

# Competition with Shared Spectrum

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**Abstract**—Sharing spectrum has emerged as a promising way to meet the exploding demand for wireless data services. In this paper, we consider a scenario in which spectrum is shared between a primary and multiple secondary service providers, all of which are competing for a common pool of customers. We adopt a similar model to one used in earlier work to study competition with unlicensed spectrum, in which users select service providers based on the sum of a congestion cost and the price announced by the service provider. Competition with unlicensed spectrum was shown to potentially decrease social welfare. In contrast, with shared spectrum, we show here that social welfare is always non-decreasing, although the welfare of the primary provider can decrease. Various models of user demand and congestion costs are considered.

## I. INTRODUCTION

One of the biggest current challenges in wireless networks is to meet the skyrocketing demand for wireless data services. Due in part to the high cost of clearing spectrum bands, sharing has been put forth as a promising approach for making more spectrum available to meet this demand. Indeed, in the U.S., promoting greater sharing is a central recommendation of the 2012 report from the President’s Council of Advisors on Science and Technology (PCAST) [1] and has been the subject of several presidential memoranda [2], [3]. Sharing has already been adopted in the TV white spaces [4] and is currently being considered for federal spectrum such as the 3.5 GHz band [5]. Sharing has also been suggested for other commercial bands including those used to offer cellular services, e.g. [6], [7]. Sharing of such commercial bands is the focus of this paper.

There has been much research on spectrum sharing in recent years, in particular on the “primary-secondary” approach to sharing in which a primary licensed holder has priority access to the spectrum while other secondary devices are allowed to share the spectrum provided that they do not degrade the primary’s service. This includes work on using spectrum sensing (e.g. [8]) or market-based approaches (e.g. [9]) to ensure the primary has acceptable performance. As in [7], we consider such primary-secondary sharing for a model where the primary is a commercial cellular provider. Other service providers may offer service in the same band as secondary users, where all providers (primary and secondary) compete for a common pool of customers. As in the TV white-spaces, we assume that secondary access is “open”, meaning that any firm can offer

such secondary service.<sup>1</sup> Such a policy lowers barriers to enter the market, since secondary providers do not need to acquire any spectrum license, and thus has the potential of increasing competition and improving overall welfare. However, since secondary access is not limited, there is also a risk that the spectrum becomes overly congested, leading to a “tragedy of the commons.” We present an analytical model to study such trade-offs.

Our approach is based on the framework in [11], [12], which considered similar questions for a model of competition among service providers with both licensed and unlicensed spectrum. This was in turn used in models for price competition with congestible resources developed in the operations and economics literature (e.g., [13], [14]). In this framework, service providers compete for customers by announcing service prices. Customers in turn select providers based on a *delivered price*, which consists of the announced service price plus a congestion cost, where the congestion cost reflects the quality of service obtained from a provider. In [11], licensed spectrum was not shared while unlicensed spectrum was shared “equally” among all providers, i.e., no provider was a primary user. A main result in [11] was to show that adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent service providers may lead to a decrease in overall social welfare. The reason for this was that when faced with competition from the unlicensed band, licensed providers have an incentive to increase their prices to drive traffic to the unlicensed band and increase its congestion, thus decreasing overall welfare.

Here, instead of a separate unlicensed band, we consider secondary providers that can operate in the same band as the primary using some form of sharing technology. We abstract this by modeling the congestion cost costs in the band differently for primary and secondary firms. Namely, a primary firm’s congestion only depends on the number of customers it serves, while a secondary firm’s congestion depends on the total customer mass served in the band by all firms. For such a model we characterize the equilibria of the resulting price setting game among the service providers and study the impact of sharing on the overall social welfare, the consumer welfare, and the service provider’s profits. Interestingly, unlike the model in [11], here, sharing never decreases overall welfare,

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<sup>1</sup>This can be contrasted with licensed secondary access approaches, in which only holders of secondary licenses can access the spectrum, e.g., [10].

but it can decrease the profits of the primary service provider.

In addition to primary-secondary sharing, this type of model is also applicable in other settings. For example, it may apply to sharing done as in the lower two tiers of the three-tier hierarchy proposed in the PCAST report [1], where at the highest tier priority is given to federal systems; the second tier corresponds to licensed secondary use, and the third tier is open access. Also the form of sharing between primary and secondary users could involve sharing “raw spectrum” or could also involve some form of infrastructure sharing as in [15], though in such cases the primary could likely exert more control over secondary users than we consider here.

The rest of the paper is organized as follows. Our models for price competition are described in Section II. We then present an analysis of the resulting welfare with and without sharing for various models of demand and congestion costs in Sections III - V. Finally, we conclude in Section VI.

## II. COMPETITION MODEL

As in [11], we consider a wireless service market in which a set of service providers (SPs) compete for a common pool of customers by setting prices for their services. We focus on a model in which there is one incumbent SP. Without sharing, this firm acts as a monopolist. With sharing, the SP will be the primary spectrum user and there will also be a set of  $N > 1$  secondary SPs, denoted by  $F_1, F_2, \dots, F_N$ , who share the primary’s spectrum subject to not degrading the performance seen by its customers. Each SP announces a price for service, which we denote by  $p$  and  $p_i^S$ , for the primary SP and the  $i$ th secondary SP, respectively. The SPs then serve all customers that accept their price. Thus if  $x$  customers accept price  $p$ , the SP’s profit is simply  $xp$ .

The service that SP’s offer is also characterized by a congestion cost,  $g(x)$ , which models various congestion effects such as increased interference or queueing delays that arise as a SP serves more customers in a given area. In the case of the primary SP, the congestion cost seen by its customers is given by  $g(x_1)$ , where  $x_1$  denotes the mass of customers it is serving. Customers of each secondary SP  $F_1, F_2, \dots, F_N$  will encounter a congestion cost of  $g_S(X)$ , where

$$X = x_1 + \sum_{k=1}^N x_k^S$$

is the overall mass of customers being served in the band, with  $x_k^S$  being the number of customers served by secondary SP  $F_k$ . This models the fact that the primary SP does not “see” any degradation from the secondaries, while the secondary firms do experience degradation due to the primary as well as each other. In general, we assume that  $g(x)$  is an increasing, convex function as shown in Figure 1, though for much of our analysis, for simplicity, we focus on the case where this is a linear function, i.e.,  $g(x) = \frac{x}{C}$ , where  $C$  represents the bandwidth or capacity of the spectrum band. For the secondary users we consider two different types of congestion costs: a model of *perfect sharing*, in which  $g_S(x) = g(x)$  and a

model of *degraded sharing*, where  $g_S(x) = g(x/\alpha)$  where  $0 < \alpha < 1$  is the degradation factor. Note as  $\alpha$  approaches 1, the degraded model approaches the model with perfect sharing. For  $\alpha < 1$ , secondary firms see greater congestion for the same customer mass. This can model a case where the secondary users experience additional congestion due for example to time spent sensing the medium to avoid interfering with the primary or being required to use a lower power level than the primary.

We assume a single mass of infinitesimal customers, and we normalize the total customer mass to be 1. Customers select SPs based on the overall *delivered price* for each SP’s service, which consists of the sum of the SP’s announced price and the congestion cost of their service. Specifically, the delivered price of the primary SP is  $p + g(x_1)$ , while the delivered price of  $F_i$  is  $p_i^S + g_S(X)$ . Each customer in the market selects the service provider who has the lowest delivered price. If more than one service provider has the same delivered price, a customer chooses one of the low price SPs uniformly at random.

Customer demand is modeled by an inverse demand function  $P(q)$ , which will be a non-increasing function that indicates the delivered price at which  $q$  customers are willing to pay for service. As shown in Fig. 1, this will in general be a concave function. However, we again focus on two special cases. First, in Section III, we assume that customers are homogeneous in their demand, i.e., all customers are willing to pay up to the same same delivered price, in which case  $P(q)$  has a “box” shape. Then in Section IV, we consider a simple case of heterogeneous demand, where  $P(q) = 1 - q$  for  $q$  between 0 and 1. The case of a general concave decreasing demand is briefly considered in V.

We view the competition among the primary firm, the secondary firms, as well as the customers as a pricing game where firms set prices,  $(p, \mathbf{p}^S)$ , simultaneously. Customers then choose one of the firms offering the lowest delivered price. Thus, given a set of prices,  $(p, \mathbf{p}^S)$ , the customers receiving service must satisfy the following Wardrop equilibrium conditions [16]:

$$\begin{aligned} p + g(x_1) &\leq P(X), & \text{(with equality if } x_1 > 0) \\ p_i^S + g_S(X) &= P(X), & \text{for } i \in N \text{ with } x_i > 0, \\ p_i^S + g_S(X) &\geq P(X), & \forall i \in N. \end{aligned}$$

These conditions specify that the delivered price of all providers serving customers are equal and no greater than  $P(X)$ .

We define a (pure strategy) *Nash equilibrium* of the overall pricing game to be a set of prices  $(p, \mathbf{p}^S)$  and demands  $(x_1, \mathbf{x}^S)$ , which satisfy these Wardrop equilibrium conditions and also have the property that no SP can increase its profit by unilaterally changing its price.

Given such an equilibrium, the firm profit,  $f_c$  is defined by the sum of the profits made by all SPs. The welfare of the  $x$ th consumer served is the difference between that consumers value for the service given by  $P(x)$  and the delivered price it pays for service; consumers that are not served receive zero

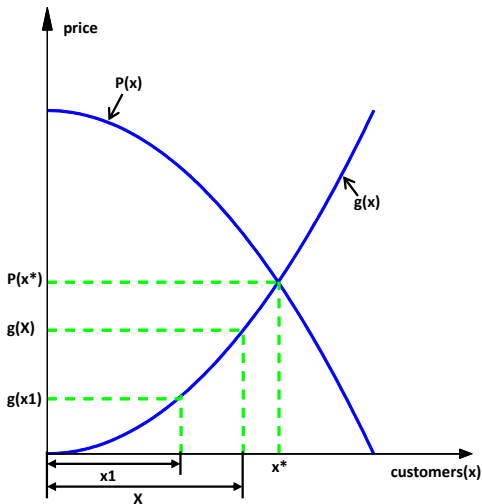


Fig. 1. Illustration of pricing game with primary and secondary firms with shared spectrum

welfare. The total *consumer welfare*,  $S_c$ , is the integral of this over all consumers. The *social welfare*,  $S$ , of the entire economy is the sum of the firm profit and the consumer welfare, i.e.,

$$S = S_c + f_c.$$

Next we give some preliminary discussions about such a game both without and with spectrum sharing.

#### A. Without Spectrum Sharing

Without spectrum sharing, the primary firm is a monopolist in the market. As there is no competition with other firms, it sets the price to maximize its own profit, i.e. it solves:

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + g(x) = P(x), \\ & 0 \leq x \leq 1. \end{aligned}$$

Under our assumptions, this will be a convex problem with a unique solution.

#### B. With Spectrum Sharing

With spectrum sharing, the primary firm now faces competition from the secondary firms, these secondary firms all see greater congestion and so must offer lower prices. Indeed, provided there are at least 2 secondary firms, the following lemma shows that this price must be zero.<sup>2</sup>

*Lemma 2.1:* In competition with shared spectrum and at least 2 secondary SPs, all secondary SPs will charge zero price to customers, i.e.,  $p_i^S = 0$  for all  $i$ .

A similar results was derived in [11] for SPs offering service using unlicensed spectrum and this result can be derived similarly. First note that all secondary SPs must charge the same price in any equilibrium since all of their customers experience the same congestion,  $g_S(X)$ . If not, then any SP

<sup>2</sup>Here we are not modeling any fixed costs for offering service. If such costs were included, then the price would be equal to the fixed cost of service.

charging a price higher than some other SP would serve no customers and thus make no profit. Further, if this common price is greater than zero, then each secondary SP has an incentive to decrease its price as this will enable it to serve the total mass of customers being served by all secondary SPs at the common non-zero price. Hence, the only possible equilibrium is for all secondary SPs to set a price of  $p_i^S = 0$ .

It follows from this lemma and the Wardrop equilibrium conditions that with secondary firms in the market, the number of customers served in the entire market must be given by the value  $x^*$  such that

$$g_S(x^*) = P(x^*). \quad (1)$$

This is the intersection of  $g_S(x)$  and  $P(x)$  as shown in Figure 1. The resulting delivered price in the market is then fixed at  $P(x^*)$ .

### III. WELFARE ANALYSIS WITH LINEAR CONGESTION AND HOMOGENEOUS DEMAND

In this section we analyze models with linear congestion costs,  $g(x) = x/C$  and homogeneous customer demands given by

$$P(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Here, we have set the price that customers are willing to pay to be 1; similar results hold for other values. For this setting, we analyze the social welfare,  $S$ , consumer welfare,  $S_c$ , and firm profit,  $f_c$  both with and without sharing. The main result from our analysis is summarized in the following theorem:

*Theorem 3.1:* When  $C > 1$ , with perfect spectrum sharing, social welfare and consumer welfare are always greater than without sharing while firm profit shrinks. When  $0 < C \leq 1$ , social welfare, consumer welfare and firm profit are the same with and without sharing.

The proof of this theorem follows from the analysis in the following two sections. Subsequently we will show an analogous result for the case of degraded sharing.

#### A. Without Spectrum Sharing

Without sharing spectrum, the primary SP's profit maximization problem in this case is:

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1, \\ & 0 \leq x \leq 1. \end{aligned}$$

To solve this optimization, we first obtain from the equality constraint that  $p = P(x) - x/C$ . Using this, the objective function can be re-written as  $P(x)x - x^2/C$ , which we can then optimize subject to  $x > 0$ , the results of this are summarized in the following lemma.

*Lemma 3.2:* For a model with with linear congestion, homogeneous demand and no-sharing, the equilibrium outcome is as follows:

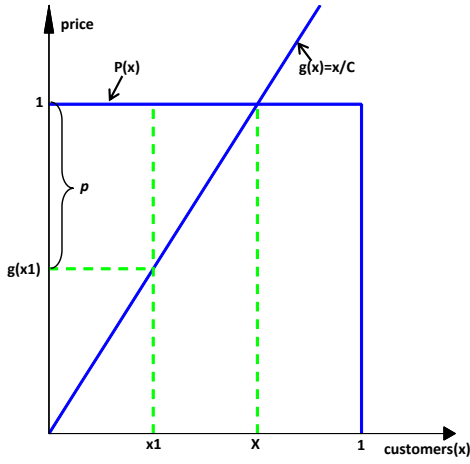


Fig. 2. Sharing and non-sharing case when  $0 \leq C \leq 1$  with linear congestion and homogeneous inverse-demand

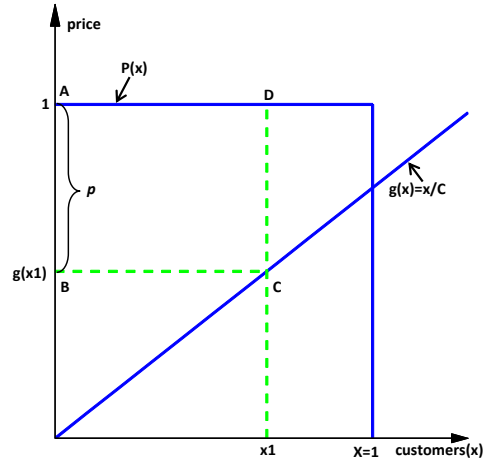


Fig. 3. Non-sharing case when  $C > 1$  with linear congestion and homogeneous inverse-demand

- i) When  $0 \leq C \leq 2$ , the primary serves a mass of  $x_1 = C/2$  customers at a price of  $p = 1/2$ , resulting in  $S = f_c = C/4$  and  $S_c = 0$ .
- ii) When  $C > 2$ , the primary serves the entire customer mass at a price of  $p = 1 - 1/C$ , resulting in  $S = f_c = 1 - 1/C$  and  $S_c = 0$ .

As stated in this lemma, the outcome in this scenario can be naturally divided into two cases depending on the spectrum bandwidth  $C$ . When  $C$  is small ( $0 < C \leq 2$ ), the monopoly firm will not serve all of the customers because the congestion cost is too high. For  $C < 1$ , as shown in Figure 2, it could not serve these customers even if it announced a price of zero; for  $1 \leq C \leq 2$ , as in Figure 3, it could serve all the customers with a small enough price, but this would not maximize its profit. The optimal price in this regime is  $1/2$  and as  $C$  increases, the SP gains more profit by the increase in number of customers. However, when  $C > 2$  is large enough as in Figure 3, the monopoly SP will serve the entire market and so will increase its profits as  $C$  increases by increasing its price. In either case, the delivered price will be equal to 1, the maximum the customers are willing to pay and so consumer welfare will be zero and social welfare will be equal to the profit of the SP. An example of how these quantities vary with  $C$  is shown by the dashed curves in Figure 5.

### B. Perfect Spectrum Sharing

Next, we turn to the case of perfect sharing, i.e.,  $g_S(x) = g(x)$ . In this case, as noted after Lemma 2.1, the delivered price is uniquely determined by the congestion and inverse demand functions. This gives the primary firm less freedom to choose its price. For example in Figure 4, these curves intersect at point E, which means the primary firm can only charge a price up to the value A. To optimize its profit, the primary tries to maximize the area of the rectangle A – B – C – D in this figure, where the height of this rectangle is the price  $p$  and the width is the number of customers it serves.

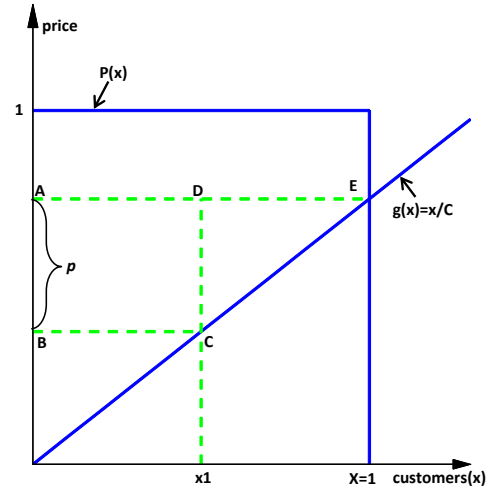


Fig. 4. Perfect sharing case when  $C > 1$  with linear congestion and homogeneous inverse-demand

When  $0 \leq C \leq 1$ , the primary's optimization problem becomes

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1, \\ & 0 \leq x \leq 1. \end{aligned}$$

In this case, sharing does not further constrain the delivered price and so the optimization is the same as in the monopoly case discussed above.

When  $C > 1$ , the delivered price is constrained and so the primary's problem becomes

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1/C, \\ & 0 \leq x \leq 1. \end{aligned}$$

The following result summarizes the solution to these optimization problems:

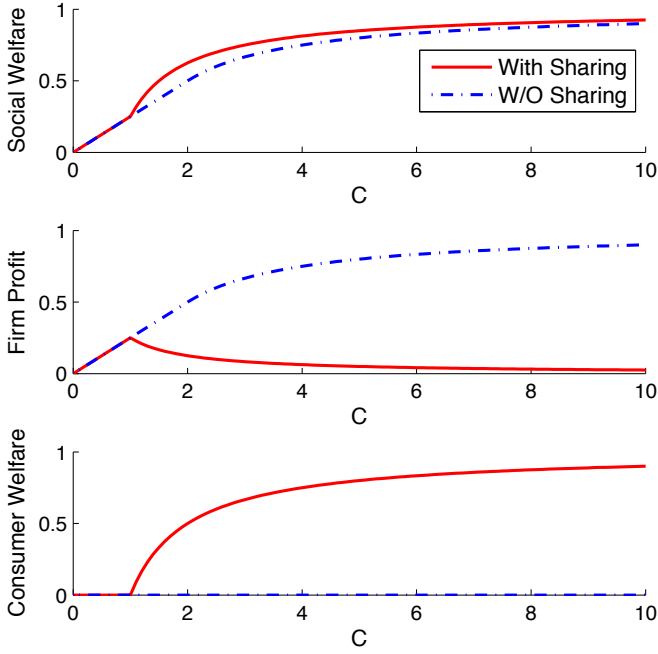


Fig. 5. Comparison of perfect sharing and non-sharing cases on welfare with linear congestion and homogeneous inverse-demand

*Lemma 3.3:* With perfect sharing, linear congestion costs and homogeneous demand, the equilibrium outcomes are characterized as follows:

- i) When  $0 \leq C \leq 1$ , the primary serves a mass of  $x_1 = C/2$  customers at a price of  $p = 1/2$ , resulting in  $S = f_c = C/4$  and  $S_c = 0$ ;
- ii) When  $C > 1$ , the primary serves a mass of  $x_1 = 1/2$  at a price of  $p = 1/(2C)$ , resulting in  $f_c = 1/(4C)$ ,  $S_c = 1 - \frac{1}{C}$  and  $S = 1 - \frac{3}{4C}$ .

As seen from Figure 5, social welfare with perfect sharing always increases when  $C$  increases. For  $0 < C \leq 1$ , social welfare increases linearly with  $C$  and is the same as in the non-sharing case. When  $C > 1$ , social welfare with sharing is always greater than in the monopoly case. Thus when the primary firm's bandwidth is large enough relative to the demand (bigger than 1 in this case), sharing spectrum will always benefit in terms of overall social welfare. Also, when  $C > 1$ , consumer welfare with sharing becomes positive. This is a dramatic change from the non-sharing case where consumer welfare stays zero no matter how large  $C$  is. With sharing, competition limits the price a primary can charge thus preventing the SPs from extracting all of the welfare. With sharing, as  $C$  increases, not only does congestion become less, but the cap on the delivered price also becomes smaller, both leading to increases in consumer welfare. As we can see, the cost of this increased social welfare and consumer welfare comes from the profit of primary firm. With the growth of  $C$ , the profit of firm first increases but once  $C$  exceeds 1, it decreases due to the ceiling on the delivered price in the market.

### C. Degraded Spectrum Sharing

Now we turn to the case where sharing is degraded, i.e.,  $g_S(x) = \frac{x}{\alpha C}$ , where the degradation factor  $\alpha$  satisfies  $0 < \alpha < 1$ . Under this assumption, for a given  $\alpha$ , with sharing, the delivered price is now fixed by the intersection of  $g_S(x)$  and  $P(x)$ , unless  $\alpha$  is too small so that the congestion cost seen by any secondary user exceeds the user's willingness to pay, in which case the secondary SPs will have no customers. The next lemma summarizes the dependence of the equilibrium outcome on both  $\alpha$  and  $C$ .

*Lemma 3.4:* With degraded spectrum sharing, linear congestion and homogeneous demand, the equilibrium outcome's dependence on  $\alpha$  is as follows:

- i) When  $0 < \alpha \leq \min\{1/2, 1/C\}$ , no sharing occurs;
- ii) When  $\min\{1/2, 1/C\} < \alpha \leq \min\{1/C, 1\}$ ,  $S = f_c = C/4$  and  $S_c = 0$ .
- iii) When  $1/C < \alpha < 1$ ,

$$S = 1 + \frac{1}{4\alpha^2 C} - \frac{1}{\alpha C},$$

$$f_c = \frac{1}{4\alpha^2 C},$$

$$S_c = 1 - \frac{1}{\alpha C}.$$

*Proof:* We will prove this by considering each of the three cases given in the theorem separately.

*Case i):*  $0 < \alpha \leq \min\{1/2, 1/C\}$ . Let  $x_M$  be the optimal customer mass served by the primary without sharing. From Lemma 3.2 we have that  $x_M = \min\{C/2, 1\}$ . For  $\alpha$  in this range it follows that  $g_S(x_M) \geq 1$ , which means that if the primary continues serving the same number of customers as it did without sharing, then the congestion of any secondary SP will exceed any user's willingness to pay. Thus, with degraded sharing, the primary can continue serving the same mass of customers at the same price, and no sharing will occur.

*Case ii):*  $\min\{1/2, 1/C\} < \alpha \leq \min\{1/C, 1\}$ . Note that this range is non-empty only when  $1 \leq C < 2$ , in which case it becomes  $1/2 \leq \alpha < 1/C$ . For  $1 \leq C < 2$ , from Lemma 3.2, the primary will not serve the entire market without sharing, but only serves a mass of  $x_M = C/2$  customers. For  $\alpha$  in the given range, it also follows that  $g_S(x_M) < 1$  and  $g_S(1) = \frac{1}{\alpha C} > 1$ , which means that if the primary continues serving  $x_M = C/2$  customers, the secondary bands will attract new customers, but will become so congested that it will not constrain the primaries delivered price and all new secondary users will receive zero welfare (see Figures 6 and 7). So although more users appear in the network, the primary firm can still act like a monopolist and the welfare and firm profit are the same as in the no sharing case. Note that in this case even if  $C > 1$  as in Figure 7, the entire market is not served; this differs from the model with perfect sharing in which when  $C > 1$  all users are served.

*Case iii):*  $1/C < \alpha < 1$ . In this case,  $g_S(1) < 1$  and so sharing constrains the total delivered price to be no more than

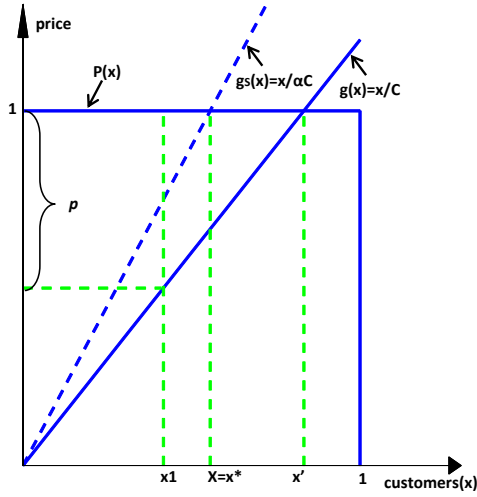


Fig. 6. An example of case (ii) for degraded sharing with linear congestion and homogeneous inverse-demand with  $C < 1$ . The primary SP serves  $x_1$  customers and the total mass of customers served in the whole market is  $X = x^*$  which is less than 1.

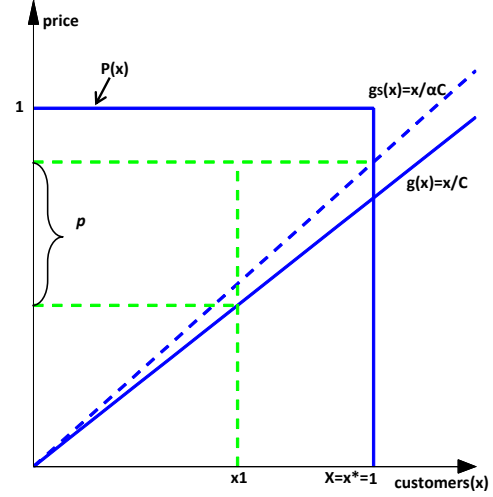


Fig. 8. An example of case (iii) for degraded sharing with linear congestion and homogeneous inverse-demand.

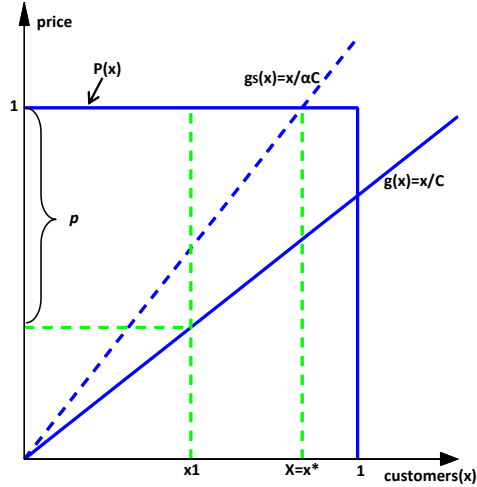


Fig. 7. An example of case (ii) for degraded sharing with linear congestion and homogeneous inverse demand where  $C > 1$ . Even in this case the total customer mass served is less than 1.

$g_S(1)$  (see Figure 8). This in turn constrains the price the primary can charge and so the primary is now faced with the following optimization problem:

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1/\alpha C, \\ & 0 \leq x \leq 1. \end{aligned}$$

Referring to Figure 9, this corresponds to maximizing the area of the square  $A - B - C - D$  that lies below the line  $A - G$  and above  $g(x)$ . The solution to this is for the primary SP to charge a price of  $p = 1/2\alpha$  as to maximize its profit and serve  $x_1 = 1/2\alpha C$  of the overall users. The entire market is served and all consumers receive a welfare of  $1 - \frac{1}{\alpha C}$ . The primary

firms profit is given by

$$px_1 = \frac{1}{4\alpha^2 C}.$$

Adding these gives the indicated total welfare. ■

Comparing Lemma 3.4 to Lemma 3.2, we have the following analog to Theorem 3.1.

**Theorem 3.5:** When  $C > 1/\alpha$ , with degraded spectrum sharing, social welfare and consumer welfare are always greater than without sharing while firm profit shrinks. When  $0 < C \leq 1/\alpha$ , social welfare, consumer welfare, and firm profit are the same with and without sharing.

This shows that the benefits of sharing depend on the product of the degradation factor  $\alpha$  and the available bandwidth  $C$ , the large  $C$  is the smaller the degradation factor that can be allowed and still see an increase in welfare. A plot of how social welfare, firm profit, and consumer welfare depend on  $\alpha$  for various fixed values of  $C$  is shown in Figure 10. Note change in the degradation factor or  $C$  has a relative small effect on firm profit, while changing  $C$  has a larger effect on consumer welfare. When  $C$  is small,  $\alpha$  has no effect on consumer welfare which is zero. For  $C$  in  $(1, 2]$ ,  $\alpha$  needs to be large enough to increase consumer welfare. When  $C$  is large enough, it dominates  $\alpha C$  as well as the delivered price, in which case, social welfare and consumer welfare will always grow with  $\alpha$  while firm profits shrinks.

#### IV. WELFARE ANALYSIS WITH LINEAR CONGESTION AND HETEROGENEOUS DEMAND

Next, we consider a model in which there is heterogeneous demand, i.e., different customers are willing to pay different amounts for service. We model this by making using a linear inverse demand function given by  $P(x) = 1 - x$  as shown in Figure 11.<sup>3</sup> Thus the only way the entire market can be served

<sup>3</sup>The analysis can be easily generalize to any linear demand of the form  $P(x) = a - bx$ , where  $a$  and  $b$  are positive constants.

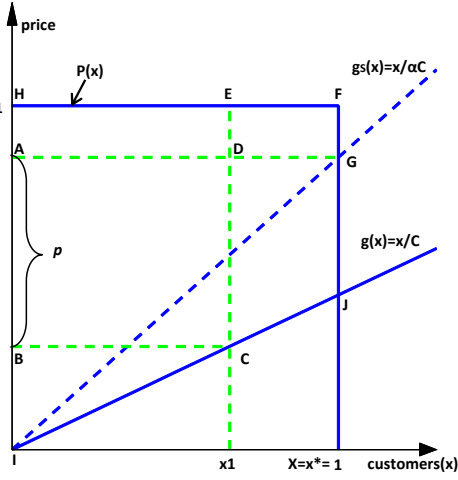


Fig. 9. Degraded sharing for  $C > 2, 1/2 < \alpha \leq 1$  with linear congestion and homogeneous inverse-demand.

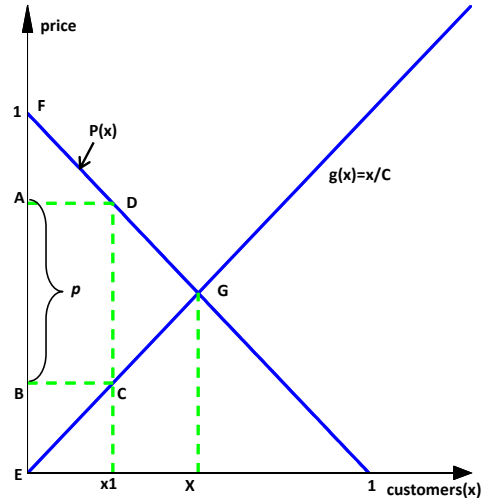


Fig. 11. Non-sharing with linear congestion and linear inverse-demand

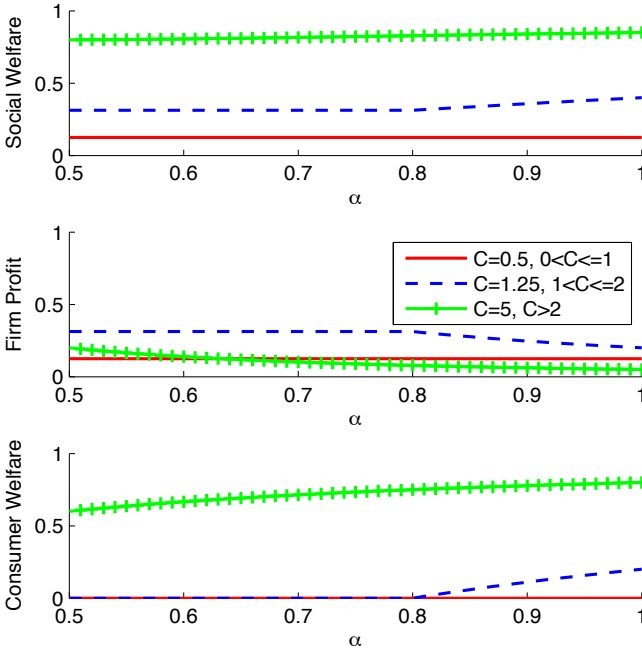


Fig. 10. The effect of  $\alpha$  on social welfare(top), firm profit(middle) and consumer welfare(bottom).

is if the delivered price goes was zero, and as the delivered price increases to one, there are fewer and fewer customers willing to pay for service. For simplicity, we still assume linear congestion  $g(x) = x/C$ . As in the previous section, we again analyze the social welfare, consumer welfare and firm profit both with and without sharing. The main result of this analysis is summarized in the following theorem:

**Theorem 4.1:** With perfect sharing, linear congestion and linear inverse demand, social welfare and consumer welfare are always greater than without sharing while firm profits shrink.

Compared to the case of homogeneous demand, now both social and consumer welfare always increase (regardless of  $C$ ). As in the case of homogeneous demand, we will prove this in the following two sections by characterizing the equilibrium both with and without sharing. We will then generalize this for imperfect sharing.

#### A. Without Spectrum Sharing

Again, without sharing the primary firm is a monopolist and simply sets the price to maximize its profit. With linear inverse demand, the corresponding optimization is given by:

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1 - x, \\ & 0 \leq x \leq 1. \end{aligned}$$

Graphically, as shown in Figure 11, this corresponds to maximizing the area off the rectangle  $A - B - C - D$  which is contained within the triangle  $F - E - G$  formed by the inverse demand and the congestion cost. By again, solving for  $p$  in terms of  $x$ , this can be written as an optimization over the single variable  $x$ , whose solution is given by:

$$x_1 = \frac{C}{2(C+1)}, \text{ and } p = 1/2. \quad (2)$$

The firms profit  $f_c$  is then given by the product  $x_1 p$  and referring to Figure 11, the consumer welfare,  $S_c$  is given by the area of the triangle  $A - F - D$ . Summing these gives the total welfare. The result of these calculations is summarized in the following lemma.

**Lemma 4.2:** For a model with linear congestion, linear heterogeneous demand, and no spectrum sharing, the equilibrium outcome is for the primary to set the price and number of

customers served as in (2) resulting in:

$$S = \frac{C + 3/2C^2}{4(C + 1)^2},$$

$$f_c = \frac{C}{4(C + 1)},$$

$$S_c = \frac{C^2}{8(C + 1)^2}.$$

Note that the number of customers served in (2) is a strictly increasing function of  $C$ , and as  $C$  becomes arbitrarily large it converges to  $1/2$ . In other words, for any finite  $C$ , the primary will always serve less than half of the market. Also note here that the price charged is always  $1/2$ , regardless of the value of  $C$ , while with homogeneous demand, the price increases with  $C$  for  $C$  large enough.

### B. Perfect Spectrum Sharing

With perfect spectrum sharing, as discussed earlier, the number of customers served in any equilibrium is given by the value  $x^*$  satisfying (1), which in this case corresponds to

$$x^* = \frac{C}{C + 1}.$$

Comparing to (2), it can be seen that  $x^*$  is exactly double the number of customers served by the primary without sharing. This corresponds to an upper bound on the delivered price of

$$P(x^*) = \frac{1}{C + 1}.$$

Given this bound the optimization faced by the primary firm is now:

$$\begin{aligned} \max \quad & px \\ \text{subject to} \quad & p + x/C = 1/(C + 1), \\ & 0 \leq x \leq 1. \end{aligned}$$

The following lemma summarizes the results of this optimization.

*Lemma 4.3:* With perfect sharing, linear congestion costs, and linear inverse demand, the equilibrium outcome is for the primary to set the price and number of customers served as

$$x_1 = \frac{C}{2(C + 1)}, \text{ and } p = \frac{1}{2(C + 1)} \quad (3)$$

resulting in:

$$S = \frac{3C^2}{4(C + 1)^2},$$

$$f_c = \frac{C^2}{4(C + 1)^2},$$

$$S_c = \frac{C^2}{2(C + 1)^2}.$$

Comparing (3) with (2) it can be seen that the primary serves the same number of customers both with and without sharing, but at a lower price with sharing. This leads to a decrease in firm profit and an increase in customer welfare. Namely, referring to Figure 12, firm profit will decrease by the area

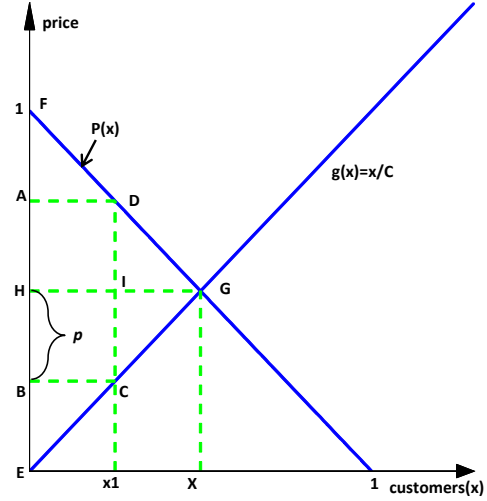


Fig. 12. Perfect sharing with linear congestion and linear inverse-demand

$A - D - I - H$  and the welfare of the customers it serves will increase by the same amount. Customer welfare will also increase due to the new customers being served by the secondary SPs, which is given by the area of the triangle  $D - I - G$  in Figure 12. Comparing the areas of these regions with the customer welfare without sharing (given by the triangle  $A - D - F$ ), we have the following lemma:

*Lemma 4.4:* With perfect spectrum sharing, linear congestion and linear demand, consumer welfare is four times that without sharing.

It follows from the above discussion that overall welfare must increase with sharing as stated in Theorem 4.1.

### C. Degraded Spectrum Sharing

We next consider degraded sharing with linear inverse demands. Again, this means that  $g_S(x) = x/\alpha C$ , for  $0 < \alpha < 1$ . As in the case of homogeneous demand, this changes the limit on the delivered price introduced by sharing (see Figure 13). The resulting equilibrium is summarized in the following lemma.

*Lemma 4.5:* With degraded spectrum sharing, linear congestion, and linear inverse demand, the equilibrium outcome's dependence on  $\alpha$  is as follows:

- i) When  $0 < \alpha \leq \frac{1}{2+C}$ , no sharing occurs and the primary serves the same customers at the same price as in the non-sharing case.
- ii) When  $\frac{1}{2+C} \leq \alpha \leq \frac{1}{2}$ , again no customers are served by secondary providers, but the primary serves  $x^*$  customers, resulting in

$$S = \frac{\alpha C(1 - \alpha) + \frac{1}{2}\alpha^2 C^2}{(\alpha C + 1)^2},$$

$$f_c = \frac{\alpha C(1 - \alpha)}{(\alpha C + 1)^2},$$

$$S_c = \frac{\alpha^2 C^2}{2(\alpha C + 1)^2}.$$



iii) When  $\frac{1}{2} < \alpha \leq 1$ , the primary serves more customers than without sharing and the secondary SPs also serve some customers, resulting in

$$S = \frac{C + 2\alpha^2 C^2}{4(\alpha C + 1)^2},$$

$$f_c = \frac{C}{4(\alpha C + 1)^2},$$

$$S_c = \frac{\alpha^2 C^2}{2(\alpha C + 1)^2}.$$

*Proof:* The first case, where no sharing occurs, again corresponds to the case where  $g_S(x_1) > P(x_1)$ , where  $x_1$  is the number of customers served by the primary without sharing as in (2). In the second and third cases, the primary SP must also account for the new constraint on the delivered price given by

$$p + \frac{x}{C} = \frac{1}{\alpha C + 1}.$$

For  $\alpha > 1/2$ , this constraint will be tight and the constraint given by the inverse demand ( $p + \frac{x}{C} = 1 - x$ ) will not be tight, resulting in the primary serving a customer mass of

$$x_1 = \frac{C}{2(\alpha C + 1)}. \quad (4)$$

This corresponds to case (iii) in the lemma and using this the indicated quantities can then be calculated.

In case (ii) both the new constraint on the delivered price and the constraint due to the inverse demand are tight, in which case the primary serves  $x^*$  customers at a price of

$$p = \frac{1 - \alpha}{\alpha C + 1},$$

from which again the indicated quantities can be calculated. ■

Note that in case (iii) from (4), it can be seen that the primary serves more customers with degraded sharing than without sharing and the number of customers served increases as sharing becomes more degraded. However once  $\alpha$  becomes smaller than  $\frac{1}{2}$ , the constraint given by the inverse demand becomes tight and the number of customers served then decreases with  $\alpha$ . In both cases, since the delivered price is smaller than without sharing, customer welfare must increase and firm profit must decrease. If the primary continued serving the same amount of customers as without sharing,  $x_M$ , then in either case (ii) or (iii), as in the case of perfect sharing, the total welfare must increase. Since in these cases, the primary always serves more than  $x_M$  customers, it follows that the customer welfare will be the same as if it continued serving  $x_M$  customers (i.e., it will always be the area of the triangle  $F - A - H$  in Figure 13). Further, since the primary is maximizing its profit, it must be that its profit in the equilibrium is greater than if it continued to serving  $x_M$  customers and so again total welfare must increase. An example of the social welfare, consumer welfare and firm profit as a function of  $C$  under different scenerios is shown in Figure 14.

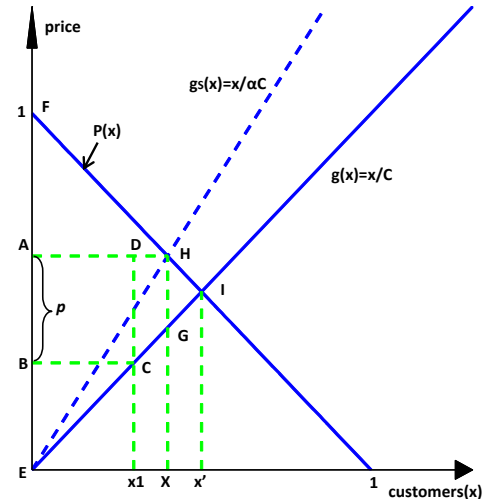


Fig. 13. Degraded sharing with linear congestion and linear inverse-demand

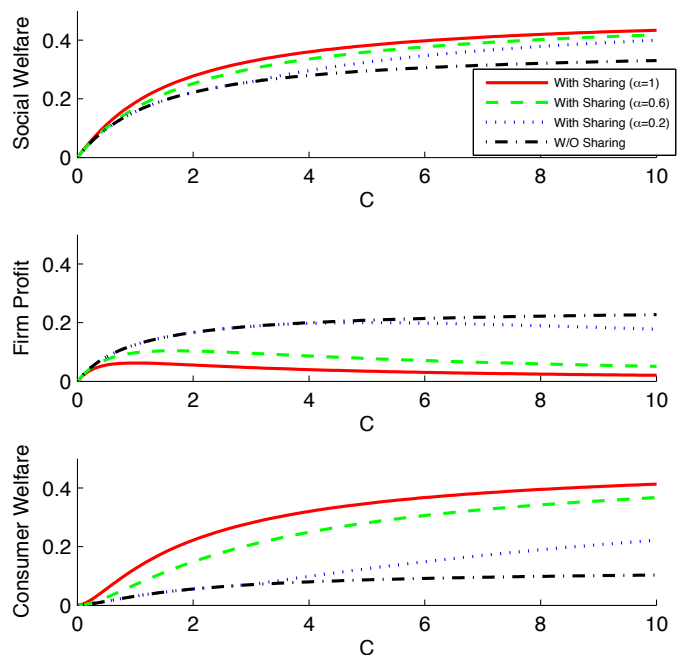


Fig. 14. Impact of  $C$  and  $\alpha$  on welfares with linear congestion and linear inverse-demand

## V. GENERAL CONGESTION AND GENERAL DEMAND

In this section we make a few comments about the more general scenario, where the congestion is any increasing, convex, differentiable function  $g(x)$  and the inverse demand function  $P(x)$  is decreasing, concave and differentiable. Additionally, we assume that  $g(x)$  decreases point-wise as bandwidth increases. Under these assumptions, the pricing problem faced by the primary will always be a convex problem with a unique solution. The following theorem shows that in such a general setting again the corresponding equilibrium consumer welfare and social welfare are non-decreasing as the bandwidth

is increases.<sup>4</sup>

**Theorem 5.1:** With perfect sharing, social welfare and consumer welfare are non-decreasing functions of bandwidth.

*Proof:* Consider two choices of bandwidth  $B < B'$ , with corresponding congestion functions  $g(x)$  and  $g'(x)$  as shown in Figure 15. Let  $x^*$  and  $x^{*'}$  be the overall customer mass served in the market with perfect sharing, where clearly  $x^* < x^{*'}$ , i.e. greater bandwidth means more customers are served. The delivered price in these two cases will be  $P(x^*)$  and  $P(x^{*'})$ , with  $P(x^*) > P(x^{*'})$  since  $P(x)$  is decreasing (these correspond the points A and H in Figure 15). From this it is clear that the consumer welfare increases with the bandwidth, since more customers are being served at a lower delivered price.

To see the result for social welfare. Referring to Figure 2, note that welfare for  $B$  and  $B'$  is given by the area of the regions  $K-A-B-C-D-N$  and  $K-H-E-F-G-M$ , respectively. To compare these two areas, we separate each of them into two pieces using the same segment  $HM$ . It is easy to see that the area above segment  $HM$  is exactly consumer welfare for  $B'$  which is larger than the corresponding area with  $B$ . Next we turn to the portion of the regions under the segment  $HM$ . For  $B$  this corresponds to the rectangle area  $H-I-C-B$ , and for  $B'$ , it corresponds to the rectangle area  $H-E-F-G$ . As rectangle  $H-E-F-G$  is the optimal profit for the primary firm, it is the largest rectangle within the region  $H-O-M$  and so it must have a larger area than that of rectangle  $H-I-C-B$ . Thus the welfare for  $B'$  must be greater than that with  $B$ , completing the proof. ■

Note in this theorem we did not say anything about the profit of the primary firm. As we have seen in the previous sections it can decrease with increasing  $B$ , but whether this occurs for a specific family of congestion costs appears to depend on the details of how the costs scale with  $B$ . The next result, shows that in general, consumer welfare and social welfare increase with sharing while firm profits decrease.

**Lemma 5.2:** With shared spectrum, social welfare and consumer welfare are greater than that without sharing while firm profit is less.

*Proof:* For the change in social welfare and consumer welfare with bandwidth, the proof is similar to Thereom 5.1 and is illustrated in Figure 16, while the proof that the primary firm profit decreases follows from the same argument as given to show this for linear inverse demands at the end of the previous section. ■

Sharing can results in either the primary serving more customers or fewer customers, the next theorem characterizes when this occurs depending of the inverse demand  $P(x)$ .

**Theorem 5.3:** When the firm shares spectrum with secondary firms, the customer mass  $x_1$  compared with non-sharing case will

- increase if  $P(x_1) + P'(x_1)x_1 < P(x^*)$
- decrease if  $P(x_1) + P'(x_1)x_1 > P(x^*)$

<sup>4</sup>We state for perfect sharing, a similar result holds for degraded sharing

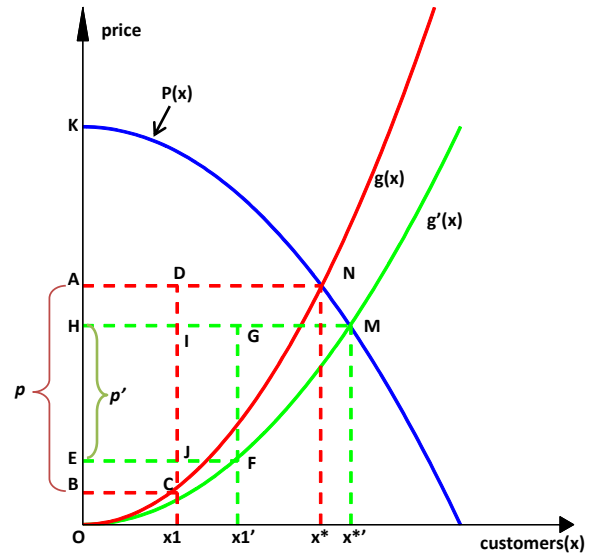


Fig. 15. Change in congestion function with respect to bandwidth with shared spectrum in a general setting. The red lines indicate the case with less bandwidth, while the green lines show the case with more bandwidth.

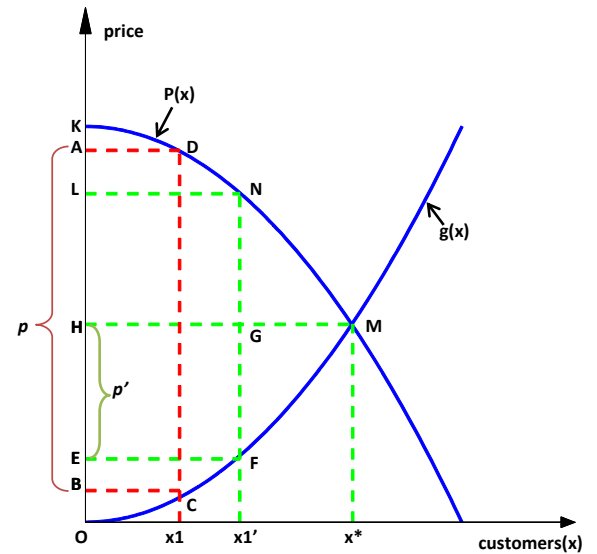


Fig. 16. Comparison between with and without sharing in a general setting. Red lines indicate the monopoly case without sharing, and green lines show the case with perfect sharing.

*Proof:* The difference between with and without perfect sharing is the constraint on the delivered price. That is,

$$\text{Without sharing } p + g(x_1) = P(x_1)$$

$$\text{With sharing } p + g(x_1) = P(x_1 + \sum x_i)$$

To maximize firm profit, i. e.,  $px_1$ , we can again solve for the price in terms of the customers served in each case, and substitute this into the objective, giving a function of only  $x_1$ . Differentiating this with respect to  $x_1$ , and setting it equal to zero we obtain

$$P(x_1) + P'(x_1)x_1 = g(x_1) + g'(x_1)x_1$$

without sharing and

$$P(x^*) = g(x_1) + g'(x_1)x_1.$$

with sharing. We know that once congestion function and demand functions are fixed,  $P(x^*)$  is a constant. The right-hand side of two previous equations is identical and strictly increasing in  $x_1$  as  $g(x)$  is increasing and convex. Hence, comparing the left-hand side of the equations will specify which case has the larger value of  $x_1$ , completing the proof. ■

## VI. CONCLUSIONS

We have studied a stylized model for sharing a licensed band of spectrum among a primary and multiple secondary service providers, all of who seek to serve a common pool of customers. We have shown that for this model, consumer welfare and overall social welfare never decreases with sharing compared to without sharing, while the profit of the primary firm never increases with sharing and may decrease. Further we have shown that as the bandwidth of the shared band increases, overall welfare and consumer welfare increases, while the primary's profits may either increase or decrease.

We note that these results are quite different from that observed in [11], where adding a separate band of shared unlicensed spectrum to a market consisting of a single primary license holder was shown to potentially decrease social welfare. Further, in [11] the social welfare could decrease as the bandwidth of the unlicensed band was increased. In [11] as in the model considered here, the addition of sharing to the market potentially places a limit on the delivered price that the primary can charge. However, when this sharing is in a separate unlicensed band, the primary can increase this limit by shifting more traffic to the unlicensed band and causing it to be more congested. Such an action is exactly the cause of the decrease in social welfare. Under the primary-secondary sharing model considered in this paper, the primary can not effect this limit on the delivered price, since it just depends on the total traffic served by it and the secondary. This gives the primary less flexibility and prevents social welfare from decreasing, but also leads to a greater decrease in the primary's profits due to sharing. This suggests that if limited spectrum is available for sharing, there might be advantages to allocating the spectrum to an incumbent and sharing it under a primary-secondary model as opposed to simply making it unlicensed, though we leave a detailed comparison of these regimes to future work.

We also considered a model with degraded sharing in which secondary users incurred an additional overhead, leading them to experience greater congestion. We showed that if this degradation was too high, then even if sharing was allowed, no secondary service providers would enter the market. This suggests that incumbents who do not want to share may have an incentive to place overly burdensome requirements on secondary users, and a good policy should seek to minimize these overheads. Here, we only modeled degradation due to sharing on the part of the secondary providers, another

potentially interesting future direction would be to consider sharing which also degrades the primary's performance, due for example to imperfect sensing on the part of secondary users.

Here we simply compared a scenario with and without sharing but did not address the primary provider's incentives to share. Given that the primary's profits decrease, it would clearly not have an incentive to share unless it was required by policy or received some compensation for doing this. Further, since the secondary users' profits are competed away, the primary could not hope to receive any compensation from them. However, since overall welfare does increase with sharing, this could suggest a policy in which the government collects a portion of the added consumer welfare (e.g. via taxes) and compensates the primary for allowing sharing.

In the model considered here, secondary sharing was open, which resulted in the secondary SPs not receiving any profits. An alternative would be for secondary access to also be licensed. If a single secondary SP was licensed, then they would be able to sustain a positive price and profit. In this case, the pricing game between the primary and secondary SP becomes more complicated, another topic we leave for future work. Also, here we did not consider the investment decisions made by the primary or secondary firms. Such issues could be introduced here using a similar model as in [12], where they were considered for the case of unlicensed spectrum. Finally, secondary SPs could serve different market segments than a primary firm and thus not be direct competitors leading to different models of competition that could be studied.

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