Correspondence

On the Average Near–Far Resistance for MMSE Detection of Direct Sequence CDMA Signals with Random Spreading

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***Abstract*—The performance of a near–far-resistant, finite-complexity, minimum mean squared error (MMSE) linear detector for demodulating direct sequence (DS) code-division multiple access (CDMA) signals is studied, assuming that the users are assigned random signature sequences. We obtain tight upper and lower bounds on the expected near–far resistance of the MMSE detector, averaged over signature sequences and delays, as a function of the processing gain and the number of users. Since the MMSE detector is optimally near–far-resistant, these bounds apply to any multiuser detector that uses the same observation interval and sampling rate. The lower bound on near–far resistance implies that, even without power control, linear multiuser detection provides near–far-resistant performance for a number of users that grows linearly with the processing gain.**

***Index Terms*—CDMA (code-division multiple access), direct sequence, interference suppression, multiuser detection, random signature sequence, spread spectrum.**

I. Introduction

It has been recently shown that linear minimum mean squared error (MMSE) receivers for direct sequence code-division multiple access (DS-CDMA) signals do not suffer from the near–far problem or the interference floor in performance exhibited by conventional matched filter reception [11]. The use of the MMSE criterion for CDMA receivers was first proposed in [26]. More recently, it was recognized by several authors [1], [11], [13], [16] that, for CDMA systems in which the signature sequences are short (i.e., the period of the signature sequence equals the symbol period), linear MMSE receivers can be implemented as adaptive tapped-delay lines with relatively low complexity. Such implementations do not require explicit knowledge of parameters such as the signature sequences and delays of the interfering users, unlike centralized multiuser detectors (see [23] for a survey of the latter). While previous performance studies of linear MMSE detection were for specific choices of signature sequences, in this correspondence, we attempt to characterize its performance averaged over randomly chosen signature sequences and randomly chosen delays. The detector considered in this correspondence is the N-tap MMSE detector proposed in [11] (N is the processing gain), which consists of an N-tap linear filter followed by a threshold device. The tap spacing is equal to the chip interval, and the taps are selected to minimize the mean squared error (MSE) between the transmitted symbol and the filter output.

We consider both synchronous and asynchronous CDMA systems. Although most systems in practice are asynchronous, consideration of a synchronous system facilitates exposition of the ideas behind the proofs of our results. We derive tight upper and lower bounds on the average near–far resistance of the MMSE detector. These bounds apply to any optimally near–far-resistant multiuser detection scheme (linear or nonlinear) that uses N-chip-spaced observations over a single symbol interval to detect each symbol. This is because, for a fixed observation interval and sampling rate, the MMSE detector, and its zero-forcing (or decorrelating) analog, have maximum near–far resistance [6], [7], [11]. The near–far resistance of the N-tap MMSE detector considered here also provides a lower bound for that of infinite-memory multiuser detectors such as the optimal (maximum-likelihood) multiuser detector [21] and the decorrelating detector [6], [7], with equality for synchronous CDMA. This leads to the approximate rule that the maximum number of strong interferers that the N-tap detector (and hence more complex multiuser detectors) can effectively suppress grows *linearly* with the processing gain N. This is in contrast to the matched-filter receiver, whose near–far resistance is zero with high probability even for *two* simultaneous users.

The random signature sequence model considered has been used to analyze the performance of the matched-filter receiver [14], [25], to obtain performance limits for matched-filter-based timing acquisition for DS-CDMA systems [12], and to derive timing acquisition schemes for single-user DS systems [3]. When the signature sequences have period much larger than the symbol interval (as in the current IS-95 DS-CDMA air interface [17]), an average with respect to signature sequences can be interpreted as an average over each user's symbol sequence, since the signature sequence restricted to each symbol interval appears random [4]. In contrast, in a system with short signature sequences (in which case the cyclostationarity of the interference permits adaptive implementation of the MMSE detector), averaging over signature sequences has the interpretation of averaging over the set of active users. Our results on average near–far resistance apply to both kinds of systems, as long as *linear* modulation is used (thus the results would not apply to the IS-95 mobile to base link, which uses orthogonal modulation).

Section II summarizes the system model and the analysis for a fixed set of signature sequences from [11]. Section III contains the bounds on near–far resistance, together with numerical results demonstrating their tightness. The derivation of these bounds is given in Section IV. Our conclusions are given in Section V.

II. Preliminaries

Consider a direct-sequence CDMA system with K simultaneous antipodal users over an additive white Gaussian noise (AWGN) real baseband channel. The received signal due to the kth user $(1 \le k \le K)$ is given by

$$r^{(k)}(t) = \sum_{n=-\infty}^{\infty} b_{k,n} A_k s_k(t - nT - \tau_k) \qquad (1)$$

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where T is the *symbol interval*, $b_{k,n} \in \{-1, 1\}$ is the nth symbol of the kth user, A_k is its amplitude, τ_k is its relative delay with respect to the receiver, and $s_k(t)$ is its *spreading waveform*, given by

$$s_k(t) = \sum_{j=0}^{N-1} a_k[j]\psi(t - jT_c). \tag{2}$$

Here $a_k[j] \in \{-1, 1\}$ is the jth element of the signature sequence for the kth user, $\psi(t)$ is the *chip waveform*, T_c is the *chip interval*, and $N = T/T_c$ is the *processing gain*. Under the random signature sequence model, $a_k[j]$, $1 \le k \le K$, $0 \le j \le N-1$, are independent random variables taking the values $+1$ and -1 with equal probability.

The net received signal is given by

$$r(t) = \sum_{k=1}^{K} r^{(k)}(t) + n(t) \tag{3}$$

where $n(t)$ is additive white Gaussian noise (AWGN). Taking the first user to be the *desired* user, our objective is to demodulate its bit sequence $\{b_{1,n}\}$.

The receiver is assumed to know the symbol and chip timing of the desired user,[1] so that we may set $\tau_1 = 0$. The received signal is passed through a chip matched filter and sampled at the chip rate. For making a decision on the nth symbol of the desired user, we consider the N samples obtained in the observation interval $(nT, (n+1)T]$ which form the received vector

$$\boldsymbol{r}_n = (r[nN], r[nN+1], \cdots, r[nN+2N-1])^T.$$

For a rectangular chip waveform, the lth chip sample is obtained as

$$r[l] = \int_{lT_c}^{(l+1)T_c} r(t)\,dt. \tag{4}$$

We now express $\boldsymbol{r}_n$ in terms of the parameters of the asynchronous CDMA model (1)–(3). Without loss of generality, let $\tau_k \in [0, T)$, and write it as $\tau_k = (n_k + \delta_k)T_c$, where n_k is an integer between 0 and $N-1$, and $\delta_k \in [0, 1)$. Let $\boldsymbol{a}_k$ denote a vector of length N consisting of the N elements of the spreading sequence of the kth user. Let $\boldsymbol{T}_L$ denote the acyclic left-shift operator, and $\boldsymbol{T}_R$ denote the acyclic right-shift operator, both operating on vectors of length N. Thus for a vector $\boldsymbol{x} = (x_0, \cdots, x_{N-1})^T$, we have $\boldsymbol{T}_L\boldsymbol{x} = (x_1, \cdots, x_{N-1}, 0)^T$ and $\boldsymbol{T}_R\boldsymbol{x} = (0, x_0, \cdots, x_{N-2})^T$. Let T_L^n, T_R^n, denote n applications of these operators, resulting in left and right shifts by n, respectively.

For each asynchronous user, two consecutive bit intervals overlap with a given observation interval of length T. Furthermore, since the system is chip-asynchronous, two adjacent chips contribute to each chip sample. The contribution of the kth user to the received vector $\boldsymbol{r}_n \in \mathcal{R}^N$ for the nth observation is, therefore, given by

$$\boldsymbol{r}_n^{(k)} = b_{k,n-1}A_k\boldsymbol{v}_k^{-1} + b_{k,n}A_k\boldsymbol{v}_k^0 \tag{5}$$

where

$$\begin{aligned}\boldsymbol{v}_k^{-1} &= (1-\delta_k)\boldsymbol{T}_L^{N-n_k}\boldsymbol{a}_k + \delta_k\boldsymbol{T}_L^{N-n_k-1}\boldsymbol{a}_k\\ \boldsymbol{v}_k^0 &= (1-\delta_k)\boldsymbol{T}_R^{n_k}\boldsymbol{a}_k + \delta_k\boldsymbol{T}_R^{n_k+1}\boldsymbol{a}_k.\end{aligned} \tag{6}$$

Remark 2.1: For general chip pulses, possibly of duration larger than T_c (e.g., the bandlimited square root raised cosine pulse used in the IS-95 standard), the received signal would be passed through a chip matched filter with impulse response $\psi_{\mathrm{MF}} = \psi^*(-t)$. Assuming that the net chip response $\phi = \psi * \psi_{\mathrm{MF}}$ is Nyquist at the chip rate, and that the receiver is synchronized to the desired user, the discrete-time response for the desired user would be the same as for the rectangular pulse. For the interference, the preceding discrete-time model holds *approximately* if $\phi(t)$ decays sufficiently rapidly with $|t|$ that a given chip makes a significant contribution to at most two adjacent chip spaced samples.

[1] MMSE interference suppression can also be used to incorporate timing recovery into the receiver [8]–[10], [18].

We consider the following cases.

Synchronous CDMA: For $1 \le k \le K$, the delays $\tau_k = 0$, so that $\boldsymbol{v}_k^{-1} = 0$ and $\boldsymbol{v}_k^0 = \boldsymbol{a}_k$.

Asynchronous CDMA: The receiver is synchronized to the desired user, so that we still have $\tau_1 = 0$, and $\boldsymbol{v}_1^{-1} = 0$ and $\boldsymbol{v}_1^0 = \boldsymbol{a}_1$. However, the interferer delays τ_k, $2 \le k \le K$, can take any value in the interval $[0, T)$. For the averaged performance measures to be considered in this correspondence, these delays are assumed to be independent random variables, uniformly distributed over $[0, T)$.

In each case, we obtain the following generic *equivalent synchronous model* for the net received vector:

$$\boldsymbol{r}_n = b_0[n]\boldsymbol{u}_0 + \sum_{j=1}^{J} b_j[n]\boldsymbol{u}_j + \boldsymbol{w}_n \tag{7}$$

where $b_0[n]$ is the *desired bit* that we wish to demodulate, $\boldsymbol{u}_0$ is the vector modulating it, and, for $1 \le j \le J$, $b_j[n]$ are interfering bits due to intersymbol interference and multiple-access interference, $\boldsymbol{u}_j$ are interference vectors modulating these bits, and $\boldsymbol{w}_n$ is white Gaussian noise with covariance $\sigma^2\boldsymbol{I}$. The correspondence between the generic model (7) and the original model (1)–(6) is as follows. The desired bit $b_0[n] = b_{1,n}$, and the desired vector $\boldsymbol{u}_0 = A_1\boldsymbol{v}_1^0$. For synchronous CDMA, the number of interference vectors $J = K-1$, with

$$\boldsymbol{u}_j = A_{j+1}\boldsymbol{v}_{j+1}^0, \qquad j = 1, \cdots, J.$$

For asynchronous CDMA, the number of interference vectors $J = 2(K-1)$, with

$$\begin{aligned}\boldsymbol{u}_{2l-1} &= A_{l+1}\boldsymbol{v}_{l+1}^{-1}\\ \boldsymbol{u}_{2l} &= A_{l+1}\boldsymbol{v}_{l+1}^0, \qquad l = 1, \cdots, J/2.\end{aligned}$$

For much of our analysis, it is convenient to work with the generic model (7), hiding the underlying structure of the signal vectors $\{\boldsymbol{u}_j\}$.

A. The Linear MMSE Detector

For the model (7), letting $\langle , \rangle$ denote the standard inner product in Euclidean space, a linear receiver produces a bit estimate

$$\hat{b}_0[n] = \mathrm{sgn}\,(\langle \boldsymbol{c}, \boldsymbol{r}_n\rangle). \tag{8}$$

The linear MMSE receiver is a correlator $\boldsymbol{c}$ that minimizes the MSE $E\{(\langle \boldsymbol{c}, \boldsymbol{r}_n\rangle - b_0[n])^2\}$ between the decision statistic and the desired bit $b_0[n]$. This receiver also maximizes the signal to interference (plus noise) ratio (SINR) among all linear receivers.

B. Near–Far Resistance and the Zero-Forcing Receiver

The *asymptotic efficiency* [22] of a multiuser detector measures the exponential rate of convergence of its error probability to zero as the noise variance $\sigma^2 \to 0$, relative to the rate in a single-user setting. The worst case asymptotic efficiency over all possible interference amplitudes is the near–far resistance [6], [7]. Let $\mathcal{S}_I$ denote the subspace spanned by the interference vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_J$. Let $\mathcal{P}_{\mathcal{S}_I}^{\perp}(\boldsymbol{u}_0)$ denote the projection of $\boldsymbol{u}_0$ orthogonal to the interference subspace. If this projection is nonzero, then a linear correlator chosen along this direction is a zero-forcing, or decorrelating, detector [6], [7], [11]. It was shown in [11] that, if the zero-forcing receiver exists (i.e., $\mathcal{P}_{\mathcal{S}_I}^{\perp}(\boldsymbol{u}_0) \ne 0$), then the MMSE detector tends to the zero-forcing detector when $\sigma^2 \to 0$. Thus the asymptotic efficiency of the MMSE detector is equal to that of the zero-forcing detector. Moreover, the asymptotic efficiency of the zero-forcing detector is independent of interference amplitudes, and therefore equals its near–far resistance. The near–far resistance and asymptotic efficiency of the MMSE and zero-forcing detectors are both given by

$$\eta = \frac{||\mathcal{P}^{\perp}_{\mathcal{S}_I}(\boldsymbol{u}_0)||^2}{||\boldsymbol{u}_0||^2}. \tag{9}$$

The near–far resistance therefore equals the fraction of the energy of the desired signal vector that remains after projecting orthogonal to the interference subspace.

C. Performance in Terms of Signal Crosscorrelations

The performance of the MMSE and zero-forcing detectors are determined by the crosscorrelations between the signal vectors $\{\boldsymbol{u}_j\}$. Denote the $J \times 1$ vector of normalized crosscorrelations between the desired signal vector and the interference vectors as ρ_I, with entries

$$\rho_I(j) = \frac{\langle \boldsymbol{u}_0, \boldsymbol{u}_j\rangle}{||\boldsymbol{u}_0|| ||\boldsymbol{u}_j||}, \qquad j = 1, \cdots, J.$$

Denote the $J \times J$ matrix of normalized crosscorrelations between the interference vectors by $\boldsymbol{R}_I$, with (i, j)th entry given by

$$R_I(i, j) = \frac{\langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle}{||\boldsymbol{u}_i|| ||\boldsymbol{u}_j||}, \qquad i, j = 1, \cdots, J.$$

It can be shown [11] that the near–far resistance of the MMSE and zero-forcing (ZF) detectors is given by

$$\eta = 1 - \rho_I^T \boldsymbol{R}_I^{\dagger} \rho_I \tag{10}$$

where $\boldsymbol{R}_I^{\dagger}$ is a generalized inverse [2] of $\boldsymbol{R}_I$, i.e., any matrix satisfying $\boldsymbol{R}_I \boldsymbol{R}_I^{\dagger} \boldsymbol{R}_I = \boldsymbol{R}_I$. Although the generalized inverse $\boldsymbol{R}_I^{\dagger}$ is unique if and only if $\boldsymbol{R}_I$ is nonsingular, it can be shown that the near–far resistance is uniquely specified regardless of the rank of $\boldsymbol{R}_I$. Nonsingularity of $\boldsymbol{R}_I$ is equivalent to the linear independence of the set of interference vectors.

III. Results

We assume that the signature sequences $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K$ are independent and identically distributed (i.i.d.) random vectors, each chosen uniformly from $\{-1, +1\}^N$. The relative amplitudes A_j are assumed to be fixed, and the relative delays for asynchronous CDMA are assumed to be uniformly distributed over a bit interval. Using this random signature sequence model, averaging the expressions for the performance measures given in the previous section leads to the results given in this section. Proofs are postponed to the succeeding section. Our results on the expected near–far resistance for the MMSE and ZF detectors are stated in Lemma 1, and Theorems 1 and 2. The matched-filter receiver is not considered, since it has zero near–far resistance with probability one for asynchronous CDMA and for synchronous CDMA with odd N, and with probability close to one for synchronous CDMA with even N. In the latter case, the probability of nonzero near–far resistance, (i.e., of the desired signature sequence being orthogonal to all other signature sequences) goes to zero exponentially fast with N.

Lemma 1: The conditional expectation of the near–far resistance, conditioning on the interference subspace $\mathcal{S}_I$ (i.e., on the interfering signature sequences and delays) and averaging over the desired signature sequence $\boldsymbol{a}_1$, is given by

$$E[\eta|\mathcal{S}_I] = 1 - d_I/N \tag{11}$$

where d_I denotes the dimension of $\mathcal{S}_I$.

The dimension d_I is a random variable taking values from 1 to J, depending on the random signature sequences and delays of the interfering users. Averaging over the latter, we obtain upper and lower bounds on the expectation of d_I, which yields the bounds on expected near–far resistance stated in the following theorem.

Theorem 1: The expected near–far resistance for *synchronous CDMA* satisfies

$$1 - \frac{K-1}{N} \le E[\eta] \le 1 - \frac{1}{N}\sum_{n=1}^{K-1} n f_{K-1}(n) \tag{12}$$

where the function f_{K-1} is computed via the following recursion:

$$f_k(m) = \frac{2^m}{2^N} f_{k-1}(m) + \left(1 - \frac{2^{m-1}}{2^N}\right) f_{k-1}(m-1), \qquad 1 \le m \le k \tag{13}$$

with initial condition

$$f_1(1) = 1 \qquad f_1(0) = 0. \tag{14}$$

Theorem 2: The expected near–far resistance for *asynchronous CDMA* satisfies

$$1 - \frac{2(K-1)}{N} \le E[\eta] \le 1 - \frac{1}{N}\sum_{n=1}^{2(K-1)} n g_{K-1}(n) \tag{15}$$

where g_{K-1} is given by the recursion

$$g_k(m) = \frac{2^m}{2^N} g_{k-1}(m) + 1.5\frac{2^m}{2^N} g_{k-1}(m-1) + \left(1 - \frac{2^{m-1}}{2^N}\right) g_{k-1}(m-2), \qquad 2 \le m \le 2k \tag{16}$$

with initial condition

$$g_1(2) = 1, \quad g_1(1) = g_1(0) = 0. \tag{17}$$

A. Numerical Results

We plot the preceding bounds for a system with processing gain $N = 31$. Fig. 1 shows the upper and lower bounds on the expected near–far resistance for synchronous and asynchronous CDMA as a function of the number of users K. In each case, the bounds are tight, so that we can infer the following rule of thumb from the lower bounds: for near–far resistant performance $(E\{\eta\} > 0)$ using the N-tap MMSE detector or its zero-forcing version, the system design should satisfy $K - 1 < N$ for synchronous CDMA, and $K - 1 < N/2$ for asynchronous CDMA. This rule refers to the number of interferers which are *strong* relative to the desired signal.

IV. Derivation of the Bounds

Consider the generic model (7), and assume first that the interference vectors are linearly independent, i.e., that $\boldsymbol{R}_I$ is invertible. From (10), the near–far resistance is given by

$$\eta = 1 - \rho_I^T \boldsymbol{R}_I^{-1} \rho_I.$$

According to the random signature sequence model, the interference vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_J$ are statistically independent of the desired vector $\boldsymbol{u}_0 = \boldsymbol{a}_1$, for both synchronous and asynchronous CDMA. Lemma 1 is obtained by averaging over $\boldsymbol{a}_1$, conditioned on the interference vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_J$. Since

$$\rho_I(j) = \frac{1}{\sqrt{N}||\boldsymbol{u}_j||}\sum_{m=0}^{N-1} a_1[m] u_j[m], \qquad 1 \le j \le J$$

we obtain that

$$E\{\rho_I(j)\rho_I(k)|\mathcal{S}_I\} = \frac{1}{N||\boldsymbol{u}_j|| \, ||\boldsymbol{u}_k||}\sum_{m=0}^{N-1} u_j[m]u_k[m] = (1/N)R_I(j,k) \tag{18}$$

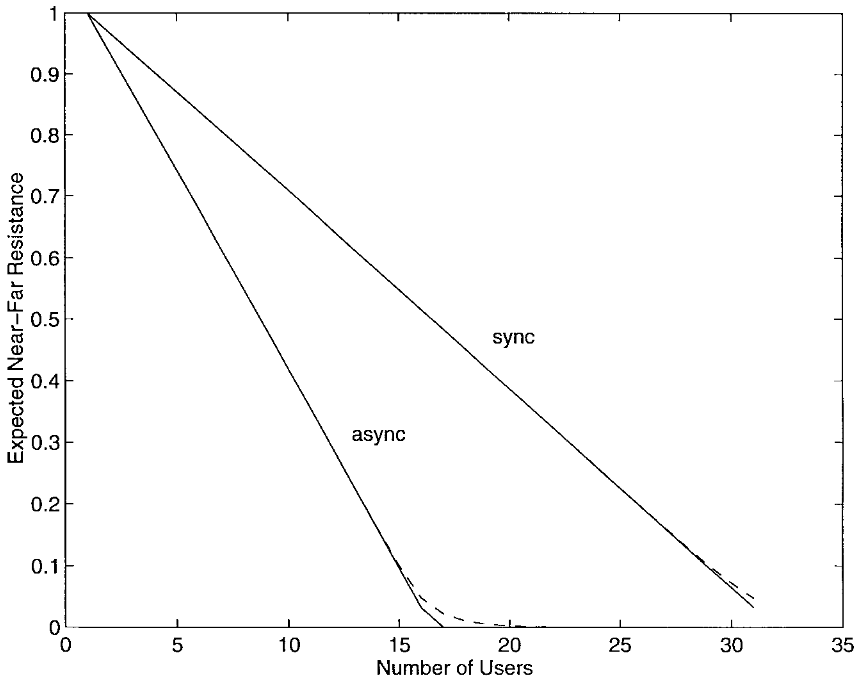

Fig. 1. Bounds on average near–far resistance for MMSE and zero-forcing detectors for asynchronous CDMA $(N = 31)$.

where we have used the fact that $E\{a_1[m]a_1[n]\} = \delta_{mn}$, and that $\boldsymbol{a}_1$ is independent of $\mathcal{S}_I$. We can now conclude that

$$\begin{aligned} E\{\rho_I^T \boldsymbol{R}_I^{-1}\rho_I|\mathcal{S}_I\} &= \sum_{j,k=1}^{J} E\{\rho_I(k)\rho_I(j)|\mathcal{S}_I\}\boldsymbol{R}_I^{-1}(j,k) \\ &= (1/N)\sum_{j,k=1}^{J} \boldsymbol{R}_I(k,j)\boldsymbol{R}_I^{-1}(j,k) \\ &= (1/N)\,\mathrm{trace}\,(\boldsymbol{R}_I^{-1}\boldsymbol{R}_I) = J/N. \end{aligned}$$

Thus if $\boldsymbol{R}_I$ is invertible, we have $E\{\eta|\mathcal{S}_I\} = 1 - J/N$.

In general, $\boldsymbol{R}_I$ need not be invertible. However, denoting the dimension of the interference space $\mathcal{S}_I$ by d_I, we can find d_I linearly independent interference vectors which span $\mathcal{S}_I$. We observe that no other interference vectors need be considered for computing the near–far resistance, since the latter depends only on (the component of $\boldsymbol{u}_0 = \boldsymbol{a}_1$ orthogonal to) $\mathcal{S}_I$. The preceding derivation is therefore applicable if J is replaced by d_I, which yields the desired expression (11).

Removing the conditioning on d_I in (11), we obtain

$$E\{\eta\} = 1 - E\{d_I\}/N. \tag{19}$$

A lower bound on $E\{\eta\}$ follows immediately upon noting that the number of interference vectors, J, is an upper bound on d_I, so that $E\{\eta\} \geq 1 - J/N$. Since $J = K - 1$ for synchronous CDMA, and $J = 2(K-1)$ for asynchronous CDMA, the lower bounds on the expected value of the near–far resistance in (12) and (15) are now immediate. The derivation of the upper bounds involves finding lower bounds on $E\{d_I\}$ using stochastic domination arguments.

A. Upper Bound on $E\{\eta\}$ for Synchronous CDMA

Since the near–far resistance depends on the directions and not the magnitudes of the signal vectors, we may set the relative amplitudes $A_j = 1$. We therefore have that the interference vectors $\boldsymbol{u}_j = \boldsymbol{a}_{j+1}$, $1 \leq j \leq J = K - 1$, are simply random signature sequences chosen uniformly from $\{-1, +1\}^N$. Let $\boldsymbol{S}_k$ denote the subspace generated by the first k interfering users, i.e., $\boldsymbol{S}_k$ is spanned by $\boldsymbol{a}_2, \cdots, \boldsymbol{a}_{k+1}$, and let D_k denote its dimension, so that $\mathcal{S}_I = \boldsymbol{S}_{K-1}$ and $d_I = D_{K-1}$. In the following, we construct a sequence of random variables F_k such that for each $1 \leq k \leq K-1$, F_k is stochastically smaller [19] than D_k. Since $E\{F_{K-1}\} \leq E\{D_{K-1}\} = E\{d_I\}$, this yields the desired upper bound on $E\{\eta\}$.

Let $\epsilon_k = D_k - D_{k-1}$ be the increase in dimension due to the kth interferer, so that $D_k = \sum_{j=1}^{k} \epsilon_j$ for $1 \leq k \leq K-1$, and $\epsilon_1 = D_1 = 1$ with probability one. For $2 \leq k \leq K-1$

$$\epsilon_k = \begin{cases} 0, & \boldsymbol{a}_{k+1} \in \boldsymbol{S}_{k-1} \\ 1, & \text{else.} \end{cases}$$

Define the conditional probability

$$p_k(0|d) = P[\epsilon_k = 0|D_{k-1} = d].$$

In the following, we find an upper bound $\overline{p}(0|d)$ on $p_k(0|d)$ which is independent of k. We use this bound to construct iteratively a sequence of random variables F_k as follows. Let $F_1 = D_1 = 1$ with probability one. For $k \geq 2$, assuming that F_{k-1} has been obtained, define the distribution of F_k by

$$F_k = \begin{cases} F_{k-1} & \text{with probability } \overline{p}(0|F_{k-1}) \\ F_{k-1}+1 & \text{with probability } [1-\overline{p}(0|F_{k-1})]. \end{cases}$$

This construction translates to the following recursion (in k) for $\{f_k(n), 1 \leq n \leq k\}$, the probability mass function of F_k:

$$f_k(m) = \overline{p}(0|m)f_{k-1}(m) + [1-\overline{p}(0|m-1)]f_{k-1}(m-1), \quad 1 \leq m \leq k \tag{20}$$

with initial condition

$$f_1(1) = 1, \quad f_1(0) = 0. \tag{21}$$

We now show that F_k is stochastically dominated by D_k for each k. This is true for $k = 1$, and we assume it is true up to $k - 1$. We must now show that for any monotone nondecreasing function f, we have $E\{f(D_k)\} \geq E\{f(F_k)\}$. Let $U(d)$ denote a Bernoulli random variable which takes value 0 with probability $\overline{p}(0|d)$. Then

$$\begin{aligned} E\{f(D_k)\} &= E\{f(D_{k-1} + \epsilon_k(D_{k-1}))\} \\ &\geq E\{f(D_{k-1} + U(D_{k-1}))\} \\ &= E\{\tilde{f}(D_{k-1})\} \end{aligned} \tag{22}$$

where we define $\tilde{f}(d) = E\{f(d + U(d))\}$. The function $\tilde{f}$ inherits the monotonicity of f, because

$$d + U(d) \leq d + 1 \leq d + 1 + U(d+1)$$

with probability one. Using the inductive hypothesis

$$\begin{aligned} E\{\tilde{f}(D_{k-1})\} &\geq E\{\tilde{f}(F_{k-1})\} \\ &= E\{f(F_{k-1} + U(F_{k-1}))\} \\ &= E\{f(F_k)\}. \end{aligned} \tag{23}$$

Combining (22) and (23) gives the desired result that

$$E\{f(D_k)\} \geq E\{f(F_k)\}.$$

The upper bound $\overline{p}(0|d)$ is given by the following proposition, which is proved in Appendix A.

Proposition 1: Let $\boldsymbol{S}$ be a subspace of $\mathcal{R}^N$ of dimension d, and let $\boldsymbol{a}$ be a random vector independent of $\boldsymbol{S}$ which is uniformly distributed over $\{-1, +1\}^N$. Then

$$P[\boldsymbol{a} \in \boldsymbol{S}] \leq \overline{p}(0|d) = \frac{2^d}{2^N}.$$

Clearly, the hypotheses of the proposition are satisfied by $\boldsymbol{a} = \boldsymbol{a}_{k+1}$ and $\boldsymbol{S} = \boldsymbol{S}_{k-1}$. Proposition 1 thus gives the desired upper bound for $p_k(0|d)$. Using $E\{F_J\}$ as a lower bound for d_I, we obtain (12)–(14) from (19)–(21). The numerical results in Section III demonstrate that the bounds on $E\{\eta\}$ are tight.

A corollary of Proposition 1 is an upper bound on the probability that the near–far resistance is zero, i.e., the probability that the desired signal vector lies in the interference space

$$P[\eta = 0] \leq E\left\{\frac{2^{d_I}}{2^N}\right\} \leq \frac{2^{K-1}}{2^N}.$$

B. Upper Bound on $E\{\eta\}$ for Asynchronous CDMA

As before, let $\boldsymbol{S}_k$ denote the subspace generated by the first k interfering users, let D_k denote its dimension, and let $\epsilon_k = D_k - D_{k-1}$. Analogous to the procedure used for synchronous CDMA, we find a sequence of random variables G_k such that G_k is stochastically smaller than D_k, and substitute $E\{G_{K-1}\}$ for $E\{d_I\}$ in (19).

As in Section IV-A, set the relative amplitudes of all users to one. Recall that each asynchronous user generates *two* linearly independent interference vectors. The subspace $\boldsymbol{S}_k$ is therefore generated by the collection of vectors $\{\boldsymbol{v}_l^{-1}, \boldsymbol{v}_l^0, 2 \leq l \leq k+1\}$, and

$$\epsilon_k = \begin{cases} 0, & \boldsymbol{v}_{k+1}^{-1} \in \boldsymbol{S}_{k-1} \text{ and } \boldsymbol{v}_{k+1}^0 \in \boldsymbol{S}_{k-1} \\ 2, & \boldsymbol{v}_{k+1}^{-1} \notin \boldsymbol{S}_{k-1} \text{ and } \boldsymbol{v}_{k+1}^0 \notin \boldsymbol{S}_{k-1} \\ 1, & \text{else.} \end{cases}$$

For $i = 0, 1, 2,$ let $q_k(i|d)$ be the probability that $\epsilon(k) = i$, conditioned on $D_{k-1} = d$. We find an upper bound $\overline{q}(0|d)$ for $q_k(0|d)$, and a lower bound $\underline{q}(2|d)$ for $q_k(2|d)$. Using these bounds, we construct the random variables G_k as follows. Under our delay model, with probability one, the relative delay of each interfering user is nonzero, so that each such user gives rise to two linearly independent interference vectors. Thus we set $G_1 = D_1 = 2$ with probability one. For $k \geq 2$, let

$$G_k = \begin{cases} G_{k-1}, & \text{with probability } \overline{q}(0|G_{k-1}) \\ G_{k-1} + 2, & \text{with probability } \underline{q}(2|G_{k-1}) \\ G_{k-1} + 1, & \text{with probability } [1 - \overline{q}(0|G_{k-1}) - \underline{q}(2|G_{k-1})]. \end{cases}$$

The corresponding recursion (in k) for $\{g_k(n), 1 \leq n \leq 2^k\}$, the probability mass function of G_k is as follows:

$$\begin{aligned} g_k(m) = {} & \overline{q}(0|m)g_{k-1}(m) + [1 - \overline{q}(0|m-1) - \underline{q}(2|m-1)] \cdot \\ & g_{k-1}(m-1) + \underline{q}(2|m-2)g_{k-1}(m-2), \quad 2 \leq m \leq 2k \end{aligned} \tag{24}$$

with initial condition

$$g_1(2) = 1 \qquad g_1(1) = g_1(0) = 0. \tag{25}$$

The proof that G_k is stochastically smaller than D_k for each k is omitted, since it is similar to the analogous proof for synchronous CDMA. The desired upper bound on the average near–far resistance given by the right-hand side of (15) is now given by

$$E\{\eta\} \leq 1 - E\{G_J\}/N = 1 - (1/N)\sum_{n=1}^{2J} n g_J(n).$$

It remains to compute the bounds $\overline{q}$ and $\underline{q}$. Writing the relative delay τ for a given asynchronous user as $(n + \delta)T_c$, recall that the assumption that τ is a random variable which is uniformly distributed over $[0, T]$ implies that n is a random variable that takes on each integer value in the interval $[0,\, N-1]$ with probability $1/N$, and that δ is a real-valued random variable which is uniformly distributed over the interval $[0,\, 1)$. The two interference vectors, say $\boldsymbol{v}^0$ and $\boldsymbol{v}^{-1}$, due to such a user are random variables depending on its random signature sequence $\boldsymbol{a}$ and its random delay parameters n and δ, as described in Section II. We can now state the following proposition.

Proposition 2: Let $\boldsymbol{S}$ be a subspace of $\mathcal{R}^N$ of dimension d. For $\boldsymbol{v}^0$ and $\boldsymbol{v}^{-1}$ as described above, corresponding to a random signature sequence and delay independent of $\boldsymbol{S}$, we have

$$\begin{aligned} P[\boldsymbol{v}^0 \in \boldsymbol{S}, \boldsymbol{v}^{-1} \in \boldsymbol{S}] &\leq \overline{q}(0|d) = \frac{2^d}{2^N} \\ P[\boldsymbol{v}^0 \notin \boldsymbol{S}, \boldsymbol{v}^{-1} \notin \boldsymbol{S}] &\geq \underline{q}(2|d) = 1 - \frac{2^{d+1}}{2^N}. \end{aligned}$$

Taking $\boldsymbol{S} = \boldsymbol{S}_{k-1}$, $\boldsymbol{v}^0 = \boldsymbol{v}_{k+1}^0$, and $\boldsymbol{v}^{-1} = \boldsymbol{v}_{k+1}^{-1}$ gives the desired bounds on $q_k(0|d)$ and $q_k(2|d)$.

For the proof of Proposition 2, we first show that it suffices to consider chip-synchronous CDMA ($\delta = 0$) in order to bound the probabilities of interest. We then find the required bounds conditioned on the integer part of the delay n, using techniques similar to those used to prove Proposition 1, and find that the bounds are independent of n. The details are given in Appendix B.

A corollary is an upper bound on the probability that the near–far resistance is zero

$$P[\eta = 0] \leq E\left\{\frac{2^{d_I}}{2^N}\right\} \leq \frac{2^{2(K-1)}}{2^N}.$$

V. Conclusions

Averaging over random signature sequences enables us to characterize the average near–far resistance of the MMSE detector as a function of the processing gain and the number of users alone. An important insight gained is that the number of simultaneous users that can be sustained without power control grows linearly with the processing gain N when either MMSE or zero-forcing detection is used, where the rate of growth depends on the required value of average near–far resistance. Of course, our results depend on our modeling assumptions, namely, a stationary user population and the absence of channel variations.

Since this manuscript was submitted, several results on performance analysis for CDMA with random spreading have appeared. Simulation results regarding the variations in SINR and near–far resistance for the model considered in this correspondence appeared in [5]. If the signature sequences consist of independent and identically distributed Gaussian random variables (rather than symmetric Bernoulli random variables as assumed here), then the *distribution* of the near–far resistance (rather than just the expectation, as considered here) can be evaluated explicitly [15] for a synchronous CDMA system. In [20], the asymptotic SINR of the MMSE and zero-forcing detectors as $N \to \infty$ with K/N fixed is characterized for a synchronous CDMA system under the general assumption of signature sequences with independent and identically distributed elements (with zero mean and finite variance). Finally, in [24], the information-theoretic capacity of synchronous CDMA with random spreading is evaluated using different detectors (including the MMSE detector) at the front end.

Appendix A
Proof of Proposition 1

Form an $N \times d$ matrix $\boldsymbol{X}$ with columns $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_d$, where these vectors form a basis for the given d-dimensional subspace, where $d < N$. The row rank of this matrix equals its column rank d. Choose d independent rows. For notational convenience, reorder the coordinates of the $\boldsymbol{x}_i$ so that these rows are numbered from 0 through $d-1$, and let $\boldsymbol{w}_i = (x_i[0], \cdots, x_i[d-1])^T$, where $1 \le i \le d$. The d-dimensional vectors $\boldsymbol{w}_i$ form the columns of a $d \times d$ matrix $\boldsymbol{W}$ obtained by deleting the last $N-d$ rows of $\boldsymbol{U}$. By construction, the row rank, and hence the column rank, of $\boldsymbol{W}$ equals d, so that the vectors $\boldsymbol{w}_i$ are linearly independent. Letting M denote the number of vectors in $\{-1,+1\}^N$ in $\boldsymbol{S}$, the probability that a randomly selected vector in $\{-1,+1\}^N$ lies in $\boldsymbol{S}$ is $M/2^N$. An upper bound on M is derived as follows.

Since the $\boldsymbol{x}_i$ are linearly independent, we have

$$M = \operatorname{card}\left\{(\lambda_1, \cdots, \lambda_d): \sum_{i=1}^{d} \lambda_i x_i[l] \in \{-1,+1\}, l = 0, \cdots, N-1\right\}. \tag{26}$$

Since $\boldsymbol{x}_i$ and $\boldsymbol{w}_i$ agree in their first d coordinates, an upper bound for M is given by

$$M_d = \operatorname{card}\left\{(\lambda_1, \cdots, \lambda_d): \sum_{i=1}^{d} \lambda_i x_i[l] = \sum_{i=1}^{d} \lambda_i w_i[l] \in \{-1,+1\}, \; l = 0, \cdots, d-1\right\} \tag{27}$$

since, compared to (26), fewer constraints are used to define the set on the right-hand side of (27). Using the linear independence of the $\boldsymbol{w}_i$, distinct $(\lambda_1, \cdots, \lambda_d)$ give rise to distinct vectors, so that M_d is bounded by the number of distinct vectors in $\{-1,+1\}^d$. Thus $M \le M_d \le 2^d$, which yields the desired upper bound $\overline{p}(0|d) = 2^d/2^N$.

Appendix B
Proof of Proposition 2

Conditioning on the delay parameters n and δ, and the signature sequence $\boldsymbol{a}$ of the asynchronous user of interest, recall from (6) that the vectors corresponding to the user are given by

$$\begin{aligned} \boldsymbol{v}^{-1} &= (1-\delta)\boldsymbol{T}_L^{N-n}\boldsymbol{a} + \delta \boldsymbol{T}_L^{N-n-1}\boldsymbol{a} \\ \boldsymbol{v}^{0} &= (1-\delta)\boldsymbol{T}_R^{n}\boldsymbol{a} + \delta \boldsymbol{T}_R^{n+1}\boldsymbol{a}. \end{aligned} \tag{28}$$

We now show that we can restrict attention to chip-synchronous CDMA. The latter is easier to analyze because, for given n, $\boldsymbol{v}^{-1}$ and $\boldsymbol{v}^0$ are linearly independent (since they are orthogonal) as well as statistically independent (since the chips $a[l]$ are independent random variables). The following lemma enables us to restrict attention to such a system.

Lemma 2: Fix the signature sequence $\boldsymbol{a}$ and the integer part of the delay n. Then

i) $\boldsymbol{v}^{-1} \in \boldsymbol{S}$ for a set of $\delta \in [0,1]$ of nonzero measure if and only $\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}$ *and* $\boldsymbol{T}_L^{N-n-1}\boldsymbol{a} \in \boldsymbol{S}$.

ii) $\boldsymbol{v}^{0} \in \boldsymbol{S}$ for a set of $\delta \in [0,1]$ of nonzero measure if and only if $\boldsymbol{T}_R^{n}\boldsymbol{a} \in \boldsymbol{S}$ *and* $\boldsymbol{T}_R^{n+1}\boldsymbol{a} \in \boldsymbol{S}$.

Proof: We prove i), since the proof of ii) is entirely analogous. The "if" part of the statement is clear from (28). For the reverse implication, suppose that there is more than one value of δ such that $\boldsymbol{v}^{-1} \in \boldsymbol{S}$. Using two distinct values of δ in (28) then gives two linearly independent simultaneous equations, each with $\boldsymbol{T}_L^{N-n}\boldsymbol{a}$ and $\boldsymbol{T}_L^{N-n-1}\boldsymbol{a}$ as unknowns on the right-hand side, and vectors in $\boldsymbol{S}$ on the left-hand side. Solving these equations, we obtain that both $\boldsymbol{T}_L^{N-n}\boldsymbol{a}$ and $\boldsymbol{T}_L^{N-n-1}\boldsymbol{a}$ lie in $\boldsymbol{S}$.

As a straightforward corollary of the lemma, we obtain, conditioning on n and averaging over δ, that

$$P[\boldsymbol{v}^{-1} \notin \boldsymbol{S}, \boldsymbol{v}^0 \notin \boldsymbol{S} | n] \ge P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \notin \boldsymbol{S}, \boldsymbol{T}_R^{n}\boldsymbol{a} \notin \boldsymbol{S}] \tag{29}$$

$$P[\boldsymbol{v}^{-1} \notin \boldsymbol{S}, \boldsymbol{v}^0 \notin \boldsymbol{S} | n] \ge P[\boldsymbol{T}_L^{N-n-1}\boldsymbol{a} \notin \boldsymbol{S}, \boldsymbol{T}_R^{n+1}\boldsymbol{a} \notin \boldsymbol{S}]. \tag{30}$$

The inequality (29) is used for $n \ge 1$, while (30) is used for $n = 0$ (since $\boldsymbol{T}_L^{N-n}\boldsymbol{a} = 0$ for $n = 0$).

Similarly, we obtain

$$\begin{aligned} P[\boldsymbol{v}^{-1} \in \boldsymbol{S}, \boldsymbol{v}^0 \in \boldsymbol{S} | n] &\le P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}, \boldsymbol{T}_R^{n}\boldsymbol{a} \in \boldsymbol{S}] \\ &= P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}] P[\boldsymbol{T}_R^{n}\boldsymbol{a} \in \boldsymbol{S}] \end{aligned} \tag{31}$$

$$\begin{aligned} P[\boldsymbol{v}^{-1} \in \boldsymbol{S}, \boldsymbol{v}^0 \in \boldsymbol{S} | n] &\le P[\boldsymbol{T}_L^{N-n-1}\boldsymbol{a} \in \boldsymbol{S}, \boldsymbol{T}_R^{n+1}\boldsymbol{a} \in \boldsymbol{S}] \\ &= P[\boldsymbol{T}_L^{N-n-1}\boldsymbol{a} \in \boldsymbol{S}] P[\boldsymbol{T}_R^{n+1}\boldsymbol{a} \in \boldsymbol{S}] \end{aligned} \tag{32}$$

where, as before, (31) is used for $n \ge 1$, and (32) for $n = 0$. The second equality in each of (31) and (32) follows from the statistical independence of the two vectors due to a chip-synchronous interferer.

Now, $\boldsymbol{T}_L^{N-n}\boldsymbol{a}$ and $\boldsymbol{T}_R^{n}\boldsymbol{a}$ span the same subspace as

$$\tilde{\boldsymbol{a}} = \boldsymbol{T}_L^{N-n}\boldsymbol{a} + \boldsymbol{T}_R^{n}\boldsymbol{a}$$

and

$$\hat{\boldsymbol{a}} = \boldsymbol{T}_L^{N-n}\boldsymbol{a} - \boldsymbol{T}_R^{n}\boldsymbol{a}$$

so that

$$\begin{aligned} P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \notin \boldsymbol{S}, \boldsymbol{T}_R^{n}\boldsymbol{a} \notin \boldsymbol{S}] &= P[\tilde{\boldsymbol{a}} \notin \boldsymbol{S}, \hat{\boldsymbol{a}} \notin \boldsymbol{S}] \\ &= 1 - P[\tilde{\boldsymbol{a}} \in \boldsymbol{S} \text{ or } \hat{\boldsymbol{a}} \in \boldsymbol{S}] \\ &\ge 1 - P[\tilde{\boldsymbol{a}} \in \boldsymbol{S}] - P[\hat{\boldsymbol{a}} \in \boldsymbol{S}] \end{aligned} \tag{33}$$

where a union bound has been used for the last inequality. Note, however, that sum vector $\tilde{\boldsymbol{a}}$ and difference vector $\hat{\boldsymbol{a}}$ are each random spreading sequences in $\{-1,1\}^N$ independent of the subspace $\boldsymbol{S}$ (though not independent of each other). Proposition 1 for synchronous CDMA can now be applied to obtain

$$P[\tilde{\boldsymbol{a}} \notin \boldsymbol{S}] \le \frac{2^d}{2^N} \qquad P[\hat{\boldsymbol{a}} \notin \boldsymbol{S}] \le \frac{2^d}{2^N}.$$

Plugging into (33), we obtain that, for $n \geq 1$

$$P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \notin \boldsymbol{S}, \boldsymbol{T}_R^n\boldsymbol{a} \notin \boldsymbol{S}] \geq 1 - \frac{2^{d+1}}{2^N}$$

which is independent of n. Substituting in (29) and (30) and averaging the left-hand side over n gives the desired lower bound on the probability that the dimension of the interference subspace increases by two

$$P[\boldsymbol{v}^{-1} \notin \boldsymbol{S}, \boldsymbol{v}^0 \notin \boldsymbol{S}] \geq 1 - \frac{2^{d+1}}{2^N} = \underline{q}(2|d). \tag{34}$$

It remains to obtain an upper bound on the probability that the dimension increase is zero, starting with (31) and (32). Given n, $\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}(-1,n)$ and $\boldsymbol{T}_R^n\boldsymbol{a} \in \boldsymbol{S}(0,n)$, where

$$\begin{aligned}\boldsymbol{S}(-1,n) = \{\boldsymbol{z} \in \mathcal{R}^N\colon\ & z[l] \in \{-1,+1\}, 0 \leq l \leq n-1 \text{ and} \\ & z[l] = 0, n \leq l \leq N-1\} \\ \boldsymbol{S}(0,n) = \{\boldsymbol{z} \in \mathcal{R}^N\colon\ & z[l] = 0, 0 \leq l \leq n-1, \text{ and} \\ & z[l] \in \{-1,+1\}, n \leq l \leq N-1\}\end{aligned}$$

are orthogonal subsets of $\mathcal{R}^N$. Let $M(-1,n)$ and $M(0,n)$ be the number of vectors in $\boldsymbol{S}(-1,n)$ and $\boldsymbol{S}(0,n)$, respectively, which lie in $\boldsymbol{S}$. We then have

$$\begin{aligned}P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}] &= \frac{M(-1,n)}{2^n} \\ P[\boldsymbol{T}_R^n\boldsymbol{a} \in \boldsymbol{S}] &= \frac{M(0,n)}{2^{N-n}}.\end{aligned} \tag{35}$$

Upper bounds on $M(-1,n)$ and $M(0,n)$ therefore yield upper bounds on the preceding probabilities.

As in the proof of Proposition 1, consider the $N \times d$ matrix $\boldsymbol{X}$ with d basis vectors for $\boldsymbol{S}$ as columns. Let $j_1 < j_2 < \cdots < j_d$ denote d independent rows. Assume that d_1 of these row basis vectors occur among the first n rows, and d_2 among the last $N-n$ rows, so that

$$0 \leq d_1 \leq n, \quad 0 \leq d_2 \leq N-n, \qquad d_1 + d_2 = d.$$

Reasoning as in the proof of Proposition 1, we have

$$\begin{aligned}M(-1,n) \leq \text{ card}\Bigg\{(\lambda_1,\cdots,\lambda_d)\colon\ & \sum_{i=1}^d \lambda_i x_i[j_l] \in \{-1,+1\}, \\ & 1 \leq l \leq d_1 \text{ and } \sum_{i=1}^d \lambda_i x_i[j_l] = 0, \\ & d_1+1 \leq l \leq d_1+d_2 = d\Bigg\} \leq 2^{d_1}\end{aligned} \tag{36}$$

and

$$\begin{aligned}M(0,n) \leq \text{ card}\Bigg\{(\lambda_1,\cdots,\lambda_d)\colon\ & \sum_{i=1}^d \lambda_i x_i[j_l] = 0, \\ & 1 \leq l \leq d_1 \text{ and } \sum_{i=1}^d \lambda_i x_i[j_l] \in \{-1,+1\}, \\ & d_1+1 \leq l \leq d_1+d_2 = d\Bigg\} \leq 2^{d_2}.\end{aligned} \tag{37}$$

Using (31), (32), and (35), we have

$$P[\boldsymbol{T}_L^{N-n}\boldsymbol{a} \in \boldsymbol{S}, \boldsymbol{T}_R^n\boldsymbol{a} \in \boldsymbol{S}] \leq \frac{2^{d_1+d_2}}{2^N} = \frac{2^d}{2^N}. \tag{38}$$

Noting that the right-hand side of (38) is independent of n, we substitute into (31) and (32) and average over n to obtain the desired upper bound

$$P[\boldsymbol{v}^{-1} \in \boldsymbol{S}, \boldsymbol{v}^0 \in \boldsymbol{S}] \leq \frac{2^d}{2^N} = \overline{q}(0|d).$$

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