n_0 + \sum_{j\neq i} w_j\right)\left(n_0 + \frac{\sum_{j\neq i} w_j}{\beta}\right) + \frac{\sum_{j\neq i} w_j}{\beta}.$$

From the assumptions on $U_i(x)$, $g_i(x) \to \infty$ as $x \to 0$. Hence, there must exist a constant $\delta > 0$ and $\beta_1 > 0$ so that

$$\min_i g_i(\beta n_0 + \delta) n_0 > M \tag{7}$$

for all $\beta < \beta_1$. Likewise, there exists a $\beta_2 > 0$ so that

$$\frac{\delta}{\beta} > M \tag{8}$$

for all $\beta < \beta_2$. Consider some $\beta < \min(\beta_1, \beta_2)$. Then either $\sum_{j\neq i} w_j \leq \delta$ or $\sum_{j\neq i} w_j > \delta$. In the first case from (7), we have $s(\beta) > M$ while in the second case, using (8) we have the same result. Hence, by an appropriate choice of β, we can make $s(\beta)$ take on any desired value. ∎

The requirement in this lemma is essentially that the Nash equilibria of $\mathcal{G}_\beta$ be continuous with respect to changes in the parameter β. To see that this requirement can be met, consider a game in which each player's utility function is given by $U_i(\gamma_i) = \log(1 + \gamma_i)$. Agent i's bid in a Nash equilibria of this game must satisfy

$$w_i = (1 - \beta n_0 - \sum_{j\neq i} w_j)^+.$$

Hence, it follows that one choice for $\mathbf{w}(\beta)$ is to set $w_i(\beta) = \frac{(1-\beta n_0)^+}{K}$ for each user i. The corresponding $s(\beta)$ is then $s(\beta) = (\frac{1}{\beta} - n_0)^+$.

IV. Price Anticipating Users

Next, motivated by [14], we consider a model in which the users are *price anticipating*. In other words, each user anticipates their effect on the price as in equation (3) when choosing their bid. We let $\mathcal{G}_{PA}$ denote the resulting game. In this game, each user again chooses any bid $w_i \geq 0$, only now each user's pay-off is given by:

$$\Pi_i(w_i, \mathbf{w}_{-i}) = \begin{cases} U_i\left(\gamma_i(w_i, \mathbf{w}_{-i})\right) - w_i, & \text{if } w_i > 0 \\ U_i(0) & \text{if } w_i = 0 \end{cases}$$

where

$$\gamma_i(w_i, \mathbf{w_{-i}}) = \frac{w_i P}{n_0 \sum_j w_j + \sum_{k\neq i} w_k P}.$$

Note that this pay-off function may be discontinuous at $w_i = 0$ and so $\mathcal{G}_{PA}$ is not a concave game and thus a different approach is required to show that this game has a Nash equilibria. We do this in the following theorem by using a "modified" utility function as in [14]. However, compared to [14], we need to use a different modification of the utility function here.

Lemma 3: Suppose that $K > 1$. Then there exists a Nash equilibrium $\mathbf{w}^*$ of $\mathcal{G}_{PA}$, if and only if $\mathbf{w}^*$ has at least two non-zero components and is also a Nash equilibrium of a game $\tilde{\mathcal{G}}_\beta$ in which there is a fixed price $\beta = \frac{\sum w_i^*}{P}$ and each user i chooses $w_i \geq 0$ to maximize $\tilde{U}_i(w_i, \mathbf{w}_{-i}) - w_i$, where $\tilde{U}_i$ is the following modified utility function:

$$\tilde{U}_i(w_i, \mathbf{w}_{-i}) = U_i(\gamma_i(w_i, \mathbf{w}_{-i})) \cdot \frac{\sum_{j\neq i} w_j (1 + \frac{n_0}{P})}{\beta n_0 + \sum_{j\neq i} w_j}. \tag{9}$$

Proof: (*sketch*) First we note that if $\mathbf{w}^*$ does not have at least two non-zero components, it can not be a Nash equilibrium for $\mathcal{G}_{PA}$. Given that at least two components of $\mathbf{w}^*$ are non-zero, then each agent's pay-off in $\mathcal{G}_{PA}$ is a continuous and concave function of w_i, and so any point $\mathbf{w}^*$ that satisfies the following first order optimality conditions for each user i will be a Nash equilibrium:

$$U_i'\left(\gamma_i(w_i, \mathbf{w}_{-i})\right) \cdot \frac{(\sum_{j\neq i} w_j)(P + n_0)P}{\left(w_i n_0 + (P + n_0)\sum_{j\neq i} w_j\right)^2} = 1, \quad \text{if } w_i > 0,$$

$$U_i'(0) \cdot \frac{(\sum_{j\neq i} w_j)(P + n_0)P}{\left((P + n_0)\sum_{j\neq i} w_j\right)^2} \leq 1, \quad \text{if } w_i = 0.$$

Next note that for any $\beta > 0$, $\tilde{G}_\beta$ is a concave game and so any $\mathbf{w}^*$ that satisfies the corresponding first order optimality conditions for $\tilde{\mathcal{G}}_\beta$ will be a Nash equilibrium of that game. The result then follows from comparing these conditions using the relation $\beta = \frac{\sum w_i^*}{P}$. ∎

The previous lemma provides a characterization of a Nash equilibrium to $\mathcal{G}_{PA}$ if one exists, but does not guarantee its existence. To study this existence question, we will consider a different modified game $\tilde{G}_{PA}(\epsilon)$ for a given (small) $\epsilon > 0$. In this game there are $K+1$ players. Again, K of the players represent the K secondary users. Each agent maximizes the modified utility function in (9) minus a bid w_i that satisfies $w_i \geq \frac{P\epsilon}{K}$. Additionally, there is another "price setting" player 0 who selects a value of $\beta \geq \epsilon$ to maximize the pay-off

$$\Pi_0(\beta, \mathbf{w}) = -\left(\frac{\sum w_i}{P} - \beta\right)^2.$$

Note that this price setting player receives its maximum pay-off when β is equal to the "correct" price $\frac{\sum w_i}{P}$.

It can be seen that for any $\epsilon > 0$, $\tilde{G}_{PA}(\epsilon)$ is a concave game and thus will have a Nash equilibrium. Furthermore, by construction at such an equilibrium, β will be set to the correct market clearing price. Given the equilibrium choice of β, the remaining K users' bids will also be a equilibrium the game $\tilde{G}_\beta$ with the restriction that their choice of bids must be no smaller than $\frac{P\epsilon}{K}$. The reason for this restriction is to avoid the discontinuities in the game near zero. If the inequality $w_i \geq \frac{P\epsilon}{K}$ is not tight for any user i, it follows from the concavity of the pay-offs, that the resulting strategy choices would also be a Nash equilibrium for $\tilde{G}_{beta}$ without this restriction on the strategy choices. Thus from Lemma 3, these would also be a equilibrium for $\mathcal{G}_{PA}$. The key difficulty in proving that $\mathcal{G}_{PA}$ always has a Nash equilibrium is to argue that as $\epsilon \to 0$, the limiting strategies to $\tilde{G}_{PA}(\epsilon)$ are also "well-behaved," even in the case where these inequalities are tight. To be more precise, we will consider a sequence of games $\tilde{G}_{PA}(\epsilon)$ as $\epsilon \to 0$. The next lemma shows that provided the strategy choices from this sequence have well defined limits, then these limiting strategies are indeed well-behaved.

Lemma 4: For each ϵ, let $(\mathbf{w}(\epsilon), \beta(\epsilon))$ be a sequence of equilibrium strategies to $\tilde{G}_{PA}(\epsilon)$. If as $\epsilon \to 0$, $(\mathbf{w}(\epsilon), \beta(\epsilon)) \to (\mathbf{w}(0), \beta(0))$ then $\mathcal{G}_{PA}$ has a Nash equilibrium.

Proof: First, suppose that $\beta(0) > 0$. Then since $\beta(\epsilon) = \frac{\sum_i w_i(\epsilon)}{P}$ for all $\epsilon > 0$, it follows that at least one component of $\mathbf{w}(0)$ is non-zero. Moreover, we next argue that in this case at least two components of $\mathbf{w}(0)$ must be non-zero. Assume that this is not true, i.e., that $\mathbf{w}(0)$ has only one non-zero component. Let user i be the corresponding agent. Then taking the limit of the left-hand side of the first order optimality condition for user i as $\epsilon \to 0$ yields

$$U_i'\left(\gamma_i(w_i, \mathbf{w}_{-i})\right) \cdot \frac{\sum_{j \neq i} w_j)(P + n_0)P}{(w_i n_0 + (P + n_0)\sum_{j \neq i} w_j)^2} \to 0.$$

This contradicts it being optimal for user i to set $w_i(\epsilon) > 0$ for ϵ small enough. This shows that if $\beta(0) > 0$ at least two-terms in the $\mathbf{w}(0)$ sequence will be non-negative and so we do not have to worry about the non-linearities in the pay-offs. It follows that $\mathbf{w}(0)$ will satisfy the conditions of Lemma 3 in $\tilde{\mathcal{G}}_\beta(0)$ and thus also be a Nash equilibrium for $\mathcal{G}_{PA}$.

Next we consider the case where $\beta(0) = 0$. In this case it must also be that $\sum_i w_i(0) = 0$. Moreover, as $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \frac{\beta(\epsilon)}{\sum_i w_i(\epsilon)} = P,$$

i.e., both $\beta(\epsilon)$ and $\sum_i w_i(\epsilon)$ must go to zero at the same rate. Additionally, we are only interested in such sequences where for every ϵ, there is at least one user i for which $w_i(\epsilon) = \frac{\epsilon P}{K}$, since otherwise, as discussed above, for the given ϵ we have already found an equilibrium to $\mathcal{G}_{PA}$. Given this, we consider the following two cases:

Case 1: $\beta(\epsilon) = \Theta(\epsilon)$, in which case it must be that $w_j(\epsilon) = \Theta(\epsilon)$ for all j.[8] Again, we consider the limit of the left-hand side of the first order condition for user i as $\epsilon \to 0$. In this case, it can be seen that this limit is unbounded, which contradicts this being an equilibrium for small enough ϵ. Hence, this case can not occur.

Case 2: $\beta(\epsilon) = \omega(\epsilon)$.[9] As noted above, there must be at least one user i for which $w_i(\epsilon) = \frac{\epsilon P}{K}$. It follows that for this user, it must be that $\sum_{j \neq i} w_j = \Theta(\beta(\epsilon))$. Again, taking the limit of the left-hand side of the first order condition for that user as $\epsilon \to 0$, it can be seen that the limit is unbounded. Hence, this case can not occur either.

From the above discussion, these two cases exhaust all possibilities when $\beta(0) = 0$ and so it follows that it must be that $\beta(0) > 0$ completing the proof. ■

[8]Here, we use the notation $f(x) = \Theta(g(x))$ to denote that $\frac{f(x)}{g(x)} \to M > 0$ as $x \to 0$.

[9]Here, $f(x) = \omega(g(x))$ denotes that $\frac{g(x)}{f(x)} \to 0$ as $x \to 0$.

V. Conclusions

We considered a proportional sharing mechanism for allocating interference power to secondary spectrum users. Our focus was on the existence of Nash equilibria for both price taking and price anticipating users. In addition to the existence of such equilibria, the performance of these in terms of total utility is also of interest. Also, here we focused on the case where the receivers of all the secondary users were co-located. Relaxing this assumption would also be of interest.

References

[1] J.M. Peha, "Approaches to spectrum sharing," *IEEE Communications Magazine,* vol. 43, no. 2, pp. 10-12, Feb. 2005.

[2] J. Huang, R. Berry and M. L. Honig, "Auction-based Spectrum Sharing," *ACM Mobile Networks and Applications Journal*, vol. 11, no. 3, June 2006.

[3] J. Bae, E. Beigman, RA. Berry, ML. Honig, and R. Vohra, "Sequential bandwidth and power auctions for distributed spectrum sharing," *IEEE JSAC,* vol. 26, no. 7, Sept. 2008.

[4] Y. Yuan, et al., "Allocating dynamic time-spectrum blocks in cognitive radio networks," In *Proceedings of MobiHoc '07*, Sept., 2007.

[5] A. Al Daoud, M. Alanyali, and D. Starobinski, "Secondary pricing of spectrum in cellular CDMA networks," in *IEEE DySPAN*, 2007.

[6] J. Zhu and K.J.R. Liu, "Dynamic Spectrum Sharing: A Game Theoretical Overview," *IEEE Communications Magazine*, Vol. 45, No.5, May 2007.

[7] S.H. Chun and R. La, "Auction Mechanism for Spectrum Allocation and Profit Sharing," in *GameNets*, 2009.

[8] M. M. Buddhikot, P. Kolodzy, S. Miller, K. Ryan, and J. Evans, "Dimsumnet: New directions in wireless networking using coordinated dynamic spectrum access," in *IEEE WoWMoM05*, June 2005.

[9] N. Mandayam, "Cognitive algorithms and architectures for open access to spectrum," in *The Conference on the Economics, Technology and Policy of Unlicensed Spectrum*, East Lansing, MI, 2005.

[10] Q. Zhao, L. Tong, A. Swami and Y. Chen, "Decentralized cognitive MAC for opportunistic spectrum access in ad hoc networks: A POMDP framework," *IEEE JSAC*, vol. 25, no. 3, April 2007.

[11] D. Cabric, S.M. Mishra, and R.W. Brodersen, "Implementation issues in spectrum sensing for cognitive radios," in *Proc. 38th. Asilomar Conf. Signals Systems, Computers*, 2004.

[12] K.N. Ramachandran, E.M. Belding, K.C. Almeroth, M.M. Buddhikot, "Interference-Aware Channel Assignment in Multi-Radio Wireless Mesh Networks," *Proc. of IEEE INFOCOM*, Barcelona, Spain, April 2006.

[13] F. Kelly, "Charging and rate control for elastic traffic," *European Transactions on Telecommunications*, vol. 9, pp. 33-37, 1997.

[14] R. Johari and J.N. Tsitsiklis, "Efficiency Loss in a Network Resource Allocation Game," *Mathematics of Operations Research*, vol. 29, no. 3, pp/ 407-435, 2004.

[15] J.B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," *Econometrica*, Vol. 33, No. 3, pp. 520-534, July, 1965.