The Impact of Unlicensed Access on Small-Cell Resource Allocation

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***Abstract*—Small-cells in licensed spectrum and unlicensed access via Wi-Fi are two commonly used options to reduce the demand for conventional macro-cellular networks and to provide expanded wireless services to low mobility users. The mix of these technologies depends on both the decisions made by wireless service providers (SPs) that seek to maximize revenue, and the allocation of licensed and unlicensed spectrum by regulators. In this paper, we study these interactions and consider heterogeneous cellular networks together with unlicensed access. Both a single monopoly SP and multiple competing SPs are investigated. The SPs split any available licensed spectrum into two separate bands for macro- and small-cells, which are then used to serve two types of users: mobile and fixed. Mobile users must be served by macro-cells only, whereas fixed users can be served by either macro- or small-cells, or alternatively by unlicensed access service. While the providers charge a (different) price per unit rate for licensed access services (macro- or small-cell), unlicensed access is free. We formulate a sequential game in which the users choose a service that yields the highest payoff, and the providers allocate bandwidth across macro-/small-cells. In general, the competition from unlicensed access results in inefficient (albeit unique) market equilibria, and in many cases all or some SPs allocate no resources to small-cell deployment. We conclude by showing how our framework can also be used to optimize the fraction of unlicensed spectrum when new bandwidth becomes available.**

I. INTRODUCTION

5G cellular networks are evolving towards heterogeneous networks (HetNets) to cope with the accelerating demand for wireless data along with variations in mobility and service requirements [2]. The primary feature of a HetNet is the deployment of multiple types of access infrastructure with different transmission powers, ranges, and spectral efficiencies, such as macro-cells targeting wide-area coverage and small-cells targeting local access. In addition to cellular infrastructure in licensed spectrum, unlicensed services (e.g., Wi-Fi) are an increasingly common alternative for users with low mobility. It is expected that HetNets will make more extensive use of unlicensed spectrum and possibly utilize technologies such as LTE-U, LAA, and MulteFire [3].[1] The mix of different technologies provide users more options to access wireless services, which in turn affects strategic decisions of SPs on pricing and network resource allocation.

While the introduction of small-cells will increase overall data capacity in a HetNet, network management and resource allocation become more complicated. In particular, an SP must allocate available spectrum resources across different cell types (small vs. macro) taking into account mobility patterns and demand for different services. This allocation then interacts with pricing strategies that can differentiate among service categories and controls revenue. Resource allocation is further complicated by the existence of unlicensed Wi-Fi networks, which can be viewed both as a complementary resource for offloading traffic, and as a competitive rival for small-cell networks in licensed spectrum.[2]

In this paper we study the effect of unlicensed spectrum on resource allocation in HetNets containing both macro- and small-cells deployed in licensed (proprietary) spectrum, and operated by a cellular SP, or multiple competing SPs. To model the demand for different services, we consider two types of users: mobile and fixed. Mobile users must be served by macro-cells only, whereas fixed users can be served by either macro- or small-cells, or via unlicensed access. We assume that the SPs charge a different price per unit rate to access their network via macro- or small-cells. In contrast, there is no access fee for the unlicensed band. In all cases the available rate is split evenly among all users sharing the band. Given this setup, the SPs wish to set prices and allocate bandwidth across macro- and small-cells to maximize their revenue.

Our model is similar to that presented in our previous work [4] to study bandwidth allocation in HetNets with competing SPs; however, here the main distinguishing feature is the presence of the unlicensed band. Our results show how the unlicensed spectrum affects the SPs' willingness to allocate resources to their small-cell networks. Moreover, the model and results can be used to quantify the utility (total welfare) offered by unlicensed spectrum, and to compare it with that generated by licensed spectrum while taking the strategic decisions of the SPs into account. The introduction of unlicensed spectrum raises several analytical challenges *vis-a-vis* prior work (e.g.,

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[1] Technologies like LTE-U and LAA are used by SPs to essentially expand their licensed services. In this paper we do not consider these types of technologies in unlicensed specturm.

[2] For example, AT&T's WiFi network helps to expand the capacity of AT&T's cellular network, whereas Comcast's WiFi service effectively competes with cellular services.

[4]), such as the existence and uniqueness of equilibrium, as well as social welfare analysis. This is due to the expanded set of choices (i.e., there are now three options for serving fixed users compared to two) and also the additional conditions that arise on equilibria due to competition from the unlicensed service; unlicensed access serves as an outside option to fixed users and an explicit comparison of net payoff needs to be made to determine the equilibria.

A. Main Contributions

We now summarize our main contributions. We first focus on a monopoly SP, and then generalize to the more complicated scenario with multiple competing SPs. In both scenarios we model SP and user actions as a two-stage game in which the SPs first partition the licensed band into separate subbands for the macro- and small-cell networks, and subsequently announce prices for services. The prices then enable the fixed users to determine their network association (macro-/small-/unlicensed). The following results are obtained by analyzing sub-game perfect equilibria, assuming a class of α-fair utility functions for tractability.

1. *HetNet Market structure:* In equilibrium the macro-cell network serves *only* mobile users. Fixed users then associate with either the small-cell or the unlicensed network. This market structure holds irrespective of whether the SPs maximize revenue or social welfare[3].

2. *Bandwidth allocation with monopoly SP:* Comparing bandwidth allocations with and without unlicensed spectrum, when maximizing revenue, a monopoly SP may allocate *more* bandwidth to small-cells when an unlicensed network is present. This occurs when the unlicensed network offers a sufficiently low rate (e.g., due to a small amount of unlicensed bandwidth). This seems counterintuitive when the unlicensed network is viewed as an additional resource; however, this is because the presence of this new resource alters the dependence of quantity (users served) with price. In contrast, when the SP maximizes social welfare, it always allocates less bandwidth to small cells when competing with unlicensed spectrum.

3. *Equilibrium with competitive SPs:* With multiple competitive SPs we prove that there always exists a unique sub-game perfect equilibrium with an associated bandwidth allocation. Furthermore, the equilibrium can fall into one of three categories: (1) all SPs allocate bandwidth to both macro- and small-cells ("Macro-Small Nash Equilibrium", or MSNE); (2) a subset of SPs allocate bandwidth to macro-cells only, and the rest allocate to both macro and small cells ("Macro-Favored Nash Equilibrium", or MFNE); and (3) all SPs allocate bandwidth only to macro-cells ("Macro-only Nash Equilibrium", or MNE). In the absence of unlicensed spectrum, for our choice of utility functions only the MSNE is possible [4]. Hence, the increased competition from the unlicensed network allows for equilibria where some SPs shun small-cells and allocate bandwidth only to the macro-cells. Extending to the asymptotic scenario where the number of SPs goes to infinity, we observe that in general, the equilibrium is not efficient for any non-zero amount of unlicensed spectrum.

4. *Dependence of social welfare on the licensed/unlicensed split:* When new spectrum becomes available, spectrum regulators, such as the FCC, determine how much spectrum should be licensed or unlicensed. We use the preceding results to show how the mix of licensed and unlicensed spectrum affects social welfare. This depends crucially on the relative spectral efficiencies associated with the small-cell and unlicensed networks. If the small-cell network has higher spectral efficiency, then allocating the entire bandwidth as licensed maximizes social welfare. Conversely, if the unlicensed network has higher spectral efficiency, then there is an optimal (positive) amount of unlicensed spectrum, which need not be efficient. Furthermore, allocating insufficient unlicensed bandwidth in this scenario can cause the social welfare to decrease below the all-licensed allocation. (This is illustrated in Fig. 6 in Sec. VII.)

[3]This result was also shown to hold without unlicensed spectrum in [4].

B. Related Work

There is extensive literature on pricing and bandwidth allocation in heterogeneous wireless networks. In [5]–[7] small-cell service is an enhancement to macro-cell service, and in [8]–[13] small- and macro-cells provide separate services (as we assume here). Optimal pricing only is studied in [5], [7], [12], whereas allocation of bandwidth only is studied in [13], [14]. Joint pricing and bandwidth allocation is studied in [6], [8]–[11]. However, in that work there is a single SP (monopoly) and no unlicensed spectrum.

In [4], [15]–[18], competition among multiple SPs providing HetNet services is investigated. References [15], [16] study pricing and service competition with fixed bandwidth allocations, while [4], [17], [18] study both pricing and bandwidth optimization. However, in [17], [18] the SPs compete to acquire the spectrum from a spectrum broker, as opposed to optimizing the bandwidth allocation across the different cell types in [4]. The preceding work does not consider any interactions with unlicensed access.

While the preceding work focuses on HetNet deployments using licensed spectrum, the interaction of unlicensed with licensed spectrum in other contexts is considered in [19]–[23]. In [19] an economic analysis of the trade-off between incremental licensed and unlicensed spectrum allocations is presented, which shows that licensed spectrum is the favored option. In [20], an intermediate model of only price competition between two SPs having a fixed licensed part of the spectrum and sharing the remaining part as unlicensed is proposed. It is shown that user (consumer) welfare increases with the proportion of unlicensed spectrum, benefiting from the decreased SP price in response to the competition from unlicensed spectrum; however, the overall social welfare always decreases when more spectrum is allocated to unlicensed access, indicating resources are used less efficiently with unlicensed access. This result is consistent with our results assuming the average spectral efficiency of unlicensed access is smaller than that of small-cells operating in licensed spectrum. Reference [21] studies social welfare when unlicensed spectrum is added to an existing allocation of licensed spectrum among incumbent competing SPs. The conclusion is that the social welfare can decrease over a significant range of unlicensed bandwidths. In [22] the authors optimize the size of licensed and unlicensed bands assuming centralized control, according to a average packet delay minimization objective. It is shown that the global minimum average delay can be achieved by a sparse allocation, where the spectrum of each radio access technology is divided into a limited segments. In contrast, [23] introduces a software defined cellular network architecture and proposes a fully distributed algorithm to optimize user association, channel usage time, and transmission power in both licensed and unlicensed bands. The authors conclude that the exploitation of licensed and unlicensed bands by integrating femto-cell and WiFi access can reduce the demand for licensed spectrum effectively. The preceding work assumes that the licensed spectrum supports a single type of service, as opposed to the mix of mobile and fixed users here.

In contrast to the previous work, a key feature in our model is that the SPs strategically choose the amount to invest in small-cells, which depends on the amount of available unlicensed spectrum and the distribution of fixed versus mobile users. An equilibrium, associated with a market outcome, then consists of an allocation across licensed small-cell and licensed macro-cell.

The paper is organized as follows. We introduce our system model in Section II. The price and user association equilibrium for a monopoly SP is presented in Section III. Optimal bandwidth allocation for revenue maximization is discussed in Section IV, and for social welfare maximization in Section V. We then consider multiple competing SPs in Section VI. The dependence of social welfare on the amount of unlicensed bandwidth is discussed in VII, and conclusions are presented in Section VIII. All proofs of the main results and several supplemental results can be be found in the appendices.

II. System Model

A. HetNet Service Model

We consider a HetNet with one or more SPs offering wireless service. Each SP potentially has a two-tier cellular network operating in licensed spectrum consisting of macro-cells and small-cells. Macro-cells have large transmission power and therefore wide-area coverage, whereas small-cells have lower transmission power and local coverage. We assume each SP has the same infrastructure deployment density and normalize the macro-cell density per SP to be one. Macro-cells and small-cells are assumed to be uniformly deployed over a given area[4] and operate in separate bands.[5] Apart from the cellular networks, there is a WiFi network operating in unlicensed spectrum in which WiFi Access Points (APs) are deployed with no access charge.

Users in the network are categorized into two types based on their mobility patterns: mobile users are highly mobile and therefore can only be served by macro-cells, whereas fixed users are relatively stationary and can associate with macro-cells, small-cells or the unlicensed WiFi network (but not multiple at the same time). Both types of users are uniformly distributed over the given area. We assume a large number of users so that we can model them as non-atomic. The density of mobile and fixed users are given by N_m and N_f, respectively.

We assume each SP i has total bandwidth B_i which it can further split into $B_{i,M}$ and $B_{i,S}$, the amount of bandwidth allocated to its macro-cell and small-cell networks, respectively. No bandwidth allocation in a cell effectively implies no deployment in that cell. Consequently, the total data rate SP i's macro-cells can provide is given by $C_{i,M} = B_{i,M}R_0$, where R_0 is the (average) spectral efficiency.[6] Similarly, the total available rate in small-cells of SP i and in the unlicensed WiFi network are defined as $C_{i,S} = \lambda_S B_{i,S} R_0$ and $C_U = \lambda_U B_U R_0$, respectively. Here λ_S and λ_U reflect the rate difference due to the combination of spectral efficiency and deployment density differences in small-cells and Wi-Fi APs compared with macro-cells. Since small-cells generally have a higher spectral efficiency and larger deployment density, we assume $\lambda_S > 1$ [25]. Wi-Fi APs typically have lower spectral efficiency than cellular deployments, but may have a higher deployment density. Therefore, we do not make any assumptions on λ_U.

Each SP i offers separate macro- and small-cell service and charges all users the same price per unit rate, denoted by $p_{i,M}$ and $p_{i,S}$, respectively. In contrast, Wi-Fi service in unlicensed spectrum is free with no service charge. We denote the density of users connected to macro-cells of SP i, small-cells of SP i and the Wi-Fi network as $K_{i,M}$, $K_{i,S}$ and K_U, respectively. Note that $K_{i,S}$ and K_U only consist of fixed users, while $K_{i,M}$ may include both mobile users and fixed users. Additionally, mobile users are assumed to have priority connecting to macro-cells. As a result, macro-cells can only accommodate fixed users if all mobile users have been served.

B. User and SP Optimization

We assume all users are endowed with the same utility function $u(r)$ which only depends on the service rate it gets from any type of service. We further restrict this to be an α-fair utility function [26] with $\alpha \in (0, 1)$:

$$u(r) = \frac{r^{1-\alpha}}{1-\alpha}, \quad \alpha \in (0, 1). \tag{1}$$

This restriction enables us to explicitly calculate many e-quilibrium quantities, which appears to be difficult for more general classes of utility, but is needed to find the multi-stage equilibrium behavior as explicit comparisons need to be made with the outside option of unlicensed access for fixed users. Further this class is widely used in both networking and economics, where it is a subset of the class of iso-elastic utility functions. Requiring $\alpha > 0$ ensures strict concavity but allows

[4] Alternatively, we can view small-cells and WiFi APs as being uniformly deployed over "hot spot" areas and restrict fixed users to these areas.

[5] Equivalently, macro- and small-cells could operate in different time-slots, e.g., using the Almost Blank Subframes (ABS) feature in LTE [24].

[6] Of course the actual rate a SP can provide at any time will depend on many factors such as the channel gains to its users and the scheduling algorithm employed. Here, we view C_i as averaging over such effects over a long enough time horizon, which is reasonable for the network planning problems we consider.

us to approach the linear case as $\alpha \to 0$. The restriction $\alpha < 1$ ensures that utility is non-negative so that a user can always "opt out" and receive zero utility. As $\alpha \to 1$, we obtain the $\log(\cdot)$ (proportionally fair) utility function.

For a given licensed service with price p,[7] each user requests a rate that maximizes their net payoff W, defined as the difference between their utility and the cost of service, i.e. they solve:

$$\underset{r \geq 0}{\text{maximize}} \quad W = u(r) - pr, \tag{2}$$

For unlicensed access, since there is no access charge, we assume all users simply share the available total rate equally. Fixed users then choose the network service (small-cell, macro-cell, or unlicensed access) with the largest payoff. This division of the total rate among subscribing users and the subsequent assessment of payoff naturally incorporates the impact of congestion-based latency; e.g., with other options available all fixed users may not exclusively use the access-cost free unlicensed band.

For α-fair utility functions, (2) has the unique solution:

$$r^* = D(p) = (u')^{-1}(p) = (1/p)^{1/\alpha}, \tag{3}$$

where $D(p)$ is the user's rate demand function and u' denotes the first derivative of u. The maximum net payoff for a user is thus:

$$W^*(p) = u(D(p)) - pD(p) = \frac{\alpha}{1-\alpha} p^{1-\frac{1}{\alpha}}. \tag{4}$$

It is easy to verify that $W^*(p)$ is decreasing in p. As a result, if a fixed user's rate demand can be satisfied by both, the user would always prefer the network service with lower price.

Each SP i decides on the bandwidth partition $(B_{i,M}, B_{i,S})$ and prices $(p_{i,M}, p_{i,S})$ to maximize its revenue S_i, i.e., the aggregate amount paid by all users choosing its macro- and small-cell services. This can be formulated as:

$$\text{maximize} \quad S_i = K_{i,M} p_{i,M} D(p_{i,M}) + K_{i,S} p_{i,S} D(p_{i,S}) \tag{5a}$$

$$\text{subject to} \quad K_{i,M} D(p_{i,M}) \leq C_{i,M}, K_{i,S} D(p_{i,S}) \leq C_{i,S} \tag{5b}$$

$$0 \leq p_{i,M}, p_{i,S} < \infty \tag{5c}$$

$$B_{i,M}, B_{i,S} \geq 0, B_{i,M} + B_{i,S} \leq B_i. \tag{5d}$$

Here the first constraint ensures that the SP can meet the rate demanded by the users, where $C_{i,M}$ and $C_{i,S}$ depend on the bandwidth allocation. Note also that $K_{i,M}$ and $K_{i,S}$ depend on the user associations, which in the case of multiple SPs will depend on the prices and bandwidths chosen by those SPs.

We therefore model the choice of bandwidth allocations and prices as a game played by the SPs. Assuming bandwidth allocations take place over a slower time-scale than price adjustments, the game then consists of two stages: First, SPs determine their bandwidth allocation between macro-cells and small-cells. Then given their bandwidth allocations, SPs announce prices for both macro- and small-cells. Users then choose services as described previously. We characterize the sub-game perfect equilibrium by first characterizing a *user association equilibrium* for a fixed set of prices and bandwidth allocation, and then study the equilibrium bandwidth allocation based on the results obtained in the first step.

C. Social Welfare

An objective of a social planner, such as the FCC, may be the social welfare provided by the network. In our model, the social welfare is the sum of the revenue over all SPs plus the net payoff of all users, i.e., the sum net utility over all users. This can be formulated as maximizing

$$\text{SW} = \sum_{i=1}^{N} K_{i,M} u(R_{i,M}) + K_{i,S} u(R_{i,S}) + K_U u(R_U) \tag{6}$$

subject to the same constraints (5b)(5c)(5d) for each SP i. Here, $R_{i,M}$, $R_{i,S}$ and R_U denote the average service rates per user in each respective service, which in turn depends on the prices and bandwidth allocations.

III. User Association and Prices: Single SP

We first focus on the user association equilibrium given a fixed bandwidth allocation between macro- and small-cells with a monopoly SP. Here we assume the SP maximizes its revenue, but consider social welfare maximization in Section V. Since mobile users can only connect to macro-cells, we need only consider the association of fixed users.

Given a fixed bandwidth allocation and announced prices, users select their service to maximize their net payoff. As indicated in Section II, given equitable macro- and small-cell service, fixed users choose the one with the lower price. For the Wi-Fi service, the user's net payoff is given by $u(\frac{C_U}{K_U})$. With non-atomic users, in a user association equilibrium the net payoffs of any services that are used by fixed users must be equal, and any service not used must have a lower net payoff.

The SP adjusts the prices p_M and p_S to maximize its revenue taking into account the user association equilibrium.[8] Two scenarios are possible:

1. *Mixed service*: Macro-cells serve both mobile users and a subset of fixed users;
2. *Separate service*: Macro-cells only serve mobile users.

Theorem 1 (User Association): Given a fixed bandwidth allocation between macro- and small-cells and revenue-maximizing prices, the market clears, i.e., all users are served, and all rate is allocated. Further, there exists a threshold $B_{S,0}$ such that if $B_S < B_{S,0}$, then the mixed service scenario holds. Otherwise, the separate service scenario holds. For the mixed service scenario, $p_M = p_S$, whereas for the separate service scenario $p_M > p_S$.

Hence, fixed users choose to associate with macro-cells only when the small-cell bandwidth is sufficiently small. In equilibrium, the mixed service scenario implies that the net

[7] We assume that the users are *price-taking* in that they do not anticipate how their selection of service or rate will effect the resulting prices. This is reasonable under our assumption of many small users.

[8] In the monopoly scenario we drop the SP subscript i.

payoff to fixed users from connecting to macro-, small-cells and Wi-Fi must be the same, and so we can write

$$u\Big(\frac{C_S}{K_S}\Big) - p_S\frac{C_S}{K_S} = u\Big(\frac{C_U}{K_U}\Big) = u\Big(\frac{C_M}{K_M}\Big) - p_M\frac{C_M}{K_M}. \quad (7)$$

Here, we are using the fact from Theorem 1 that all of the rate is allocated. For separate service the same equality will hold only for small-cells and unlicensed. With α-fair utilities, the rate per user $\frac{C_S}{K_S}$ must satisfy (3). Combining with (28c) yields the following lemma that gives a useful comparison of the equilibrium rates of each service.

Lemma 1: In equilibrium fixed users in licensed spectrum (macro-/small-cells) achieve $\frac{1}{\kappa}$ times the average rate of users in unlicensed spectrum where $\kappa = \alpha^{1/(1-\alpha)}$.

Note that $1/\kappa > 1$, which accounts for the access price charged in licensed spectrum. Interestingly, with α-fair utilities this ratio is independent of the other system parameters. We can express the bandwidth threshold in Theorem 1 as

$$B_{S,0} = \max\left(\kappa N_f B_M R_0 - N_m C_U, 0\right) / (\kappa N_m \lambda_S R_0). \quad (8)$$

From Theorem 1, since all users are served we have $K_U + K_S + K_M = N_m + N_f$. Using this and Lemma 1, we have the following expressions for the equilibrium prices, service rates, and densities of users for each type of service in the mixed service scenario:

$$K_U = (N_f + N_m)\frac{C_U}{C_U + \kappa(C_M + C_S)}, R_U = \frac{C_U}{K_U}, \quad (9a)$$

$$K_M = (N_f + N_m)\frac{\kappa C_M}{C_U + \kappa(C_M + C_S)}, R_M = \frac{C_M}{K_M},$$
$$p_M = 1/({R_M}^{\alpha}), \quad (9b)$$

$$K_S = (N_f + N_m)\frac{\kappa C_S}{C_U + \kappa(C_M + C_S)}, R_S = \frac{C_S}{K_S},$$
$$p_S = 1/({R_S}^{\alpha}). \quad (9c)$$

Similarly, for the separate service scenario, we have:

$$K_U = N_f\frac{C_U}{\kappa C_S + C_U}, R_U = \frac{C_U}{K_U}, \quad (10a)$$

$$K_M = N_m, R_M = \frac{C_M}{K_M}, p_M = 1/({R_M}^{\alpha}), \quad (10b)$$

$$K_S = N_f\frac{\kappa C_S}{\kappa C_S + C_U}, R_S = \frac{C_S}{K_S}, p_S = 1/({R_S}^{\alpha}). \quad (10c)$$

IV. Bandwidth Allocation with a Single SP

We now characterize the revenue-maximizing bandwidth allocation given the prices and user association equilibrium in the preceding section. In the mixed service scenario, the problem is:

$$\begin{aligned} \text{maximize} \quad & S = (B_M + \lambda_S B_S)R_0 u'\Big(\frac{(B_M + \lambda_S B_S)R_0}{K_M + K_S}\Big) \\ \text{subject to} \quad & B_M, B_S \geq 0, B_M + B_S \leq B, B_S < B_{S,0}. \end{aligned}$$

where K_M, K_S are given in (9b)(9c), and $B_{S,0}$ is defined in (8).

In the separate service scenario, the objective becomes

$$S = B_M R_0 u'\Big(\frac{B_M R_0}{K_M}\Big) + \lambda_S B_S R_0 u'\Big(\frac{\lambda_S B_S R_0}{K_S}\Big) \quad (11)$$

where K_M, K_S are given in (10b)(10c), and the constraint $B_S \geq B_{S,0}$ applies.

Theorem 2 (Monopoly Bandwidth Allocation): The optimal bandwidth allocation is unique and always corresponds to the separate service scenario.

The optimal bandwidth allocation $(B_S^{\text{rev}}, B_M^{\text{rev}})$ satisfies the necessary conditions:

$$\begin{cases} \Big[\big(\frac{\kappa\lambda_S B_S^{\text{rev}} R_0 + C_U}{KN_f}\big)^{-\alpha} - \alpha\big(\frac{\kappa\lambda_S B_S^{\text{rev}} R_0 + C_U}{KN_f}\big)^{-\alpha} \\ \quad \times \frac{\kappa\lambda_S B_S^{\text{rev}} R_0}{\kappa\lambda_S B_S^{\text{rev}} R_0 + C_U}\Big] = \frac{(1-\alpha)}{\lambda_S}\big(\frac{B_M^{\text{rev}} R_0}{N_m}\big)^{-\alpha} \\ B_S^{\text{rev}} + B_M^{\text{rev}} = B, \qquad B_S^{\text{rev}}, B_M^{\text{rev}} \geq 0. \end{cases} \quad \text{(P1)}$$

Although the solution to equation (P1) must be computed numerically, some general properties are easily established. First, $B_M^{\text{rev}} > 0$ (always), but $B_S^{\text{rev}} > 0$ if and only if $C_U < C_U^{\text{rev}}$ (otherwise $B_S^{\text{rev}} = 0$), where

$$C_U^{\text{rev}} = \frac{\kappa N_f B R_0}{N_m}\Big(\frac{\lambda_S}{1-\alpha}\Big)^{\frac{1}{\alpha}}. \quad (12)$$

When $\alpha \to 0^+$ or $\alpha \to 1^-$, $C_U^{\text{rev}} \to +\infty$. That is, as the utility function becomes either linear or logarithmic, the SP always allocates some bandwidth to small-cells.

The optimal bandwidth allocation with no unlicensed spectrum can be determined by setting $C_U = 0$. We will denote all associated quantities for this case with a tilde, i.e., $(\tilde{B}_S^{\text{rev}}, \tilde{B}_M^{\text{rev}})$ is the optimized bandwidth with $C_U = 0$. With no unlicensed spectrum, it is straightforward to show that $\tilde{B}_S^{\text{rev}} = \tilde{\beta}B$ where $\tilde{\beta} = \frac{N_f}{N_f + N_m\lambda_S^{1-1/\alpha}}$, so that $\tilde{B}_S^{\text{rev}} > 0$ and $\tilde{B}_M^{\text{rev}} > 0$. In contrast, with unlicensed spectrum, the additional competition can cause a revenue-maximizing SP to abandon small-cells altogether. We will see a more drastic example of this effect when we consider competing SPs.

The next theorem compares the amount of bandwidth allocated to small-cells with and without unlicensed spectrum.

Theorem 3 (Bandwidth Allocation Comparison): There exists a threshold C_U^{th} such that when $C_U < C_U^{th}$, $B_S^{\text{rev}} > \tilde{B}_S^{\text{rev}}$, and when $C_U > C_U^{th}$, $B_S^{\text{rev}} < \tilde{B}_S^{\text{rev}}$, where $C_U^{th} = \beta^*\kappa\lambda_S\tilde{B}_S^{\text{rev}}R_0$ with β^* being the unique strictly positive solution of

$$(1-\alpha)(1+\beta)^{1+\alpha} - \beta = 1-\alpha. \quad (13)$$

Furthermore, the SP's revenue is always less with unlicensed spectrum than without.

The last part of Theorem 3 is expected since unlicensed access competes with the SP for fixed users. The first part can be explained as follows. Compared to the case without unlicensed spectrum, if the SP keeps its bandwidth allocation the same when unlicensed spectrum is added, then it will clearly lose revenue as fewer users will use its small-cell service. To increase its revenue it could decrease B_S, shifting more resources to mobile users to increase revenue from them, or it could increase B_S to make its small-cell service more attractive relative to the unlicensed network. However, increasing B_S also results in a decrease in p_S as the service rate per user increases. (There is also a loss in revenue from the mobile users.) When C_U is small, this decrease in p_S is small as more users will switch to the small-cells making the second option more attractive. When C_U is large enough, fewer users will switch to the small-cells per unit of additional bandwidth, making the first option more profitable.

When $\alpha \to 0^+$ (approaching a linear utility function), $\kappa \to 0^+$ and $\beta^* \to 0^+$, and therefore, $C_U^{th} \to 0^+$. This implies that the SP should always invest less bandwidth in small-cells compared with the scenario without unlicensed spectrum. However, in this case the SP should still allocate almost all bandwidth to small-cells, i.e., $B_S^{\text{rev}} \to B$. This is because $\alpha = 0$ effectively corresponds to maximizing the sum rate. (This can be seen directly from the revenue function.)

When $\alpha \to 1^-$, the utility function becomes logarithmic, $\kappa \to \mathrm{e}^{-1}$, and $\beta^* \to \infty$, so that $C_U^{th} \to \infty$. Hence, when unlicensed spectrum is added the SP should allocate more bandwidth to small-cells. Again, in the limit the SP allocates all bandwidth to small-cells, i.e., $B_S^{\text{rev}} \to B$. We can see this by rewriting the revenue function as follows:

$$\begin{aligned} S &= B_M R_0 u'\Big(\frac{B_M R_0}{K_M}\Big) + \lambda_S B_S R_0 u'\Big(\frac{\lambda_S B_S R_0}{K_S}\Big) \\ &= K_M R_M u'(R_M) + K_S R_S u'(R_S). \end{aligned} \tag{14}$$

For a log-utility function the revenue per mobile user $R_M u'(R_M)$ and revenue per fixed user $R_S u'(R_S)$ become constants and are equal. Therefore maximizing revenue is equivalent to maximizing K_M and K_S. As a result, the SP allocates an arbitrarily small amount of bandwidth to the macro-cells to guarantee that all mobile users are served, and the remaining bandwidth to small-cells to maximize K_S. In contrast, without unlicensed access $\tilde{\beta}$ converges to $\frac{N_f}{N_f+N_m} < 1$ and $\lim_{\alpha \to 1^-} \tilde{B}_S^{\text{rev}} < B$.

V. Social Welfare Maximization

Now we change the objective function to social welfare and analyze the corresponding prices, user association equilibrium and bandwidth allocation.

Theorem 4 (Social Welfare Maximization): Given a single social welfare maximizing SP, the equilibrium has the following properties:
1. The prices and user association are the same as for revenue maximization as stated in Theorem 1.
2. The optimal bandwidth allocation is unique and corresponds to separate service.
3. Compared to the scenario without unlicensed spectrum, with unlicensed spectrum the SP always allocates less bandwidth to small-cells and more bandwidth to macro-cells.

The optimal bandwidth allocation $(B_S^{\text{sw}}, B_M^{\text{sw}})$ satisfies the necessary conditions:

$$\begin{cases} \frac{(N_f)^\alpha \lambda_S \left((\kappa^\alpha+\kappa)C_U + \kappa^{\alpha+1}\lambda_S B_S^{\text{sw}} R_0\right)}{(\kappa\lambda_S B_S^{\text{sw}} R_0 + C_U)^{\alpha+1}} = \big(\frac{B_M^{\text{sw}} R_0}{N_m}\big)^{-\alpha}, \\ B_S^{\text{sw}} + B_M^{\text{sw}} = 1, \qquad B_S^{\text{sw}}, B_M^{\text{sw}} \geq 0. \end{cases} \tag{P3}$$

It can be shown that $B_M^{\text{sw}} > 0$ (always), and $B_S^{\text{sw}} > 0$ if and only if $C_U < C_U^{\text{sw}}$ (otherwise $B_S = 0$), where

$$C_U^{\text{sw}} = \frac{\kappa N_f B R_0 \left[(\alpha+1)\lambda_S\right]^{\frac{1}{\alpha}}}{N_m}. \tag{15}$$

As $\alpha \to 0$, $C_U^{\text{sw}} \to +\infty$, i.e., in the limiting case of a linear utility function, the SP always allocates some bandwidth to small-cells, even if C_U is large. As for revenue maximization, this is because this maximizes the total rate. When $\alpha \to 1$, $C_U^{\text{sw}} \to 2\mathrm{e}^{-1}\lambda_S \frac{N_f B R_0}{N_m}$ (as opposed to infinity for revenue maximization). With no unlicensed spectrum, $\tilde{B}_S^{\text{sw}} > 0$ and $\tilde{B}_M^{\text{sw}} > 0$.

The third item in Theorem 4 is due to the additional resources in the unlicensed network. Since fixed users are better off with unlicensed spectrum, to maximize the sum utility over all users, the SP allocates more bandwidth to the macro-cells.

We emphasize that with unlicensed spectrum, Theorems 3 and 4 imply that the optimal bandwidth allocation is different for revenue maximization versus social welfare maximization. That is, the corresponding necessary conditions generally have different solutions. In contrast, without unlicensed access, revenue and social welfare maximization give the same bandwidth allocation for α-fair utility functions [11].

Figure 1 shows an example of the optimal bandwidth allocation for different α's as the rate offered by the unlicensed network increases. Curves are shown for both revenue and social welfare maximization. The system parameters are $N_f = N_m = 50$, $R_0 = 50$, and $\lambda_S = 4$. For revenue maximization, the curve initially increases for small C_U, and then decreases. In contrast, for social welfare maximization, the curve is monotonically decreasing.

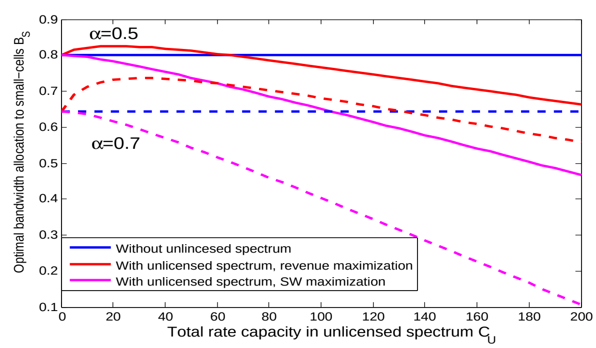

Fig. 1. Optimal bandwidth allocation to small-cells for a single SP versus total unlicensed capacity.

VI. Service Competition Among Multiple SPs

In this section we study service competition among $N > 1$ SPs and investigate the corresponding sub-game perfect equilibrium with unlicensed spectrum. Users choose the service (macro-/small-/unlicensed) which yields the largest net payoff, and they fill the corresponding capacity accordingly. In licensed spectrum, if multiple services offer the same price, then the users are allocated across them in proportion to the capacities. Once a particular service's capacity is exhausted, the leftover demand continues to fill the remaining services in the same fashion. In unlicensed spectrum, fixed users always get an average rate equal to the total rate divided by the mass of fixed users associated with that network.

We again consider a sub-game perfect Nash equilibrium consisting of: (i) A price equilibrium given a fixed bandwidth allocation; and (ii) A bandwidth allocation equilibrium given that prices are set according to (i). As for the scenario with a single SP, given a set of prices and bandwidth allocations, the user association equilibrium falls in one of two categories: a mixed service equilibrium in which all macro-cells serve both mobile and a subset of fixed users, and a separate service equilibrium in which the macro-cells serve only mobile users. The next theorem generalizes Theorem 1 to multiple SPs.

Theorem 5 (Price Equilibrium with Multiple SPs): Given fixed bandwidth allocations for all SPs, there is a unique price equilibrium which clears the market. Further, if

$$\sum_{i=1}^{N} B_{i,S} < \frac{\max\left(\kappa N_f \sum_{i=1}^{N} B_{i,M} R_0 - N_m C_U, 0\right)}{\kappa N_m \lambda_S R_0}, \quad (16)$$

then the mixed service equilibrium holds. Otherwise, the separate service equilibrium holds.

For the mixed service scenario, $p_{i,M} = p_{i,S} = p$ for each SP i; in the separate service case, all SPs i charge the same $p_{i,M}$ and $p_{i,S}$, with $p_{i,M} > p_{i,S}$.

The next theorem characterizes the equilibrium for the bandwidth allocation stage and thus the overall sub-game perfect Nash equilibrium for the game. Before stating this we define the following types of equilibria:

- Macro-Small Nash Equilibrium (MSNE): all SPs allocate bandwidth to both macro- and small-cells.
- Macro-Favored Nash Equilibrium (MFNE): some SPs allocate bandwidth to both macro- and small-cells and the remaining SPs allocate bandwidth to macro-cells only.
- Macro-only Nash Equilibrium (MNE): all SPs allocate bandwidth to macro-cells only.

Theorem 6 (Nash Equilibrium): There always exists a unique sub-game perfect Nash equilibrium and it corresponds to the separate service scenario. In equilibrium fixed users in small-cells achieve a higher average rate than mobile users in macro-cells. Moreover, the equilibrium can only be one of the following types: MSNE, MFNE or MNE.

If there is no unlicensed spectrum, then for α-fair utilities only an MSNE exists and it is always efficient, i.e., it maximizes social welfare [4]. Here, the presence of unlicensed access can cause a subset of the SPs to provide macro-service only. Further, for any number of SPs, it can be shown that, in general, none of the equilibrium categories (including MSNE) are efficient.

Figure 2 illustrates the Nash equilibrium regions for two SPs as a function of the available bandwidths B_1 and B_2. When B_1 and B_2 are sufficiently large, then the equilibrium is an MSNE, whereas if B_1 and/or B_2 become sufficiently small, the equilibrium transitions so that at least one SP serves only mobile users.

Proposition 1 (MSNE Properties): Assuming an MSNE, for any two SPs i and j with total bandwidth B_i and B_j, the following properties hold:

1) *Symmetry*: If $B_i = B_j$, then each SP's bandwidth allocation must be the same, i.e., $B_{i,S} = B_{j,S}, B_{i,M} = B_{j,M}$.

2) *Monotonicity*: If $B_i > B_j$, then SP i allocates more bandwidth to both macro- and small-cells than SP j, i.e., $B_{i,S} > B_{j,S}, B_{i,M} > B_{j,M}$.

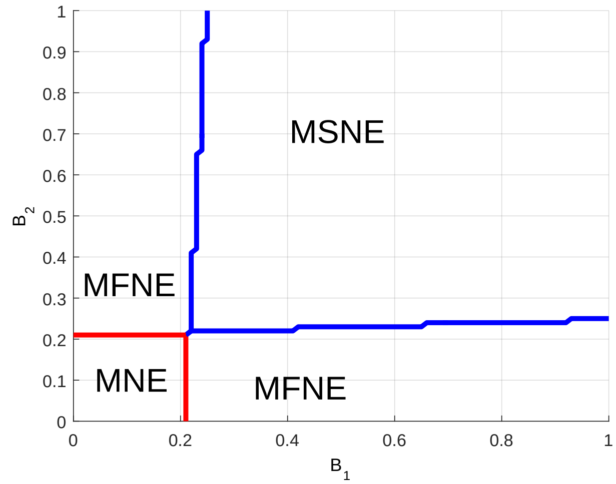

Fig. 2. Nash equilibrium regions for 2 SPs. The system parameters are $\alpha = 0.5, N_m = N_f = 50, R_0 = 50, \lambda_S = 4, \lambda_W = 3, B_U = 1$.

Denote the total bandwidth allocated to small-cells by all SPs with and without unlicensed spectrum as B_S and $\tilde{B_S}$, respectively.

Proposition 2 (MSNE Bandwidth Comparison): If an MSNE exists with unlicensed access, then $B_S < \tilde{B_S}$.

Hence, for the MSNE case, competing SPs reduce the bandwidth allocation to small-cells when unlicensed spectrum is introduced.

Proposition 3 (MNE Conditions): An MNE holds if and only if

$$C_U \geq \frac{R_0 \sum_{i=1}^{N} B_i}{\left(1 - \alpha \frac{B_{\max}}{\sum_{i=1}^{N} B_i}\right)^{\frac{1}{\alpha}}} \frac{\kappa N_f \lambda_S^{\frac{1}{\alpha}}}{N_m}, \quad (17)$$

where $B_{\max} = \max\{B_i : i \in \mathcal{N}\}$.

For $N = 1$ (monopoly), this condition yields the threshold in (12) so that the SP allocates no bandwidth to small-cells.

Proposition 3 has the following corollary in the symmetric setting where all SPs have the same bandwidths, i.e., $B_i \equiv B$ for all $i \in \mathcal{N}$.

Corollary 1: If all SPs have the same bandwidths B and

$$C_U \geq R_0 N B \left(\frac{\lambda_S}{1 - \frac{\alpha}{N}}\right)^{\frac{1}{\alpha}} \frac{\kappa N_f}{N_m}, \quad (18)$$

then we have an MNE. Otherwise we have an MSNE in which all SPs have the same bandwidth allocation satisfying

$$\left(1 - \frac{\alpha}{N}\right)\left(\frac{N_m}{N B_M R_0}\right)^{\alpha} = \left(\frac{\kappa N_f}{\kappa N B_S \lambda_S R_0 + C_U}\right)^{\alpha} \times \lambda_S \left(1 - \alpha \frac{\kappa \lambda_S B_S R_0}{\kappa N B_S \lambda_S R_0 + C_U}\right) \quad (19a)$$

$$B_S + B_M = B, \qquad B_S, B_M > 0. \quad (19b)$$

Next we consider the asymptotic case of an infinite number of SPs. Specifically, we consider two different scenarios:

1) The number of SPs, $N \to \infty$, and each SP has the same bandwidth B. We also linearly scale the mass of fixed users, mobile users and the total bandwidth in unlicensed spectrum.

That is, $N_f^N = a_1 N, N_m^N = a_2 N, B_U^N = a_3 N$ for some positive variables a_1, a_2, and a_3.
2) The number of SPs, $N \to \infty$, where the total amount of bandwidth in licensed spectrum B, the bandwidth in unlicensed Wi-Fi network B_U and the mass of fixed users N_f and mobile users N_m are all fixed. Each SP gets bandwidth $\frac{B}{N}$.

The following theorem characterizes the asymptotic social welfare performance under both these scenarios.

Theorem 7 (Asymptotic Social Welfare Performance): For both asymptotic scenarios, in general, any limiting MSNE, MFNE or MNE is not efficient.

Theorem 7 can be explained by observing that adding SPs only increases competition in licensed spectrum, but does not result in an efficient allocation of users across the licensed and unlicensed bands.

For the second asymptotic scenario, using Corollary 1 we also present the following characterization of the equilibrium as $N \to \infty$.

Proposition 4: If

$$B_U \lambda_U > B \frac{\kappa N_f \lambda_S^{\frac{1}{\alpha}}}{N_m}, \tag{20}$$

then there exists an $N^*(B_U\lambda_U)$ such that for all $N > N^*(B_U\lambda_U)$ we have an MNE. Otherwise, we always have an MSNE ($B_{i,M} > 0$ and $B_{i,S} > 0 \ \forall \ i \in \mathcal{N}$, and $\forall \ N$).

VII. Unlicensed Bandwidth and Social Welfare

In this section we study the impact of increasing unlicensed bandwidth on social welfare. Using the preceding framework, we can determine the specific mix of unlicensed/licensed spectrum such that the market equilibrium yields the same social welfare as that achieved by a social planner. This is motivated by the scenario in which a spectrum regulator, such as the FCC, must determine how much of newly available spectrum will be licensed or unlicensed. We assume a total available bandwidth B and consider the following scenarios:

1) *Efficient allocation*: A social planner determines the bandwidth allocation to macro-cells B_M, small-cells B_S, and unlicensed network B_U that maximizes total utility. We will denote the optimal allocation as $B_M^{\text{opt}}, B_S^{\text{opt}}, B_U^{\text{opt}}$, and use the corresponding social welfare as a benchmark.

2) *Market equilibrium*: Here a social planner determines the bandwidth assigned to licensed spectrum B_L and unlicensed spectrum B_U. Each of the N SPs operating in licensed spectrum obtains the same portion of total bandwidth $B_i \equiv \frac{B_L}{N}$, and then further determines the split of B_i between B_{i_M} and B_{i_S} to maximize its revenue. This scenario corresponds to the more practical setting in which the regulator sets aside part of the available bandwidth as unlicensed, and grants licenses for the remainder. We will denote the bandwidths that maximize social welfare in this scenario as B_L^*, B_U^*.

In the first scenario the social planner determines the bandwidth assignment without explicit pricing. The optimal bandwidth allocation equalizes the marginal utility for mobile and fixed users. It is easy to verify that the efficient allocation corresponds to separate service. In the market equilibrium scenario, we will use the results from Sections IV and VI to determine the optimal bandwidth assignments. We will also present results for the asymptotic regime of many SPs, i.e., as $N \to \infty$.

Then, given newly available bandwidth B, the optimal split into licensed and unlicensed subbands depends on the relative values of λ_S and λ_U. We have the following cases:
a) $\lambda_S > \lambda_U$: In this case an efficient allocation by a social planner would assign all spectrum to macro- and small-cells, i.e., there is no unlicensed network. The optimal bandwidth assignment is then:

$$B_M^{\text{opt}} = \frac{N_m B}{N_m + \mu_S N_f}, B_S^{\text{opt}} = \frac{\mu_S N_f B}{N_m + \mu_S N_f}, B_U^{\text{opt}} = 0, \tag{21}$$

where $\mu_S := \lambda_S^{\frac{1}{\alpha}-1}$. This is also true for the market equilibrium since it is shown in [4], [11] that without unlicensed spectrum, maximizing revenue is the same as maximizing social welfare for α-fair utility functions, independent of N. Hence in this case the market equilibrium achieves the efficient allocation.
b) $\lambda_S = \lambda_U$: In this case the social planner only needs to consider the bandwidth assigned to macro-cells; the bandwidth split between small-cells and unlicensed access can be arbitrary. Here the optimal assignment satisfies

$$B_M^{\text{opt}} = \frac{N_m B}{N_m + \mu_S N_f}, B_S^{\text{opt}} + B_U^{\text{opt}} = \frac{\mu_S N_f B}{N_m + \mu_S N_f}, \tag{22}$$

where $\mu_S = \lambda^{\frac{1}{\alpha}-1}$. This includes the two extremes ($B_S^{\text{opt}} > 0, B_U^{\text{opt}} = 0$) and ($B_S^{\text{opt}} = 0, B_U^{\text{opt}} > 0$). Hence for the market equilibrium case, an optimal bandwidth assignment strategy is to allocate all bandwidth as licensed. As in case a), this achieves the efficient allocation. However, here there may exist another efficient allocation in which the fixed users are served by the unlicensed network. For the competing SPs, that corresponds to an MNE, i.e., the SPs only allocate their licensed bandwidth to the macro-cells ($B_{i,S} = 0$ for all $i \in \mathcal{N}$).

The bandwidth allocation corresponding to this second optimal assignment assignment with the condition for its existence is

$$B_L^* = \frac{N_m B}{N_m + \mu_S N_f}, B_U^* = \frac{\mu_S N_f B}{N_m + \mu_S N_f},$$
$$\text{if } N \geq 2 \text{ or } N = 1 \text{ and } 0 < \alpha \leq 0.5.$$

For $N = 1$, if $\alpha \in (0.5, 1)$, then this second optimal point does not exist and the unique optimal bandwidth allocation corresponds to no unlicensed spectrum. All other bandwidth assignments yield lower social welfare.
c) $\lambda_S < \lambda_U$: In this case a social planner assigns spectrum to the macro-cell and unlicensed networks only; there is no small-cell network. The optimal allocation is:

$$B_M^{\text{opt}} = \frac{N_m B}{N_m + \mu_U N_f}, B_S^{\text{opt}} = 0, B_U^{\text{opt}} = \frac{\mu_U N_f B}{N_m + \mu_U N_f}, \tag{23}$$

where $\mu_U = \lambda_U^{\frac{1}{\alpha}-1}$. For the market equilibrium, allocating all bandwidth to licensed spectrum no longer achieves the efficient allocation. The only possibility for achieving an efficient allocation is to allocate B_U^{opt} to unlicensed access.

This achieves the efficient allocation if and only if the corresponding equilibrium is an MNE. The corresponding bandwidth assignment, along with the condition that guarantees its existence are:

$$B_L^* = \frac{N_m B}{N_m + \mu_U N_f}, B_U^* = \frac{\mu_U N_f B}{N_m + \mu_U N_f},$$
$$\text{if } N \geq 2 \text{ or } N = 1 \text{ and } 0 < \alpha \leq \alpha_0,$$

where $0 < \alpha_0 < 1$ is the unique solution to $\kappa^\alpha \frac{\lambda_S}{\lambda_U} + \alpha = 1$. All other bandwidth assignments yield lower social welfare. Also, for $N = 1$, if $\alpha \in (\alpha_0, 1)$, then the optimal bandwidth allocation is not efficient.[9]

To summarize, in contrast to the results of [4], [11], with the addition of unlicensed spectrum it is possible to achieve efficiency only with a specific split of licensed and unlicened spectrum, even as $N \to \infty$ (perfect competition). Moreover, when $\lambda_S < \lambda_U$, the optimal bandwidth assignment (when it exists) is an MNE.

Figures 3-6 illustrate the preceding observations. They show social welfare versus the amount of unlicensed bandwidth for total bandwidth $B = 2$, $N_f = N_m = 50$, $R_0 = 50$, and $\lambda_S = 4$. Each figure shows four plots corresponding to the efficient allocation (straight line), monopoly SP, two competitive SPs, and perfect competition ($N \to \infty$).

Figure 3 illustrates case a), where all scenarios achieve the maximum social welfare when all bandwidth is allocated to licensed spectrum if $\lambda_S > \lambda_U$. This is because when $B_U = 0$, from [4], [11] maximizing revenue is the same as maximizing social welfare for any N.

Figure 4 illustrates case b), where $\alpha = 0.5$ is chosen so that two different bandwidth assignments give the efficient allocation for all scenarios. We also compute $B_U^{\text{opt}} = 1.6$, $\frac{C_U^{\text{sw}}}{\lambda_U R_0} =: B_U^{\text{sw}} \approx 1.39$, and $\frac{C_U^{\text{rev}}}{\lambda_U R_0} =: B_U^{\text{rev}} = 1.6$. A social welfare maximizing monopolist would start ignoring small-cells when $B_U \approx 1.39$, while a revenue maximizing monopolist takes the same action exactly at the value of B_U where social welfare is maximized.

Figure 5 illustrates case c) where a monopolist can be efficient ($\frac{\lambda_S}{\lambda_U} = 0.4, \alpha = 0.8$). Here $B_U^{\text{rev}} \approx 1.16$, $B_U^{\text{opt}} \approx 1.28$ and $B_U^{\text{sw}} \approx 0.56$, so that both a social welfare or revenue maximizing monopolist abandon small-cells for small enough values of B_U. Figure 6 illustrates case c) where a monopolist is not efficient ($\frac{\lambda_S}{\lambda_U} = \frac{8}{9}, \alpha = 0.8$). Here $B_U^{\text{rev}} \approx 1.51$, $B_U^{\text{opt}} \approx 1.19$ and $B_U^{\text{sw}} \approx 0.92$. Note that the revenue maximization objective makes the monopolist abandon small-cells only for large B_U, well after B_U^{opt}.

We make several observations from the figures. First, the three curves corresponding to the market scenarios ($N = 1, 2$, and $N \to \infty$) all have a "kink", or turning point after which the curves are concave in B_U. This corresponds to the transition from an MSNE to MNE (macro-cells only). That is, to the right of the turning point, bandwidth is allocated only to macro-cells and unlicensed access, and the social welfare is always concave in B_U. For the MNE, the social welfare is the same for the three market scenarios, so that the three curves overlap in this region, i.e., when B_U becomes sufficiently large. This common strictly concave function gives the sum utility as function of B_U if there are no small-cells. It can be extended to all $B_U \in (0, B)$ with a unique maximizer in $(0, B)$.

When $\lambda_S = \lambda_U$, our results show that for $\alpha = 0.5$ there are two optimal points. Therefore in Fig. 4 the curves first decrease and then increase to the second optimal point as part of the concave function we previously described. When $\lambda_S < \lambda_U$, with a small amount of bandwidth allocated to unlicensed spectrum, we obtain slightly more rate, but this does not maximize social welfare, and so the social welfare decreases. However, when we increase B_U, the fixed users' utility increases with rate, and this effect dominates even though we are not allocating the rate efficiently. As a result, the social welfare goes up again. As B_U approaches B, mobile users in macro-cells suffer and therefore the social welfare decreases again.

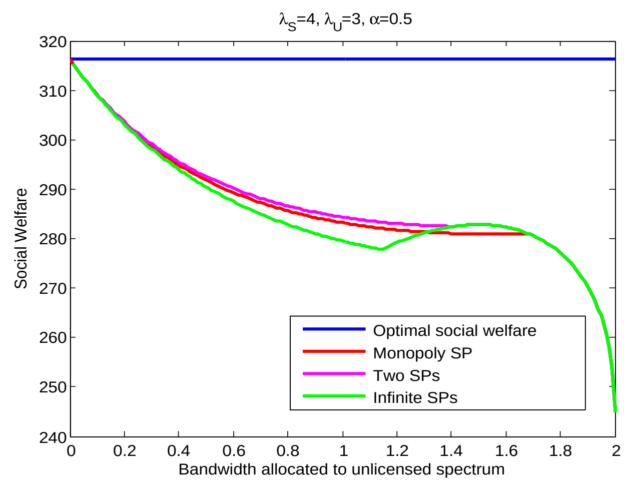

Fig. 3. Social welfare versus unlicensed bandwidth with $\lambda_S > \lambda_U$.

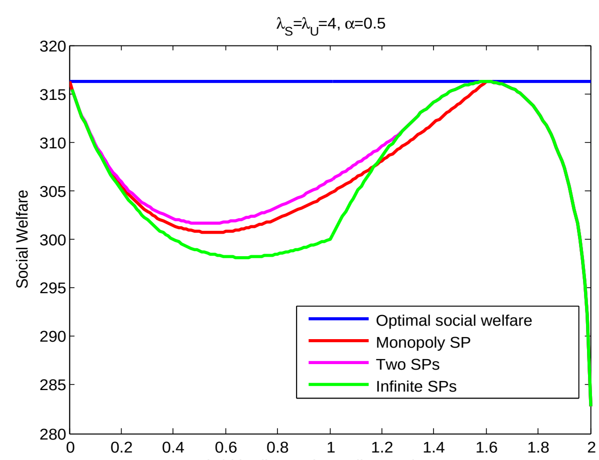

Fig. 4. Social welfare versus unlicensed bandwidth with $\lambda_S = \lambda_U$.

[9]The market equilibrium does achieve the efficient allocation when $B_U = 0$ [4], [11]. Here we have that for the efficient allocation $B_U > 0$.

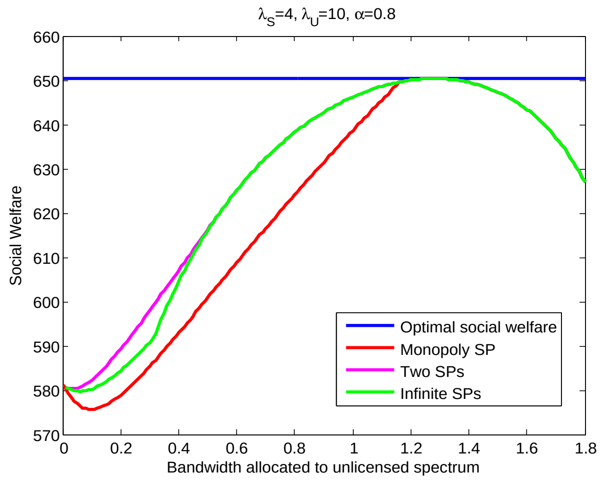

Fig. 5. Social welfare versus unlicensed bandwidth with $\lambda_S < \lambda_U$. Here a monopolist can be efficient.

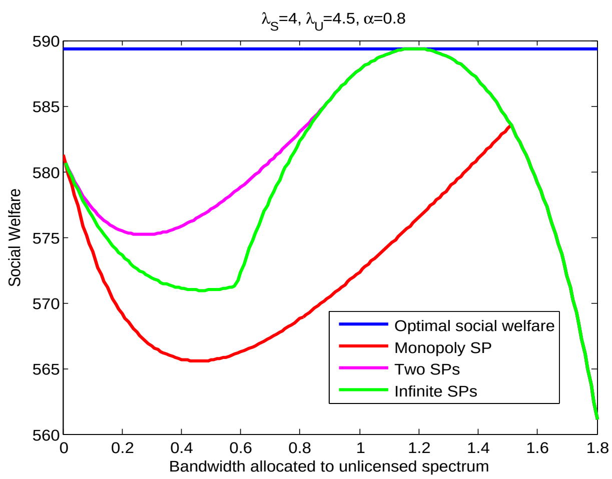

Fig. 6. Social welfare versus unlicensed bandwidth with $\lambda_S < \lambda_U$. Here the monopolist is always inefficient.

VIII. Conclusions

We have presented a model for allocating bandwidth in a HetNet with both licensed and unlicensed spectrum, taking into account the pricing strategies of the SPs. Our results characterize the equilibrium allocations assuming that the SPs maximize revenue or social welfare. For a monopoly SP that maximizes revenue, we show that the presence of a small amount of unlicensed spectrum may cause the SP to allocate more bandwidth to its competing small-cell network. However, when maximizing social welfare the SP always allocates less small-cell bandwidth with unlicensed spectrum. With multiple competing SPs the (unique) equilibrium is one of three different types, depending on the system parameters, including one in which the SPs do not allocate any bandwidth to a small-cell network (e.g., when the available bandwidth is small). We observe that in general, these equilibria do not achieve the maximum social welfare even when the number of competing SPs is large. In contrast, without unlicensed spectrum, the equilibrium is always efficient for the class of α-fair utility functions considered here.

We have used this framework to analyze the effect of unlicensed bandwidth on social welfare. If the small-cell network offers higher spectral efficiency than the unlicensed network, then according to our model, allocating all of the bandwidth as licensed is efficient. Otherwise, if the unlicensed network offers higher spectral efficiency, then we observe that there is a unique mix of unlicensed and licensed spectrum that maximizes social welfare, but that mix may or may not correspond to an equilibrium when the licensed spectrum is allocated to revenue-maximizing SPs.

In practice, given the same density of access points, a managed licensed network would likely have the highest spectral efficiency, in which case, according to our model, allocating bands as licensed will maximize the social welfare. Of course, this assumes that the utility of a band depends only on the offered rate, whereas various other factors have to be taken into account in real-world deployment. Some of these, such as the investment costs and spectrum constraints associated with small-cell deployment, are investigated in [27], which concludes that adding these practical considerations significantly changes the small-cell resource allocation strategies.

Although unlicensed access might yield less spectral efficiency compared to managed licensed access, advocates of unlicensed spectrum have pointed to other properties, which are not taken into account here, such as open access for spectrum sharing, lower entry barriers and the potential for developing new technologies and business models.[10]

Spectrum regulators consider all these aspects when determining the spectrum policy. For example, recent allocations in the 3.5 and 6 GHz bands are intended to support a mix of licensed small-cell (via low power constraints) and unlicensed services [30] [31]. The model presented here can be extended to account for heterogeneous types of services and traffic, in order to provide additional insights into the tradeoffs associated with these allocations.

Appendix A
Proof of Theorem 1

We only discuss the case of $B_S, B_M > 0$. The sub-cases of either or both the variables being 0 follow a similar logic with the obvious restriction of no users being served in the bands with no bandwidth.

1. We first show that the market always clears, i.e., all users would be served and the total rate supply is equal to the total rate demand.

Since the WiFi network on unlicensed spectrum is free to use, it's obvious that all fixed users would be served. If there are some mobile users that are not served yet, the SP can increase the price in macro-cells so that users in macro-cells would request less rate, leading to the SP having some redundant rate to serve more mobile users. The SP can thus use up its rate in macro-cells at a higher price since the unserved mobile users would fill in, which then leads to larger revenue.

On the other hand, if all users are served but there is still some redundant rate available in macro-cells or small-cells,

[10] See, for example, [28]. The relative benefits of unlicensed versus licensed spectrum continue to be debated, e.g., see [29].

the SP can decrease the price in corresponding cells so that users now request a higher rate. Since $ru'(r)$, which is the revenue gained per user, increases with r, it's easy to see the SP can gain more revenue by doing so.

As a result of the market clearing property we have the following important conclusion. In both the macro-cells and the small-cells, the per-user rate equals the allocated total rate (bandwidth times the spectral efficiency) divided by the mass of customers associated with macro-cells and the same-cells. The price for access is given precisely by the inverse of the demand function $D(\cdot)$ at this per-user rate.

2. We then prove the price choice and user association equilibrium in the two different scenarios, given the fixed bandwidth allocation.

Assume macro-cells only serve mobile users. This then implies that $R_S \geq R_M$ so that the boundary point would correspond to the point at which $R_S = R_M$ holds. It can be determined using the following steps.

$$K_M = N_m \tag{24a}$$

$$u\Big(\frac{\lambda_S B_S R_0}{K_S}\Big) - \frac{\lambda_S B_S R_0}{K_S} u'\Big(\frac{\lambda_S B_S R_0}{K_S}\Big) = u\Big(\frac{C_U}{N_f - K_S}\Big) \tag{24b}$$

$$R_S = \frac{\lambda_S B_S R_0}{K_S} = R_M = \frac{B_M R_0}{K_M} \tag{24c}$$

The above equation simplify to

$$B_S = \frac{\kappa N_f B_M R_0 - N_m C_U}{\kappa \lambda_S N_m R_0} =: B_S^0. \tag{25}$$

Therefore if B_S is larger than B_S^0, it's easy to see that the user equilibrium must be such that macro-cells only serve mobile users. In contrast, if B_S is smaller than B_S^0, then $R_S < R_M$ holds assuming macro-cells only serve mobile users. As a result, some fixed users would have the incentive to associate with macro-cells, and next we will prove that in this case at user equilibrium it's indeed the case that some fixed users would associate with macro-cells.

Suppose $B_S < B_S^0$ and if macro-cells only serve mobile users such that the market clears, then we have $p_M < p_S$. The SP can then increases the price to $p_M^{'}$, where $p_M < p_M^{'} < p_S$, so that the mobile obtain a smaller rate creating some spare capacity. As a result, some fixed users in small-cells and WiFi network would switch to macro-cells, denote the total mass as δ and the mass of fixed users from small-cells switching being $\delta' \leq \delta$. Note that before the price change $K_M = N_m$. The resulting revenue of the SP would then be :

$$S = B_M R_0 u'\Big(R_M \frac{N_m}{N_m + \delta}\Big) + \lambda_S B_S R_0 u'\Big(R_S \frac{K_S}{K_S - \delta'}\Big) \tag{26}$$

By Lemma 1 we can rewrite the revenue as:

$$S = B_M R_0 u'\Big(R_M \frac{N_m}{N_m + \delta}\Big) + \lambda_S B_S R_0 u'\Big(R_S \frac{N_f}{N_f - \delta}\Big) \tag{27}$$

Then we have:

$$\begin{aligned}\frac{\partial S}{\partial \delta} &= -{R_M^{'}}^2 u''(R_M^{'}) + \frac{\lambda_S B_S R_0}{N_f - \delta} R_S^{'} u''(R_S^{'}) \\ &= -{R_M^{'}}^2 u''(R_M^{'}) + \frac{K_S}{N_f} \frac{\lambda_S B_S R_0}{K_S} \frac{N_f}{N_f - \delta} R_S^{'} u''(R_S^{'}) \\ &> -{R_M^{'}}^2 u''(R_M^{'}) + {R_S^{'}}^2 u''(R_S^{'}),\end{aligned}$$

where $R_S' = R_S \frac{K_S}{K_S - \delta'} = R_S \frac{N_f}{N_f - \delta}$ and $R_M' = R_M \frac{N_m}{N_m + \delta}$ are the new per user rates in the small-cells and macro-cells, respectively, after the shift of δ mass of fixed users to macro-cells.

Based on our assumptions, $r^2 u''(r) = -\alpha r^{1-\alpha}$ decreases with r, therefore as long as $R_S^{'} < R_M^{'}$, i.e., $p_M^{'} < p_S^{'}$, S always increases with δ. As a result, it's always better for macro-cells to serve some fixed users in this case and the optimal price choice is $p_M = p_S$.

In mixed service scenario, we therefore have the following equations:

$$u\Big(\frac{C_S}{K_S}\Big) - p_S \frac{C_S}{K_S} = u\Big(\frac{C_U}{K_U}\Big) = u\Big(\frac{C_M}{K_M}\Big) - p_M \frac{C_M}{K_M}, \tag{28a}$$

$$K_U + K_S + K_M = N_m + N_f, \tag{28b}$$

$$D(p_M) = R_M = D(p_S) = R_S = \frac{C_S + C_M}{K_M + K_S}. \tag{28c}$$

Using Lemma 1 we can get:

$$K_U = (N_f + N_m) \frac{C_U}{C_U + \kappa(C_M + C_S)}, \tag{29a}$$

$$K_M = (N_f + N_m) \frac{\kappa C_M}{C_U + \kappa(C_M + C_S)}, \tag{29b}$$

$$K_S = (N_f + N_m) \frac{\kappa C_S}{C_U + \kappa(C_M + C_S)}. \tag{29c}$$

which gives the number of active users in terms of the network capacities.

Similarly, for the separate service scenario, writing the analogous conditions to (28c) we can get:

$$K_M = N_m, \quad K_S = N_f \frac{\kappa C_S}{\kappa C_S + C_U}, \tag{30a}$$

$$K_U = N_f \frac{C_U}{\kappa C_S + C_U}. \tag{30b}$$

Appendix B
Proof of Theorem 2

1. We first prove that the optimal bandwidth allocation cannot occur at mixed service scenario. The revenue of the SP under the mixed service scenario is:

$$\begin{aligned} S =& (B_M + \lambda_S B_S) R_0 u'\Big(\frac{(B_M + \lambda_S B_S) R_0}{K_M + K_S}\Big) \quad (31a)\\ =& (B_M + \lambda_S B_S) R_0 u'\Big(\frac{C_U + \kappa(B_M + \lambda_S B_S) R_0}{\kappa(N_m + N_f)}\Big) \quad (31b)\\ =& (N_m + N_f) R u'(R) - \frac{C_U}{\kappa} u'(R) \quad (31c)\end{aligned}$$

where $R := \frac{C_U + \kappa(B_M + \lambda_S B_S) R_0}{\kappa(N_m + N_f)}$ is the average rate in both macro-cells and small-cells.

Based on our assumptions, $Ru'(R)$ increases with R and $u'(R)$ decreases with R, therefore it's always beneficial to allocate more bandwidth to small-cells: since $\lambda_S > 1$, R increases with B_S. This means the optimal point cannot exist at a mixed service scenario.

2. We then prove the optimal bandwidth allocation scheme in separate service scenario. The revenue of the SP at a separate service equilibrium is:

$$S = B_M R_0 u'\Big(\frac{B_M R_0}{K_M}\Big) + \lambda_S B_S R_0 u'\Big(\frac{\lambda_S B_S R_0}{K_S}\Big) \tag{32a}$$

$$= N_m R_M u'(R_M) + N_f R_S u'(R_S) - \frac{C_U}{\kappa} u'(R_S) \tag{32b}$$

where $R_M = \frac{B_M R_0}{N_m}, R_S = \frac{\lambda_S B_S R_0}{K_S} = \frac{\kappa \lambda_S B_S R_0 + C_U}{\kappa N_f}$ are the average rate in macro-cells and small-cells, respectively.

It's easy to verify for α-fair utility functions, $R_M u'(R_M), R_S u'(R_S)$ are concave increasing functions with respect to B_M and B_S, respectively. Furthermore, $-u'(R_S)$ is also a concave increasing function with respect to B_S. As a result, S is a increasing with either B_S or B_M. Therefore at optimal point the SP uses up all its total bandwidth and we have $B_S + B_M = 1$. This further means S is strictly concave with B_S. As a result, the optimal point occurs at the point which uses up the total bandwidth and equalizes the marginal revenue increase with respect to per unit bandwidth increase in both macro-cells and small-cells, which can be achieved by the straightforward calculation given below:

$$\begin{aligned} \frac{1-\alpha}{\lambda_S}\Big(\frac{B_M^{\text{rev}} R_0}{N_m}\Big)^{-\alpha} = &\Bigg[\Big(\frac{\kappa \lambda_S B_S^{\text{rev}} R_0 + C_U}{\kappa N_f}\Big)^{-\alpha} \\ &- \alpha\Big(\frac{\kappa \lambda_S B_S^{\text{rev}} R_0 + C_U}{\kappa N_f}\Big)^{-\alpha} \frac{\kappa \lambda_S B_S^{\text{rev}} R_0}{\kappa \lambda_S B_S^{\text{rev}} R_0 + C_U}\Bigg], \\ B_S^{\text{rev}} + B_M^{\text{rev}} = B, &\qquad B_S^{\text{rev}}, B_M^{\text{rev}} \geq 0. \end{aligned} \tag{P1}$$

APPENDIX C
PROOF OF THEOREM 3

We first make the following definitions:

$$\begin{aligned} A_1 = &\Big[-\frac{\alpha \kappa \lambda_S B_S}{\kappa \lambda_S B_S + \lambda_U B_U}\Big(\frac{\kappa \lambda_S B_S + \lambda_U B_U}{\kappa N_f} R_0\Big)^{-\alpha} \\ &+ \Big(\frac{\kappa \lambda_S B_S + \lambda_U B_U}{\kappa N_f} R_0\Big)^{-\alpha}\Big]\lambda_S \end{aligned} \tag{33a}$$

$$A_2 = (1-\alpha)\lambda_S \Big(\frac{\lambda_S B_S R_0}{N_f}\Big)^{-\alpha} \tag{33b}$$

We only need to compare A_1 and A_2 at $\tilde{B}_S^{\text{rev}}$. By explicit calculation, this is given by:

$$A_1 - A_2 = \frac{M}{(\kappa \lambda_S \tilde{B}_S^{\text{rev}} R_0 + C_U)^{\alpha+1} (\lambda_S \tilde{B}_S^{\text{rev}} R_0)^{\alpha}} \tag{34}$$

where $M = (1-\alpha)\big[(\kappa \lambda_S \tilde{B}_S^{\text{rev}} R_0)^{\alpha+1} - (\kappa \lambda_S \tilde{B}_S^{\text{rev}} R_0 + C_U)^{\alpha+1}\big] - C_U (\kappa \lambda_S \tilde{B}_S^{\text{rev}} R_0)^{\alpha}$.

It's easy to verify that M first increases with C_U and then decreases with C_U. Moreover, $M = 0$ when $C_U = 0$. Therefore we only need to determine the other zero-crossing point C_U^{th} by letting $M = 0$. The conclusions in Theorem 3 then follow in a straightforward manner.

APPENDIX D
PROOF OF THEOREM 4

1. We first show that the market always clears, i.e., all users would be served and the total rate supply is equal to the total rate demand.

Since the WiFi network is free to use, again it's obvious that all fixed users would be served. If there are some mobile users that are not served yet, the SP can increase the price in macro-cells so that users in macro-cells would request less rate and therefore it has some redundant rate to serve the mobile users. Since the function $K_M u(\frac{C_M}{K_M})$ increases with K_M, the social welfare is thus increased.

On the other hand, if all users are served but there is still some redundant rate available in macro-cells or small-cells. The SP can decrease the price in corresponding cells so that users now request higher rate, which naturally leads to higher social welfare.

2. We then prove the price choice and users association equilibrium in two different scenarios, depending on the fixed bandwidth allocation.

Similar to the analysis for revenue maximization, we first prove that at certain fixed bandwidth allocation, if macro-cells only serve mobile users and mobile users achieve larger rate than fixed users in small-cells, then the final user association should be such that some fixed users associate with macro-cells and the price in macro-cells and small-cells should be the same.

Suppose macro-cells only serve mobile users and we have $p_M < p_S$, and $R_M > R_S$. The SP can increase the macro-cell price to $p_M^{'}$, where $p_M < p_M^{'} < p_S$. As a result, some fixed users in small-cells and WiFi network would switch to macro-cells, denote the mass of these fixed users as δ and $N_t = N_m + N_f = K_M + K_S + K_U$. By Lemma 1 we can show that the social welfare would then be :

$$\begin{aligned} \text{SW} = &(K_M + \delta) u\Big(R_M \frac{K_M}{K_M + \delta}\Big) \\ &+ K_S \frac{N_t - K_M - \delta}{N_t - K_M} u\Big(R_S \frac{N_t - K_M}{N_t - K_M - \delta}\Big) \\ &+ K_U \frac{N_t - K_M - \delta}{N_t - K_M} u\Big(R_U \frac{N_t - K_M}{N_t - K_M - \delta}\Big). \end{aligned} \tag{35}$$

Then we have:

$$\begin{aligned} \frac{\partial \text{SW}}{\partial \delta} = &u(R_M^{'}) - R_M^{'} u'(R_M^{'}) - \frac{K_S}{N_t - K_M}\Big[u(R_S^{'}) - \\ &R_S^{'} u'(R_S^{'})\Big] - \frac{K_U}{N_t - K_M}\Big[u(R_U^{'}) - R_U^{'} u'(R_U^{'})\Big] \end{aligned} \tag{36}$$

For the α-fair utility functions we use, it's easy to verify that $u(r) - r u'(r)$ increases with r. We also have $R_M^{'} > R_S^{'} > R_U^{'}$, thus

$$\begin{aligned} \frac{\partial \text{SW}}{\partial \delta} &> u(R_M^{'}) - R_M^{'} u'(R_M^{'}) - \big[u(R_S^{'}) - R_S^{'} u'(R_S^{'})\big] \\ &> 0. \end{aligned}$$

Therefore the social welfare increases with δ in mixed service scenario as long as $p_M < p_S$. As a result, in this case some fixed users would associate with macro-cells, and the optimal price choice will be $p_M = p_S$.

3. Next, we show that the optimal bandwidth allocation is unique and only occurs at separate service scenario. The social welfare at mixed service scenario is:

$$\text{SW} = (K_M + K_S)u(R) + K_U u(R_U) \tag{37}$$

where $R = \frac{[\lambda_U B_U + \kappa(B_M + \lambda_S B_S)]R_0)}{\kappa(N_m+N_f)}$, $R_U = \frac{[\lambda_U B_U + \kappa(B_M + \lambda_S B_S)]R_0)}{N_m+N_f}$ are the average user rate in licensed and unlicensed spectrum, respectively. Therefore we have:

$$\begin{aligned} \text{SW} =& (K_M + K_S)u(R) + K_U u(\kappa R) && \text{(38a)}\\ =& (N_m + N_f)u(R) - K_U(u(R) - u(\kappa R)) && \text{(38b)}\\ =& (N_m + N_f)u(R) - C_U \frac{u(R) - u(\kappa R)}{\kappa R} && \text{(38c)} \end{aligned}$$

It's easy to see that the first term is increasing with R. For α-fair utility functions we use, it's easy to verify the second minus term is decreasing with R. Therefore the social welfare increases with R, which leads to the conclusion that it's always better to invest more bandwidth to small-cells. As a result, the optimal point cannot occur at mixed service scenario.

We then prove that the optimal bandwidth allocation is unique at separate service scenario.

$$\begin{aligned} &\text{SW} = N_m u(R_M) + K_S u(R_S) + K_U u(R_U) && \text{(39a)}\\ &R_S = \frac{\kappa\lambda_S B_S R_0 + C_U}{\kappa N_f}, R_U = \kappa R_S && \text{(39b)}\\ &K_S = \frac{\lambda_S B_S R_0}{R_S}, K_U = \frac{C_U}{R_U} && \text{(39c)} \end{aligned}$$

We then have:

$$\text{SW} = N_m u(R_M) + N_f u(R_S) + \frac{C_U}{\kappa}[\frac{u(\kappa R_S) - u(R_S)}{R_S}] \tag{40}$$

Since $N_m u(R_M)$ is concave increasing with R_M and both $N_f u(R_S)$, $\frac{u(\kappa R_S) - u(R_S)}{R_S} = -R_S^{-\alpha}$ are concave increasing with R_S. It's easy to verify that social welfare is increasing with either B_S or B_M and therefore at optimal point we have $B_S + B_M = B$. By this we can further show that it is also a strictly concave function with B_S and therefore the optimal point is unique. At this optimal point, SP uses up the total bandwidth and equalizes the marginal social welfare increase of mobile users and fixed users with respect to per unit of bandwidth increase in macro-cells and small-cells. The optimal bandwidth allocation can be calculated as follows:

$$\begin{cases} \frac{(N_f)^\alpha \lambda_S \left((\kappa^\alpha+\kappa)C_U + \kappa^{\alpha+1}\lambda_S B_S^{\text{sw}} R_0\right)}{(\kappa\lambda_S B_S^{\text{sw}} R_0 + C_U)^{\alpha+1}} = (\frac{B_M^{\text{sw}} R_0}{N_m})^{-\alpha}, \\ B_S^{\text{sw}} + B_M^{\text{sw}} = 1, \qquad B_S^{\text{sw}}, B_M^{\text{sw}} \geq 0. \end{cases} \tag{P3}$$

4. Last, we prove $B_S^{\text{sw}} < \tilde{B}_S^{\text{sw}}$. We first define some notations.

$$\begin{aligned} A_3 &= \frac{(N_f)^\alpha \lambda_S \left[(\kappa^\alpha + \kappa)C_U + \kappa^{\alpha+1}\lambda_S \tilde{B}_S^* R_0\right]}{(\kappa\lambda_S \tilde{B}_S^* R_0 + C_U)^{\alpha+1}} && \text{(41a)}\\ A_4 &= \lambda_S (\frac{\lambda_S \tilde{B}_S^{**} R_0}{N_f})^{-\alpha} && \text{(41b)} \end{aligned}$$

We only need to compare A_3 and A_4 at $\tilde{B}_S^{\text{sw}}$. It turns out that:

$$\begin{aligned} &\frac{A_3 - A_4}{\lambda_S R_0 (N_f)^\alpha} = \frac{M}{(\lambda_S \tilde{B}_S^{\text{sw}} R_0)^\alpha (\kappa\lambda_S \tilde{B}_S^{\text{sw}} R_0 + C_U)^{\alpha+1}} && \text{(42a)}\\ &M = (\kappa^\alpha + \kappa)C_U(\lambda_S \tilde{B}_S^{\text{sw}} R_0)^\alpha + \kappa^{\alpha+1}(\lambda_S \tilde{B}_S^{\text{sw}} R_0)^{\alpha+1}\\ &- (\kappa\lambda_S \tilde{B}_S^{\text{sw}} R_0 + C_U)^{\alpha+1} && \text{(42b)} \end{aligned}$$

We then have:

$$\begin{aligned} \frac{\partial M}{\partial C_U} =& (\alpha+1)\Big[(\kappa\lambda_S \tilde{B}_S^{\text{sw}} R_0)^\alpha - (\kappa\lambda_S \tilde{B}_S^{\text{sw}} R_0 + C_U)^\alpha\Big]\\ <& 0, \quad \forall C_U > 0. \end{aligned}$$

Since when $C_U = 0$, $A_3 = A_4$. Therefore $A_3 < A_4$ when $C_U > 0$. As a result, compared with the scenario without unlicensed spectrum, the SP should always invest less bandwidth to small-cells in terms of social welfare maximization.

Appendix E
Proof of Theorem 5

The proof is essentially the same as the proof for the price choice and user association equilibrium for monopoly service provider scenario. The only difference is that we need to prove the prices must be equal across all small-cells or macro-cells with multiple SPs.

Suppose one SP i has lower small-cell price $p_{i,S}$ than the other SP j. SP i can then increase the price to $p'_{i,S}$ satisfying $p_{i,S} < p'_{i,S} < p_{j,S}$. As a result, SP i would attract some users who previously associated with SP j's small-cell or some users from unlicensed access and still use up all the rate with a higher price. Therefore SP i can increase its revenue by doing so. Thus, at the price equilibrium all small-cell price must be equal. The proof for all macro-cell prices being equal holds in a similar manner.

Appendix F
Proof of Theorem 6

We prove the theorem in the following steps.

1. First, we prove that no Nash equilibrium exists at mixed service scenario. It's easy to see the revenue of SP i in this

case is:

$$S_i = (B_{i,M} + \lambda_S B_{i,S})R_0 u'\Big[\frac{\sum_{j=1}^{N}(B_{j,M} + \lambda_S B_{j,S})R_0}{K_M + K_S}\Big] \tag{43a}$$

$$= (K_M + K_S)Ru'(R) - \sum_{j\neq i}^{N}(B_{j,M} + \lambda_S B_{j,S})R_0 u'(R) \tag{43b}$$

$$K_M + K_S = N_m + N_f - \frac{(N_m + N_f)C_U}{C_U + \kappa \sum_{j=1}^{N}(B_{j,M} + \lambda_S B_{j,S})R_0} \tag{43c}$$

$$R = \frac{C_U + \kappa \sum_{j=1}^{N}(B_{j,M} + \lambda_S B_{j,S})R_0}{\kappa(N_m + N_f)} \tag{43d}$$

It's easy to see that R and $K_M + K_S$ both increase with $B_{i,S}$. Meanwhile, $Ru'(R)$ increases with R and $u'(R)$ decreases with R. As a result, S_i increases with $B_{i,S}$ and every SP would have incentive to invest more bandwidth to small-cells, which would finally push the equilibrium to separate service scenario. Therefore no Nash equilibrium exists in a mixed service scenario.

2. We next prove that there always exists a Nash equilibrium in the separate service scenario. The revenue of SP i in this case is:

$$S_i = S_{i,M} + S_{i,S} \tag{44a}$$

$$= B_{i,M} R_0 u'(R_M) + \lambda_S B_{i,S} R_0 u'(R_S) \tag{44b}$$

$$= B_{i,M} R_0 u'\Big(\frac{\sum_{j=1}^{N} B_{j,M} R_0}{N_m}\Big) + \lambda_S B_{i,S} R_0 u'\Big(\frac{\sum_{j=1}^{N} \lambda_S B_{j,S} R_0}{K_S}\Big) \tag{44c}$$

It's easy to verify:

$$\begin{aligned}\frac{\partial S_{i,M}}{\partial B_{i,M}} =& R_0\big[u'(R_M) + \frac{B_{i,M}R_0}{N_m}u''(R_M)\big] \\ =& R_0\big[u'(R_M) + R_M u''(R_M) - \frac{\sum_{j\neq i} B_{j,M}R_0}{N_m}u''(R_M)\big] \\ >& 0\end{aligned}$$

$$\begin{aligned}\frac{\partial S_{i,S}}{\partial B_{i,S}} =& \lambda_S R_0\big[u'(R_S) + \frac{\lambda_S B_{i,S}R_0}{N_f}u''(R_S)\big] \\ =& \lambda_S R_0\big[u'(R_S) + \frac{\sum_{j=1}^{N}\lambda_S B_{j,S}R_0}{N_f}u''(R_S) \\ & - \frac{\sum_{j\neq i}\lambda_S B_{j,S}R_0}{N_f}u''(R_S)\big] \\ =& \lambda_S R_0\big[u'(R_S) + R_S u''(R_S) \\ & - \frac{\lambda_S C_U}{\kappa N_f}u''(R_S) - \frac{\sum_{j\neq i}\lambda_S B_{j,S}R_0}{N_f}u''(R_S)\big] > 0\end{aligned}$$

Since $u'(r) + ru''(r)$ decreases with r and $u''(r)$ increases with r, it's easy to verify $S_{i,M}$ and $S_{i,S}$ are both concave increasing with $B_{i,M}$ and $B_{i,S}$ respectively. This also indicates all SPs would always use up their total bandwidth, i.e., $B_{i,S} + B_{i,M} = B_I$. By this we can further show S_i is a concave function with $B_{i,S}$. Moreover, the constraint on separate service scenario are linear with $B_{i,S}$. We can then apply Rosen's theorem on concave games [32] to prove the existence of Nash equilibrium.

3. For the third step, we prove that fixed users in small-cells achieve higher average rate than mobile users in macro-cells. This is equivalent to say $R_S > R_M$ at equilibrium.

We only need to rule out the possibility that $R_S = R_M$ since we already showed that the Nash equilibrium falls into the separate service scenario. Denote the group of SPs that only allocate bandwidth to small-cells (macro-cells) as $G_S(G_M)$ and the group of SPs that allocate bandwidth to both cells as G_{MS}, then we have:

$$\begin{aligned}&\forall i \in G_S, B_{i,S} = B_i, B_{i,M} = 0 \\ &\forall j \in G_M \cup G_{MS}, \frac{\partial S_j}{B_{j,S}} \leq 0\end{aligned} \tag{45a}$$

If $R_S = R_M = R$ holds, then $\forall j \in G_M \cup G_{MS}$, we have:

$$\begin{aligned}\frac{\partial S_j}{\partial B_{j,S}} =& \lambda_S R_0\Big[u'(R_S) + \frac{\lambda_S B_{j,S}R_0}{N_f}u''(R_S)\Big] \\ & - R_0\Big[u'(R_M) + \frac{B_{j,M}R_0}{N_m}u''(R_M)\Big] \\ =& (\lambda_S - 1)u'(R) + \Big[\frac{\lambda_S B_{j,S}R_0}{N_f} - \frac{B_{j,M}R_0}{N_m}\Big]u''(R) \\ \leq& 0\end{aligned} \tag{46}$$

Therefore $\forall j \in G_M \cup G_{MS}$, we have:

$$\frac{\lambda_S B_{j,S}R_0}{N_f} \geq \frac{B_{j,M}R_0}{N_m} \tag{47}$$

As a result, the following holds:

$$R_S = \sum_{j\in G_M\cup G_{MS}} \frac{\lambda_S B_{j,S}R_0}{N_f} + \sum_{i\in G_S}\frac{\lambda_S B_i R_0}{N_f} \tag{48a}$$

$$> \sum_{j\in G_M\cup G_{MS}} \frac{B_{j,M}R_0}{N_m} = R_M \tag{48b}$$

Therefore we have a contradiction.

4. We then show that at Nash equilibrium it is impossible that one SP only allocates bandwidth to macro-cells while the other SP only allocates bandwidth to small-cells.

Suppose SP i only allocates bandwidth to macro-cells while SP j only allocates bandwidth to small-cells. Then we have:

$$\begin{aligned}\frac{\partial S_i}{\partial B_{i,S}} =& \lambda_S R_0 u'(R_S) - R_0\Big[u'(R_M) + \frac{B_{i,M}R_0}{N_m}u''(R_M)\Big] \\ \leq& 0\end{aligned} \tag{49a}$$

$$\begin{aligned}\frac{\partial S_j}{\partial B_{j,S}} =& \lambda_S R_0\Big[u'(R_S) + \frac{\lambda_S B_{j,S}R_0}{N_f}u''(R_S)\Big] - R_0 u'(R_M) \\ \geq& 0\end{aligned} \tag{49b}$$

which would yield:

$$-\frac{B_{i,M}R_0}{N_m}u''(R_M) \leq \lambda_S \frac{\lambda_S B_{j,S} R_0}{N_f} u''(R_S) \tag{50}$$

Clearly we have a contradiction then.

5. After step 4, it's clear that there are only five possible Nash equilibrium types:

1) Small-only Nash equilibrium(SNE): All SPs only allocate bandwidth to small-cells.
2) Macro-only Nash equilibrium(MNE): All SPs only allocate bandwidth to macro-cells.
3) Macro-Small Nash Equilibrium (MSNE): All SPs allocate bandwidth to both macro- and small-cells.
4) Macro-Favored Nash Equilibrium (MFNE): Some SPs allocate bandwidth to both small– and macro-cells while the other SPs only allocate bandwidth to macro-cells.
5) Small-Favored Nash Equilibrium (SFNE): Some SPs allocate bandwidth to both small– and macro-cells while the other SPs only allocate bandwidth to small-cells.

We next prove that SNE cannot exist.

The marginal revenue increase with respect to per unit bandwidth increase in macro-cells for SP i is given by:

$$\frac{\partial S_i}{\partial B_{i,M}} = R_0\Big[u'(R_M) + \frac{B_{i,M}R_0}{N_m}u''(R_M)\Big] \tag{51}$$

It can be easily shown that for α-fair utility functions, the marginal revenue increase with respect to per unit bandwidth increase in macro-cells goes to infinity when B_M is near 0. As a result, SNE cannot exist.

However, the above argument doesn't apply to MNE. At first glance, it seems that when all SPs only allocate bandwidth to macro-cells, the marginal revenue increase with respect to per unit bandwidth in small-cells also goes to infinity when B_S is near 0. Actually, the marginal revenue increase does go to infinity when R_S is near 0. However, R_S doesn't go to 0 when B_S goes to 0. In fact, R_S is discontinuous at the point 0. We have the following:

$$R_S = \frac{\kappa\lambda_S B_S R_0 + C_U}{\kappa N_f}, \qquad B_S > 0; \tag{52a}$$

$$R_S = 0, \qquad B_S = 0. \tag{52b}$$

Therefore as $B_S \to 0^+, R_S \to \frac{C_U}{\kappa N_f} > 0$.

6. The fact that SFNE cannot exist requires more work and in this part we would focus on it. At SPNE, we know that there exist SPs i, j such that:

$$\frac{\partial S_i}{\partial B_{i,S}} \geq 0, B_{i,S} = B_i, B_{i,M} = 0 \tag{53a}$$

$$\frac{\partial S_j}{\partial B_{j,S}} = 0, B_{j,S} > 0, B_{j,M} > 0 \tag{53b}$$

We therefore have:

$$\lambda_F\Big[u'(R_S) + \frac{\lambda_S B_{i,S}R_0}{N_f}u\prime\prime(R_S)\Big] \geq u'(R_M) \tag{54a}$$

$$\lambda_S\Big[u'(R_S) + \frac{\lambda_S B_{i,S}R_0}{N_f}u\prime\prime(R_S)\Big] = u'(R_M) + \frac{B_{i,M}R_0}{N_m}u''(R_M) \tag{54b}$$

From the first inequality above, we can easily conclude that:

$$\lambda_S u'(R_S) > u'(R_M) \tag{55}$$

Next, we consider the group of service providers that allocate bandwidth to both cells, denoted as G_{MS} and assume $|G_{MS}| = L$.

$$\forall j \in G_{MS}, \frac{\partial S_j}{\partial B_{j,S}} = 0, B_{j,S} > 0, B_{j,M} > 0 \tag{56}$$

We then have:

$$\lambda_S\Big[Lu'(R_S) + \frac{\lambda_S \sum_{j\in G_{MS}} B_{j,S}R_0}{N_f}u\prime\prime(R_S)\Big] = Lu'(R_M) + \frac{\sum_{j\in G_{MS}} B_{j,M}R_0}{N_m}u''(R_M) \tag{57}$$

It's easy to get:

$$\lambda_S\Big[Lu'(R_S) + \frac{\lambda_S \sum_{j\in \mathcal{N}} B_{j,S}R_0}{N_f}u\prime\prime(R_S)\Big] < Lu'(R_M) + R_M u''(R_M) \tag{58}$$

Together with inequality (55), we can get the second inequality:

$$\lambda_S\Big[u'(R_S) + \frac{\lambda_S \sum_{j\in \mathcal{N}} B_{j,S}R_0}{N_f}u''(R_S)\Big] < u'(R_M) + R_M u''(R_M) \tag{59}$$

For α-fair utility functions, we have:

$$u'(R_M) + R_M u''(R_M) = (1-\alpha)u'(R_M) \tag{60}$$

However, we also have:

$$\begin{aligned} &\lambda_S\Big[u'(R_S) + \frac{\lambda_S \sum_{j\in \mathcal{N}} B_{j,S}R_0}{N_f}u''(R_S)\Big] - (1-\alpha)\lambda_S u'(R_S) \\ &= \lambda_S\Big[\alpha u'(R_S) + \frac{\lambda_S \sum_{j\in \mathcal{N}} B_{j,S}R_0}{N_f}u''(R_S)\Big] \\ &> \lambda_S\Big[\alpha u'(R_S) + R_S u''(R_S)\Big] \\ &= \lambda_S \alpha R_S^{-\alpha} - \lambda_S \alpha R_S^{-\alpha} = 0 \end{aligned} \tag{61}$$

Therefore the two inequalities lead to a contradiction. As a result, SPNE cannot exist.

7. We next prove that MSNE, MNE and MFNE cannot coexist. First it's easy to verify that at MSNE, MNE and MFNE, we have the following:

$$\text{MSNE}: \quad \lambda_S\Big[Nu'(R_S) + \frac{\lambda_S \sum_{i\in \mathcal{N}} B_{i,S}R_0}{N_f}u''(R_S)\Big] = Nu'(R_M) + R_M u''(R_M) \tag{62a}$$

$$\text{MNE or MFNE}: \quad \lambda_S\Big[Nu'(R_S) + \frac{\lambda_S \sum_{i\in \mathcal{N}} B_{i,S}R_0}{N_f}u''(R_S)\Big] \leq Nu'(R_M) + R_M u''(R_M) \tag{62b}$$

We can show that the LHS is a decreasing function with R_S, since we have:

$$R_S = \frac{C_U + \kappa\lambda_S \sum_{i\in\mathcal{N}} B_{i,S}R_0}{\kappa N_f} \tag{63}$$

Therefore we have:

$$\begin{aligned}\text{LHS} =& \lambda_S\big[u'(R_S) + R_S u''(R_S)\big] + (N-1)\lambda_S u'(R_S)\\ &- \frac{\lambda_S C_U}{\kappa N_f} u''(R_S)\end{aligned} \tag{64}$$

which is decreasing with R_S.

Similarly, RHS is also a decreasing function with R_M.

Now suppose for the same set of parameters, we have one MNE or MFNE and another MSNE, denote the corresponding bandwidth allocation profile as $\mathbf{B}$ and $\bar{\mathbf{B}}$, respectively. Then we must have:

$$\bar{R}_S \le R_S, \bar{R}_M \ge R_M \tag{65}$$

Now it is clearly shown that MNE and MSNE cannot coexist since for MNE we have $R_M > \bar{R}_M$.

If $\mathbf{B}$ corresponds to MFNE, we can conclude that at MSNE, some SPs must have less bandwidth allocation to small-cells than that of MFNE. Denote this group of SPs as G_{S-} and assume $|G_{MS-}| = L$, we have:

$$\forall j \in G_{S-}, \frac{\partial S_j}{\partial B_{j,S}} = 0, \frac{\partial \bar{S}_j}{\partial \bar{B}_{j,S}} \le 0 \tag{66}$$

Summing up, we get:

$$\lambda_S\Big[Lu'(R_S) + \frac{\lambda_S \sum_{j\in G_{S-}} B_{j,S}R_0}{N_f} u''(R_S)\Big] = Lu'(R_M) + \frac{\sum_{j\in G_{S-}} B_{j,M}R_0}{N_m} u''(R_M) \tag{67a}$$

$$\lambda_S\Big[Lu'(\bar{R}_S) + \frac{\lambda_S \sum_{j\in G_{S-}} \bar{B}_{j,S}R_0}{N_f} u''(\bar{R}_S)\Big] \le Lu'(\bar{R}_M) + \frac{\sum_{j\in G_{S-}} \bar{B}_{j,M}R_0}{N_m} u''(\bar{R}_M) \tag{67b}$$

Rearranging some of the terms, we have:

$$\begin{aligned}&\lambda_S\Big[Lu'(R_S) + \frac{\lambda_S \sum_{j\in\mathcal{N}} B_{j,S}R_0}{N_f} u''(R_S)\Big] =\\ &Lu'(R_M) + R_M u''(R_M) + \lambda_S \frac{\lambda_S \sum_{j\notin G_{S-}} B_{j,S}R_0}{N_f} u''(R_S)\\ &- \frac{\sum_{j\notin G_{S-}} B_{j,M}R_0}{N_m} u''(R_M)\end{aligned} \tag{68a}$$

$$\begin{aligned}&\lambda_S\Big[Lu'(\bar{R}_S) + \frac{\lambda_S \sum_{j\in\mathcal{N}} \bar{B}_{j,S}R_0}{N_f} u''(\bar{R}_S)\Big] \le\\ &Lu'(\bar{R}_M) + \bar{R}_M u''(\bar{R}_M) + \lambda_S \frac{\lambda_S \sum_{j\notin G_{S-}} \bar{B}_{j,S}R_0}{N_f} u''(\bar{R}_S)\\ &- \frac{\sum_{j\notin G_{S-}} \bar{B}_{j,M}R_0}{N_m} u''(\bar{R}_M)\end{aligned} \tag{68b}$$

However, we also have:

$$\forall j \notin G_{S-}, \bar{B}_{j,S} \ge B_{j,S}, \bar{B}_{j,M} \le B_{j,M} \tag{69}$$

We already showed that LHS decreases with R_S and the first two terms on RHS decreases with R_M. For α-fair utility functions, $u''(r) < 0$ increases with r. Combining with the fact that $\bar{R}_S \le R_S, \bar{R}_M \ge R_M$ and noticing that at least one of the inequalities must be strict, we can also conclude:

$$\begin{aligned}&\lambda_S\Big[Lu'(\bar{R}_S) + \frac{\lambda_S \sum_{j\in\mathcal{N}} \bar{B}_{j,S}R_0}{N_f} u''(\bar{R}_S)\Big] >\\ &Lu'(\bar{R}_M) + \bar{R}_M u''(\bar{R}_M) + \lambda_S \frac{\lambda_S \sum_{j\notin G_{S-}} \bar{B}_{j,S}R_0}{N_f} u''(\bar{R}_S)\\ &- \frac{\sum_{j\notin G_{S-}} \bar{B}_{j,M}R_0}{N_m} u''(\bar{R}_M)\end{aligned} \tag{70}$$

Clearly we have a contradiction then. As a result, MSNE and MFNE cannot coexist.

We then only need to show MNE and MFNE cannot coexist. It can be proved in a similar way as we proved MSNE and MFNE cannot coexist introduced above. We now focus on the group of SPs which decrease bandwidth allocation to small-cells and apply the same procedures to get a contradiction.

8. Finally, we need to show that within MSNE, MNE or MFNE, the Nash equilibrium is unique.

The uniqueness of MNE is trivial.

The uniqueness of MFNE can be proved in a similar way in which we proved MSNE and MFNE cannot coexist. Here we don't repeat the steps.

The uniqueness of MSNE can also be proved similarly. However, here we use another method. It's easy to see that at MSNE, we have the following system of equations.

$$\begin{cases} \lambda_S[Nu'(R_S) + \frac{\lambda_S \sum_{i\in\mathcal{N}} B_{i,S}R_0}{N_f} u''(R_S)] \\ = Nu'(R_M) + R_M u''(R_M) \\ \frac{\lambda_S^2 \Delta B_{ij,S}}{N_f} u''(R_S) = \frac{\Delta B_{ij,M}}{N_m} u''(R_M) \end{cases} \tag{P4}$$

where $\Delta B_{ij,S} = B_{i,S} - B_{j,S}$ is the difference of bandwidth allocation to small-cells between SP i and SP j, the same for $\Delta B_{ij,M}$.

By the monotonicity of both LHS and RHS with respect to R_S and R_M, the first equation we can uniquely determine $\sum_{i=1}^{N} B_{i,S}$. While the second equation characterizes the relationship of $B_{i,S}$ between any pair of service providers, as a result the above equation system is essentially a linear equation system with N unknowns and N independent linear equations. Thus if there is a solution to the equation system, it must be unique.

Appendix G
Proof of Proposition 1

The bandwidth allocation at an MSNE can be computed via the following system of equations:

$$\begin{cases} \lambda_S[Nu'(R_S) + \frac{\lambda_S \sum_{i\in\mathcal{N}} B_{i,S}R_0}{N_f}u''(R_S)] \\ = Nu'(R_M) + R_M u''(R_M) \\ \frac{\lambda_S^2 \Delta B_{ij,S}}{N_f}u''(R_S) = \frac{\Delta B_{ij,M}}{N_m}u''(R_M) \end{cases} \tag{P4}$$

where $\Delta B_{ij,S} = B_{i,S} - B_{j,S}$ is the difference of bandwidth allocation to small-cells between SP i and SP j, the same for $\Delta B_{ij,M}$.

Then it's easy to prove the symmetry and monotonicity properties of MSNE.

Appendix H
Proof of Proposition 2

The equilibrium equations for MSNE in scenarios with and without unlicensed spectrum are given below:

$$\text{With : } \lambda_S\Big[Nu'(R_S) + \frac{\lambda_S B_S R_0}{N_f}u''(R_S)\Big] =$$
$$Nu'(R_M) + R_M u''(R_M) \tag{71a}$$
$$\text{Without : } \lambda_S\Big[Nu'(R_S) + R_S u''(R_S)\Big] =$$
$$Nu'(R_M) + R_M u''(R_M) \tag{71b}$$

We only need to compare the LHS for two scenarios:

$$A_1 = N\Big(\frac{\kappa N_f}{C_U + \kappa\lambda_S B_S R_0}\Big)^\alpha - \alpha\frac{\lambda_S B_S R_0}{N_f}\Big(\frac{\kappa N_f}{C_U + \kappa\lambda_S B_S R_0}\Big)^{\alpha+1} \tag{72a}$$
$$A_2 = N\Big(\frac{N_f}{\lambda_S B_S R_0}\Big)^\alpha - \alpha\Big(\frac{N_f}{\lambda_S B_S R_0}\Big)^\alpha \tag{72b}$$

Doing the calculation of $A_1 - A_2$, the resulting ratio's numerator Z can be simplified to:

$$Z = (\kappa\lambda_S B_S R_0)^\alpha (C_U + \kappa\lambda_S B_S R_0) - \alpha(\kappa\lambda_S B_S R_0)^{\alpha+1} - (N-\alpha)(C_U + \kappa\lambda_S B_S R_0)^{\alpha+1} \tag{73}$$

Taking the derivative with respect to C_U, we have:

$$\frac{\partial Z}{\partial C_U} = (\kappa\lambda_S B_S R_0)^\alpha - (N-\alpha)(1+\alpha)(C_U + \kappa\lambda_S B_S R_0)^\alpha \tag{74}$$

Evaluate the expression at $C_U = 0$:

$$\begin{aligned} \frac{\partial Z}{\partial C_U}\Big|_{C_U=0} &= (\kappa\lambda_S B_S R_0)^\alpha (1 - (N-\alpha)(1+\alpha)) \\ &\le (\kappa\lambda_S B_S R_0)^\alpha (1 - (2-\alpha)(1+\alpha)) \\ &= (\kappa\lambda_S B_S R_0)^\alpha \Big[(\alpha - \frac{1}{2})^2 - \frac{5}{4}\Big] < 0 \end{aligned} \tag{75}$$

Therefore we can conclude that $\frac{\partial Z}{\partial C_U} < 0, C_U \ge 0$. On the other hand, we have $Z = 0$ when $C_U = 0$. As a result, $A_1 < A_2$ always holds.

Thus, we can conclude at MSNE, the optimal bandwidth allocation to small-cells with unlicensed spectrum is always less than that of without unlicensed spectrum.

Appendix I
Proof of Proposition 3

At MNE, we need to have:

$$\frac{\partial S_i}{\partial B_{i,S}} \le 0 \text{ as } R_S \to 0^+, \quad \forall i \in \mathcal{N}. \tag{76}$$

In previous sections we know:

$$\begin{aligned} \frac{\partial S_j}{\partial B_{j,S}} =& \lambda_S R_0\Big[u'(R_S) + \frac{\lambda_S B_{j,S} R_0}{N_f}u''(R_S)\Big] \\ &- R_0\Big[u'(R_M) + \frac{B_{j,M} R_0}{N_m}u''(R_M)\Big] \end{aligned} \tag{77a}$$

$$\text{as } R_S \to 0^+, R_S \to \frac{C_U}{\kappa N_f}, R_M \to \frac{\sum_{i=1}^N B_i R_0}{N_m} \tag{77b}$$

Substituting the values of R_S and R_M as $R_S \to 0^+$, we can therefore get the inequality condition characterized in Proposition 3.

Appendix J
Proof of Theorem 7

At MSNE, we have:

$$\begin{aligned} &\lambda_S[Nu'(R_S) + \frac{\lambda_S \sum_{i\in\mathcal{N}} B_{i,S}R_0}{N_f}u''(R_S)] \\ &= Nu'(R_M) + R_M u''(R_M) \end{aligned} \tag{78}$$

At MFNE, we have:

$$\begin{aligned} &\lambda_S[Nu'(R_S) + \frac{\lambda_S \sum_{i\in\mathcal{N}} B_{i,S}R_0}{N_f}u''(R_S)] \\ &\le Nu'(R_M) + R_M u''(R_M) \end{aligned} \tag{79}$$

Asymptotically, for both scenarios described we would have the following:

$$\text{MSNE: } \lambda_S u'(R_S) = u'(R_M) \tag{80a}$$
$$\text{MFNE: } \lambda_S u'(R_S) \le u'(R_M) \tag{80b}$$

However, generally these are not the optimality conditions for maximizing social welfare as we discussed in equation P3 in Section V.

We need to consider an MNE separately. Equation (15) gives the threshold of allocating all bandwidth to macro–cells at social welfare maximization.

$$C_U^{\text{sw}} = \frac{\kappa N_f B R_0\big[(\alpha+1)\lambda_S\big]^{\frac{1}{\alpha}}}{N_m} \tag{81}$$

Meanwhile, equation (17) gives the condition for MNE.

$$\lambda_S\Big(\frac{C_U}{\kappa N_f}\Big)^{-\alpha} \le \Big(\frac{\sum_{i=1}^N B_i R_0}{N_m}\Big)^{-\alpha} - \alpha\frac{B_{i_{\max}} R_0}{N_m}\Big(\frac{\sum_{i=1}^N B_i R_0}{N_m}\Big)^{-\alpha-1} \tag{82}$$

Therefore the threshold of allocating all bandwidth to macro–cells C_U^{sp} at competitive scenario is:

$$\lambda_S\left(\frac{C_U^{sp}}{\kappa N_f}\right)^{-\alpha} = \left(\frac{\sum_{i=1}^{N} B_i R_0}{N_m}\right)^{-\alpha} - \alpha\frac{B_{i_{\max}} R_0}{N_m}\left(\frac{\sum_{i=1}^{N} B_i R_0}{N_m}\right)^{-\alpha-1} \tag{83a}$$

$$\lambda_S\left(\frac{C_U^{sp}}{\kappa N_f}\right)^{-\alpha} = \left(1-\frac{\alpha}{N}\right)\left(\frac{\sum_{i=1}^{N} B_i R_0}{N_m}\right)^{-\alpha} \tag{83b}$$

$$C_U^{sp} = \frac{\kappa N_f B R_0\left(\frac{\lambda_S}{1-\frac{\alpha}{N}}\right)^{\frac{1}{\alpha}}}{N_m}, B = \sum_{i=1}^{N} B_i \tag{83c}$$

When $N \to +\infty$, $C_U^{sp} \to \frac{\kappa N_f B R_0 \lambda_S^{\frac{1}{\alpha}}}{N_m} < C_U^{\text{sw}}$. Therefore the limiting MNE is not necessarily optimizing the social welfare.

Appendix K
Proof of Results in Section VII

The ideal case is a simple allocation optimization and we only need to equalize the marginal utility increase with respect to bandwidth increase for both mobile users and fixed users. Since both WiFi network in unlicensed spectrum and small-cells in licensed spectrum are able to serve fixed users, the social planner should always choose the one which can lead to more rate to invest bandwidth. As a result, when $\lambda_S > \lambda_U$, $B_U^{\text{opt}} = 0$, when $\lambda_S < \lambda_U$, $B_S^{\text{opt}} = 0$, when $\lambda_S = \lambda_U$, what matters is only the total bandwidth allocated to small-cells and unlicensed spectrum, while the split between them is arbitrary.

When $\lambda_S > \lambda_U$, the optimal bandwidth allocation strategy is given by:

$$\begin{cases} \lambda_S u'\left(\frac{\lambda_S B_S^{\text{opt}} R_0}{N_f}\right) = u'\left(\frac{B_M^{\text{opt}} R_0}{N_m}\right) \\ B_S^{\text{opt}} + B_M^{\text{opt}} = B \end{cases} \tag{P5}$$

When $\lambda_S < \lambda_U$, the optimal bandwidth allocation strategy is given by:

$$\begin{cases} \lambda_U u'\left(\frac{\lambda_U B_U^{\text{opt}} R_0}{N_f}\right) = u'\left(\frac{B_M^{\text{opt}} R_0}{N_m}\right) \\ B_U^{\text{opt}} + B_M^{\text{opt}} = B \end{cases} \tag{P6}$$

When $\lambda_S = \lambda_U = \lambda$, the optimal bandwidth allocation strategy is given by:

$$\begin{cases} \lambda u'\left(\frac{\lambda (B_S^{\text{opt}} + B_U^{\text{opt}}) R_0}{N_f}\right) = u'\left(\frac{B_M^{\text{opt}} R_0}{N_m}\right) \\ B_S^{\text{opt}} + B_U^{\text{opt}} + B_M^{\text{opt}} = B \end{cases} \tag{P7}$$

For the practical scenario, when $\lambda_S \geq \lambda_U$, it's always optimal for the social planner to allocate all bandwidth to licensed spectrum because in the case of α-fair utility functions, maximizing revenue is exactly the same as maximizing social welfare with a monopoly SP. In competitive scenario, it's also easy to verify for α-fair utility functions, equation (71b) would also yield the same bandwidth allocation that maximizes social welfare.

When $\lambda_S = \lambda_U = \lambda$, another optimal allocation occurs at the point that the social planner allocate $B_U = B_S^{\text{opt}} + B_U^{\text{opt}}$ to unlicensed spectrum and $B_L = B_M^{\text{opt}}$ to licensed spectrum. We can prove that in this case the SP(s) would only allocate the bandwidth B_L to macro-cells under certain conditions, which therefore achieves the same social welfare as the benchmark optimal case. To prove this, we notice that the condition for the SP(s) to only allocate bandwidth to macro-cells is given by equation (17):

$$\lambda_S\left(\frac{C_U}{\kappa N_f}\right)^{-\alpha} \leq \left(\frac{\sum_{i=1}^{N} B_i R_0}{N_m}\right)^{-\alpha} - \alpha\frac{B_{i_{\max}} R_0}{N_m}\left(\frac{\sum_{i=1}^{N} B_i R_0}{N_m}\right)^{-\alpha-1} \tag{84}$$

When $B_U = B_S^{\text{opt}} + B_U^{\text{opt}}$, $B_L = B_M^{\text{opt}}$, we have:

$$\frac{C_U}{N_f} = \lambda^{\frac{1}{\alpha}} \frac{B_M^{\text{opt}} R_0}{N_m} \tag{85}$$

Therefore we have:

$$\lambda_S\left(\frac{C_U}{\kappa N_f}\right)^{-\alpha} = \kappa^{\alpha}\left(\frac{B_M^{\text{opt}} R_0}{N_m}\right)^{-\alpha} \tag{86a}$$

$$\left(\frac{B_L R_0}{N_m}\right)^{-\alpha} - \alpha\frac{B_{i_{\max}} R_0}{N_m}\left(\frac{B_L R_0}{N_m}\right)^{-\alpha-1} = \left(1-\frac{\alpha}{N}\right)\left(\frac{B_L R_0}{N_m}\right)^{-\alpha} \tag{86b}$$

which means (84) holds the following holds:

$$\kappa^{\alpha} + \frac{\alpha}{N} \leq 1 \tag{87}$$

It's easy to verify when $N \geq 2$, the above condition is always satisfied for $\alpha \in (0, 1)$. When $N = 1$, it is satisfied when $\alpha \in (0, 0.5]$.

We can also prove that when $\lambda_S < \lambda_U$, one possible way to achieve the optimal benchmark social welfare is to allocate $B_U = B_U^{\text{opt}}$ to unlicensed spectrum and $B_L = B_M^{\text{opt}}$ to licensed spectrum. As a result, under certain conditions, the SP(s) would also allocate all bandwidth B_L only to macro-cells, which therefore leads to the same social welfare as in scenario 1). By similar argument, the conditions for this to hold is the following:

$$\kappa^{\alpha}\frac{\lambda_S}{\lambda_U} + \frac{\alpha}{N} \leq 1 \tag{88}$$

It's easy to verify when $N \geq 2$, the above condition is always satisfied for $\alpha \in (0, 1)$. When $N = 1$, it is satisfied when $\alpha \in (0, \alpha_0]$, where α_0 is the unique solution to the following equation:

$$\kappa^{\alpha_0}\frac{\lambda_S}{\lambda_U} + \alpha_0 = 1.$$

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