# ON WAITING FOR SIMULTANEOUS ACCESS TO TWO RESOURCES

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*ABSTRACT*

This paper studies the $M/G/1$ queue where a special (test) customer can get service only if he has simultaneous access to the server and a second resource. All other customers only need access to the server. The second resource becomes available after an exponentially distributed amount of time. The ordinary customers are served according to the FIFO discipline. The test customer has the freedom to leave his place in the queue at any time and join the end of the queue. If he reaches the server before the second resource becomes available, he then must return to the back of the queue.

We derive the waiting time distribution of the test customer given that he always maintains his position in the queue until he reaches the server. A number of conditions are given under which this "move-along" policy is optimal, i.e., minimizes the test customer's mean delay until service. These conditions depend on the amount of information and freedom of action available to the test customer.

## *1. PROBLEM DESCRIPTION, AND OUTLINE*

The problem studied in this paper is derived from the scheduling problems which occur in a service system with multiple resources, where a customer can get service only when all the resources it needs are simultaneously available. The prototype multi-resource service system can be thought of as a multi-processor computer system or a database system with locking mechanisms for integrity protection. The simplified problem considered here is described in [Gopinath, 1984], and has become known as the "Waiting for Godot" problem. This simplification still captures some of the effects of waiting for simultaneous availability of multiple resources, and, as is seen in this paper, is reasonably tractable.

In the model studied there is one special customer, called the test customer, who is waiting for simultaneous availability of two resources: the server and an "extraneous" resource. All other customers only need the server. The other customers arrive according to a Poisson process with intensity $\lambda$ and have service times (with the server) which are i.i.d. random variables with distribution function $F(\cdot)$, Laplace Stieltjes Transform (LST) $\phi(\cdot)$ and expected value $m$. For these other customers the service discipline is First In, First Out (FIFO). The time until the extraneous resource becomes available is an independent, exponentially distributed random variable with parameter $\alpha$ (expected value $\alpha^{-1}$). At the time the extraneous resource becomes available we say that the event $E$ occurs, and once $E$ has occurred the extraneous resource remains constantly available.

At any time the test customer has two options. He can either maintain his place in the queue, or he can voluntarily leave his place in the queue and move to the end of the queue. Whenever he reaches the server after event $E$ has occurred, service starts. If the test customer reaches the server before event $E$ has occurred, however, he must go back to the end of the queue. An interesting question, which will be partially answered, is whether and why it is ever profitable for the test customer to use the option of moving to the end of the queue without being forced to do so.

For a further study of waiting for simultaneous access to multiple resources it will be necessary to consider situations where there is competition for the external resources, or where many other customers are also waiting for their own (possibly different and independent) resources, or both.

As stated before, the test customer has the freedom to give up his place in the queue and go back to the end of the line even when he is not facing the server. The *move-along policy* is the policy where the test customer never uses that option. With that policy he maintains his place in the queue until he reaches the server. At that point either service starts (if in the meantime event $E$ has occurred), or he goes back to the end of the line.

Our first results, stated in Section 2, are expressions for the distribution (in fact its LST) and the expected value of the time until service starts for the test customer, given that he uses the move-along policy, and given that at time $t = 0$ there is a random variable $X$ representing the total amount of work in front of him in the queue, and a random variable $Y$ representing the total amount of work behind him in the queue. These expressions of course are in terms of the joint distribution of $X$ and $Y$. Other results, also stated in Section 2, describe under what circumstances the move-along policy is better than competing policies. The competing policies depend on the information and degree of freedom available to the test customer. The test customer always knows $\lambda$ and $F(\cdot)$. A policy is said to be optimal if it minimizes the expected value of the test customer's delay until service, starting from an arbitrary state. The situations considered are:

*1.1 Complete Information and Freedom*

In these policies the test customer always knows the (remaining) service times for all customers in the system. He therefore exactly knows the total amount of work in front of him in the queue ($t_f$) and the total amount of work behind him in the queue ($t_b$). At any time he can decide (based on $t_f$ and $t_b$) to give up his place and join the end of the line. It will be proved that as long as

$$\lambda \leq \frac{1}{m} = \frac{1}{abs \phi'(0) abs} \tag{1.1}$$

the move along policy is the best among all such "complete information and freedom" policies.

*1.2 Partial Information and Complete Freedom*

In these policies the test customer knows, at any time, the (remaining) service times of the customers in front of him, but only the number of customers behind him. As a result he knows the total amount $t_f$ of work in front of him and the total number $j$ of customers behind him. At any point in time he can decide (based on $t_f$ and $j$) to give up his place and join the end of the queue. It will be proved that if

$$\lambda \leq \frac{\alpha}{1-\phi(\alpha)} \tag{1.2}$$

then the move-along policy is the best among all such "partial information and complete freedom" policies.

*1.3 Minimal Information and Complete Freedom*

In these policies the test customer only knows the numbers $i$ and $j$ of customers in front of him, respectively behind him, and for the customer currently being served he also knows the elapsed service time $\tau$. At any point in time he can decide (based on $i$, $j$, and $\tau$) to give up his place and join the end of the queue. It is clear that if (1.2) holds then the move along policy is the best of all "minimal information and complete freedom " policies. It is possible (because of the smaller amount of information available) to replace (1.2) with a weaker (larger) upper bound for $\lambda$ (see (2.26)).

*1.4 Minimal Information and Limited Freedom*

In these policies the test customer only knows the numbers $i$ and $j$ of customers in front of him and behind him, and is allowed to leave his place and go to the end of the queue only at service completion epochs. This situation is called the *discrete-epoch* situation. It will be proved that if

$$\lambda \leq \frac{1}{abs\phi'(\alpha)abs} \tag{1.3}$$

then the move along policy is the best among all discrete-epoch policies.

It is shown in [Li, 1987] that the move-along policy is the best among all discrete-epoch policies if $\lambda \leq 1/[\bar{m}\phi(\alpha)]$ where $\bar{m}$ is the expected service time of a customer given that event $E$ did not occur during (or before) his service. Since

$$\bar{m} = \frac{abs\phi'(\alpha)abs}{\phi(\alpha)} , \tag{1.4}$$

the two results are equivalent, although the proof given by Li is different from that given in Section 3.

In [Honig, 1987] it is shown for deterministic service that there exists a threshhold $\lambda_0$, which depends on $\alpha$ and $m$, such that for $\lambda > \lambda_0$, the move along policy is not optimal. In Section 3 it will be proved that in the case of a general service time distribution there exists a threshhold $\lambda_0^*$ such that for $\lambda > \lambda_0^*$ the move along policy is not the optimal discrete-epoch policy. This result is easily explained by the following simple intuitive argument. Assume that $\lambda$ is quite large (e.g., 200), $m = 1$, and that $\alpha = 0.1$, so that the expected time until $E$ occurs is on the order of 10 service times. Suppose that initially there is one person ahead of the test customer. While the test customer is waiting for the customer ahead of him to finish service, new arrivals are rapidly joining the queue behind him. Consequently, if the test customer chooses to maintain his position until he reaches the server, he will most likely have to wait in back of all of the (approximately 200) new arrivals. Alternatively, if the test customer decides to join the back of the queue after, say, the first 100 new arrivals, which most likely occurs during the service epoch, he will almost certainly reduce his delay.

The previous results suggest the following conjecture:

**Conjecture:** Given $\alpha$ and $F(\cdot)$ there exist critical levels $\lambda_k^*$, $1 \le k \le 4$, $k = 1$ for "complete information, complete freedom", $k = 2$ for "partial information, complete freedom", $k = 3$ for "minimal information and complete freedom", and $k = 4$ for "minimal information and limited freedom", with

$$\frac{1}{m} < \lambda_1^* \le \lambda_2^* \le \lambda_3^* \le \lambda_4^* \,, \tag{1.5}$$

such that in situation $k$ ( $k$=1,2,3,4) the move-along policy is the best of all situation $k$ policies if and only if

$$\lambda \le \lambda_k^* \,. \tag{1.6}$$

From (1.3) it is apparent that $\lambda_4^* \ge \dfrac{1}{abs\phi'(\alpha)abs}$.

In Section 2 we give, without proofs, the main results of this paper. The proofs of these results are given in Section 3. Section 4 discusses a related problem, where the test customer can decide to wait outside the queue before joining the end of the line. Finally, some specific service distributions are considered in Section 5, and Section 6 discusses some other related problems.

## *2. NOTATION AND THE MAIN RESULTS*

At time zero the test customer has a total amount of work $X$ in front of him and a total amount of work $Y$ behind him in the queue. The joint distribution of $X$ and $Y$ is given by

$$P(x, y) = Pr\{X \le x, Y \le y\} \tag{2.1}$$

and the marginal distributions of $X$ and $Y$ are denoted as

$$G(x) = Pr\{X \le x\}\ ,$$
$$H(y) = Pr\{Y \le y\}\ . \tag{2.2b}$$

Given any distribution $R(\cdot)$, the LST of $R$ is denoted by $\psi_R(\cdot)$. In particular,

$$\psi_P(s_1, s_2) = \int_{0^-}^{\infty}\int_{0^-}^{\infty} \exp(-s_1 x - s_2 y)\ P(dx, dy)\ ,$$
$$\psi_G(s) = \int_{0^-}^{\infty} e^{-sx}\ dG(x), \qquad \psi_H(s) = \int_{0^-}^{\infty} e^{-sy}\ dH(y)\ . \tag{2.3b}$$

The distribution of $T$, the time until the test customer starts service, of course depends on $P(\cdot,\cdot)$ and on the policy used. $\eta_P(\cdot)$ denotes the LST of $T$ given that the move along policy is used:

$$\eta_P(s) = E\left[e^{-sT}\ vbar\ P(x, y), \text{move along}\right]. \tag{2.4}$$

If $X$ and $Y$ are independent we denote this as $\eta_{G,H}(s)$. By a (hopefully not confusing) abuse of notation we define

$$\eta_{t_f,t_b}(s) = E\left[e^{-sT}\ vbar\ X = t_f, Y = t_b, \text{move along}\right], \tag{2.5}$$

and

$$\eta_{i,j}(s) = \eta_{F^{*i},F^{*j}}(s) \tag{2.6}$$

(where $F^{*i}(\cdot)$ denotes the $i$-fold convolution of $F(\cdot)$). Finally, we define:

$$\eta_i(s) = \eta_{i,0}(s) \tag{2.7}$$

(where $\eta_{i,0}(s)$ is defined as in (2.6)), and we write $\eta_G(s)$ for $\eta_{G,H}(s)$ when $Pr\{Y = 0\} = 1$ (no customers are behind the test customer).

*2.1 Waiting Time Distribution and Move-Along Mean Delay*

Theorems 1 and 2, which follow, give explicit expressions for $\eta_i(s)$ and $\eta_P(s)$ in terms of $\lambda$, $m$, $\alpha$, $\phi(s)$, and $\psi_P(s_1, s_2)$, and will be proved in Section 3. The basic idea of the proofs is as follows: If there is a deterministic amount $t_f$ of work in front of the test customer, and a random amount $Y$ of work behind him, then the test customer first waits for an amount of time $t_f$. If by that time event $E$ has occurred, $T = t_f$. If not, at time $t_f$ the test customer is at the end of the queue with an amount of work in front of him equal to $Y$ plus the service times of all customers who arrived in the time interval $[0, t_f]$. Averaging over the distribution of $t_f$ expresses $\eta_{G,H}(s)$ in terms of the sequence $\{\eta_{HstarF^{stark}}(s)\}_{k=0}^{\infty}$.

Choosing $Y = 0$ and $G = F^{stari}$ expresses $\eta_i(s)$ in terms of $\{\eta_k(s)\}_{k=0}^{\infty}$, and makes it possible to compute $\eta_i(s)$ and thus prove Theorem 1. Theorem 2 is then proved by repeated use of the same idea. Section 3 not only contains the proofs of Theorems 1 and 2, but also a number of intermediate results such as expressions for $\eta_{G,H}(s)$ and $\eta_{t_f,\,t_b}(s)$. Some readers may prefer to read Section 3 before reading the remainder of this section.

The results in this section are expressed in terms of the sequences $\{x_k(s)\}$, $\{xTilde_k(s)\}$, $\{y_k(s)\}$, and $\{yTilde_k(s)\}$, which are defined as:

$$x_0(s) = s, \qquad xTilde_0(s) = 0, \tag{2.8a}$$

$$x_{k+1}(s) = s + \alpha + \lambda\{1 - \phi[x_k(s)]\}, \qquad xTilde_{k+1}(s) = s + \alpha + \lambda\{1 - \phi[xTilde_k(s)]\}, \tag{2.8b}$$

$$y_k(s) = \phi[x_k(s)], \qquad yTilde_k(s) = \phi[xTilde_k(s)]\ . \tag{2.8c}$$

For $s \ge 0$ it is easily shown that

$$0 = xTilde_0(s) \le x_0(s) < xTilde_1(s) \le x_1(s) < \cdots < xTilde_\infty(s) = x_\infty(s) < \infty, \tag{2.9a}$$

and

$$1 = yTilde_0(s) \ge y_0(s) > yTilde_1(s) \ge y_1(s) > \cdots > yTilde_\infty(s) = y_\infty(s) > 0, \tag{2.9b}$$

where the ‘less than or equal to’ signs are equalities if and only if $s = 0$. The sequences $x_k(s)$ and $y_k(s)$ are shown graphically in Figure 1. $x_\infty(s)$ and $y_\infty(s)$ satisfy

$$\begin{aligned} x_\infty(s) &= s + \alpha + \lambda\{1 - \phi[x_\infty(s)]\}, \\ y_\infty(s) &= \phi\{s + \alpha + \lambda[1 - y_\infty(s)]\} = \beta(s + \alpha), \end{aligned} \tag{2.10b}$$

where $\beta(s)$ is the LST of the lengths of the busy periods in the $M/G/1$ queue with $\lambda$, $F(\cdot)$, $\phi(\cdot)$. In section 3 a number of results related to (2.9) and (2.10) will be derived which show that the infinite series in Theorems 1, 2, and 3 below converge uniformly for $\mathrm{Re}(s) \ge 0$.

**Theorem 1:**

$$\begin{aligned} \eta_i(s) &= \eta_0(s)y_\infty^i(s) + \sum_{k=0}^{\infty}[y_k^i(s) - yTilde_{k+1}^i(s)] \\ &= \phi^i(s) - [1 - \eta_0(s)]y_\infty^i(s) - \sum_{k=1}^{\infty}[yTilde_k^i(s) - y_k^i(s)], \end{aligned} \tag{2.11}$$

where

$$\eta_0(s) = 1 - \frac{\sum_{k=0}^{\infty}[x_k(s) - xTilde_k(s)]}{x_\infty(s)} \ . \tag{2.12}$$

**Theorem 2:**

$$\eta_P(s) = \eta_0(s)\psi_P[x_\infty(s), x_\infty(s)] + \sum_{k=0}^{\infty} (\psi_P[x_k(s), x_{k-1}(s)] - \psi_P[xTilde_{k+1}(s), xTilde_k(s)])$$

$$= \psi_P(s, 0) - [1 - \eta_0(s)]\psi_P[x_\infty(s), x_\infty(s)] - \sum_{k=0}^{\infty} (\psi_P[xTilde_{k+1}(s), xTilde_k(s)] - \psi_P[x_{k+1}(s), x_k(s)]), \tag{2.13}$$

where $x_{-1}(s) \equiv 0$.

The expected value of the time until the test customer starts service, assuming the move-along policy is adopted and that the joint distribution of work in front of and in back of the observer is given by (2.1), is denoted as

$$\bar{T}_P = -\frac{d}{ds}\,\eta_P(s)vbar_{s=0} \ . \tag{2.14}$$

In analogy with the notation introduced before, $\bar{T}_{G,H}$ denotes the expected delay when $P(x, y) = G(x)H(y)$, $\bar{T}_{t_f,t_b}$ denotes the expected delay given that $X = t_f$ and $Y = t_b$, $\bar{T}_{ij}$ denotes the expected delay when $G(t) = F^{*i}(t)$ and $H(t) = F^{*j}(t)$, and $\bar{T}_i$ denotes the expected delay when $G(t) = F^{*i}(t)$ and $Pr\{Y = 0\} = 1$. Taking the derivative of the expression in (2.13) gives the next Theorem.

**Theorem 3:**

$$\bar{T}_P = \bar{T}_0\psi_P(x_\infty, x_\infty) + \bar{X}$$

$$+ \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \left[ -\frac{\partial\psi_P(s_1, s_2)}{\partial s_2}\, vbar_{s_1=x_{k+1}, s_2=x_k} + \lambda\phi'(x_k)\frac{\partial\psi_P(s_1, s_2)}{\partial s_1}\, vbar_{s_1=} \right. \tag{2.15}$$

where

$$\bar{T}_0 = \frac{1}{x_\infty}\left[1 + \sum_{k=0}^{\infty} \lambda^{k+1} \prod_{m=0}^{k} abs\phi'(x_m)abs\right] \tag{2.16}$$

is the mean delay delay given that $X = Y = 0$. $\bar{X}$ is the expected value of $X$, and $x_k \equiv x_k(0)$, where $x_k(s)$ is defined in (2.8).

If $G(t) = F^{*i}(t)$ and $H(t) = F^{*j}(t)$, i.e., there are $i$ customers ahead of the test customer, none of whom have received any service yet, and $j$ customers in back of the test customer, (2.15) becomes

$$\overline{T}_{ij} = \overline{T}_0 y_\infty^{i+j} + im + \sum_{k=0}^{\infty} \left( \lambda^k \prod_{m=0}^{k} abs\phi'(x_m) abs \left( j y_k^{j-1} y_{k+1}^i + i\lambda abs\phi'(x_{m+1}) abs y_k^j y_{k+1}^{i-1} \right) \right) \tag{2.17}$$

where $y_k \equiv y_k(0)$, and $y_k(s)$ is defined in (2.8).

*2.2 Conditions for Move-Along Optimality*

A *policy* is a sequence of *actions* which the test customer may take, and in general each action depends on the entire history of states visited. The only allowable action the test customer may take is to give up his current position in the queue, and move to the back of the queue. A policy is said to be *optimal* if it minimizes (over the entire class of allowable policies) the test customer's expected delay until the start of service, given some arbitrary initial state.

Let $D(X, Y; \Pi)$ denote the expected time until the start of service for the test customer given that initially the amounts of work in front of him and behind him are $X$, respectively $Y$, and given that consistently policy $\Pi$ is used. Let $\Pi_{MA}$ denote the move along policy. The maximum principle from dynamic programming suggests that $\Pi_{MA}$ is optimal if and only if

$$D(X, Y; \Pi_{MA}) \le D(X + Y, 0; \Pi_{MA}) \tag{2.18}$$

for all (nonnegative) random variables $X$ and $Y$. This is a consequence of well known results in Markov Decision theory. Intuitively, it can be seen as follows: suppose there is a time $t = L$ such that for $t > L$ the move-along policy will be used. The problem is to find the optimal policy for $t \le L$. But this now is a finite horizon dynamic programming problem, and (2.18) implies that the move-along policy is always optimal. By choosing $L$ sufficiently large, the probability that $T > L$, where $T$ is the time until the test customer starts service, can be made arbitrarily small. This implies that the "end effect" of what happens after time $L$ becomes irrelevant, and that the move-along policy is optimal. This argument can be made rigorous by observing that in the worst case ($\rho > 1$), the expected amount of work in the system at time $t$ grows linearly with $t$, while the probability that the extraneous resource is not yet available at time $t$ is $e^{-\alpha t}$, and $\lim_{L \to \infty} \int_L^\infty cte^{-\alpha t} dt = 0$ for any constant $c$.

If $X = t_f$ and $Y = t_b$, then (2.18) becomes

$$\overline{T}_{t_f,t_b} \le \overline{T}_{t_f+t_b,0} \ . \tag{2.19}$$

For the case $G(t) = F^{*i}(t)$ and $H(t) = F^{*j}(t)$, the condition (2.18) becomes

$$\overline{T}_{ij} \le \overline{T}_{i+j} \tag{2.20}$$

for all positive integers $i$ and $j$. It is therefore of interest to study $\overline{T}_{t_f+t_b,0} - \overline{T}_{t_f,t_b}$ as a function of $\lambda$, $\alpha$, and $m$. The next corollary is obtained from Theorem 3.

**Corollary 1:**

$$\overline{T}_{t_f+t_b,0} - \overline{T}_{t_f,t_b} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \tag{2.21}$$
$$\times \left( t_b e^{-x_k t_b} (e^{-x_k t_f} - e^{-x_{k+1} t_f}) - \lambda abs\phi'(x_k) abs t_f e^{-x_{k+1} t_f} (e^{-x_k t_b} - e^{-x_{k+1} t_b}) \right)$$

Averaging over $t_f$ and $t_b$ for the case where $X$ and $Y$ are independent gives

$$\overline{T}_{G*H} - \overline{T}_{G,H} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \tag{2.22}$$
$$\times \left( abs\psi_H{}'(x_k)abs[\psi_G(x_k) - \psi_G(x_{k+1})] - \lambda\phi'(x_k)\psi_G{}'(x_{k+1})[\psi_H(x_k) - \psi_H(x_{k+1})] \right).$$

If $G(t) = F^{*i}(t)$ and $H(t) = F^{*j}(t)$, (2.22) becomes

$$\overline{T}_{i+j} - \overline{T}_{ij} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k} abs\phi'(x_m)abs \right) \left[ j y_k^{j-1} (y_k^i - y_{k+1}^i) - \lambda i abs\phi'(x_{k+1}) abs y_{k+1}^{i-1} (y_k^j - y_{k+1}^j) \right]. \tag{2.23}$$

The expressions (2.21)-(2.23) are used to prove the following four Theorems, which imply the results stated in Sections 1.1 - 1.4. The condition on $\lambda$ in each case ensures that *every* term in the corresponding sum in (2.21)-(2.23) is positive. The following theorems therefore give sufficient, but not necessary, conditions for the move-along policy to be optimal.

**Theorem 4:** If $\lambda m \le 1$, the condition $\overline{T}_{t_f,t_b} \le \overline{T}_{t_f+t_b,0}$ holds for all $t_f$, $t_b \ge 0$.

Theorem 4 applies to the "complete information and freedom" situation and establishes the statement made in subsection 1.1.

**Theorem 5:** If $\lambda \le \dfrac{\alpha}{1 - \phi(\alpha)}$, the condition $\overline{T}_{G,H} \le \overline{T}_{G*H}$ holds for all $G$, $H$ such that $X = t_f$ and

$H(t) = F^{*j}(t)$, for any $t_f > 0,\ j \geq 0$.

Theorem 5 establishes the statement made in subsection (1.2). To prove the statement in subsection (1.3) some more notation is needed. Let

$$F_\tau(t) = Pr(W \leq t + \tau \ abs\ W \geq \tau) = \frac{F(t+\tau^+) - F(\tau^-)}{1 - F(\tau^-)} \ , \tag{2.24}$$

(where superscript "+" denotes limit from the right and superscript "-" denotes limit from the left), i.e., $F_\tau(t)$ is the probability distribution of the remaining service time given that the customer has been in service $\tau^-$ time units, and let

$$\phi_\tau(s) = \int_{0^-}^{\infty} e^{-st} dF_\tau(t) \ . \tag{2.25}$$

Suppose that initially there are $i$ customers ahead of the test customer, and $j$ customers behind the test customer, and that the elapsed time since the customer at the front of the queue started service is $\tau$. For this case $\psi_G(s) = \phi^{i-1}(s)\phi_\tau(s)$ and $\psi_H(s) = \phi^j(s)$. The next theorem gives a weaker condition on $\lambda$ than that given in Theorem 5.

**Theorem 6.** If there are initially $i$ customers in front of the test customer, and $j$ customers in back of the test customer, then the condition $\overline{T}_{G,H} \leq \overline{T}_{G*H}$, where $\psi_H(s) = \phi^j(s)$, $\psi_G(s) = \phi^{i-1}(s)\phi_\tau(s)$, and $\phi_\tau(s)$ is defined in (2.25), holds for all $i$, $j$, and $\tau$ if

$$\lambda \leq \frac{1}{1 - \phi(\alpha)} \inf_{i>0,\ \tau\geq 0} \frac{1 - \psi_G(\alpha)}{abs \psi_G{}'(\alpha) abs} \ . \tag{2.26}$$

This condition on $\lambda$ is weaker than the condition stated in Theorem 5 since $\dfrac{1 - \psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} \geq \alpha$ for any distribution $G(t)$ over $[0, \infty)$. If $\tau$, the elapsed time since the customer at the front of the queue started service, is taken to be zero, then $\psi_G(s) = \phi^i(s)$, and the upper bound on $\lambda$ in (2.26) can be evaluated to give:

**Corollary 2:** If $\lambda \leq \dfrac{1}{abs\phi'(\alpha)abs}$, the condition $\overline{T}_{ij} < \overline{T}_{i+j}$ holds for all positive $i$ and $j$.

The move-along policy is therefore optimal for the discrete-epoch problem if $\lambda \leq 1/abs\phi'(\alpha)abs$.

It will be shown in the next section that

$$\frac{1}{m} < \frac{\alpha}{1-\phi(\alpha)} \le \frac{1}{1-\phi(\alpha)} \inf_{i>0,\tau\ge 0} \frac{1-\psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} \le \frac{1}{abs\phi'(\alpha)abs} \ , \tag{2.27}$$

so that Theorems 4, 5, 6, and Corollary 2 give progressively weaker conditions on $\lambda$ corresponding to less information or freedom available to the test customer.

We also have the following conjecture about the expression in (2.26):

**Conjecture:**

$$\inf_{i\ge 1,\ \tau\ge 0} \frac{1-\psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} = \inf_{\tau\ge 0} \frac{1-\phi_\tau(\alpha)}{abs\phi_\tau{}'(\alpha)abs} \ . \tag{2.28}$$

Namely, we believe that for $\tau \ge 0$ fixed the expression $[1-\psi_G(\alpha)]/abs\psi_G{}'(\alpha)abs$ is increasing in $i$. We have not succeeded in proving this. Neither have we succeeded in proving the even stronger statement (which probably is not always true) that $[1-\psi_G(\alpha)]/abs\psi_G{}'(\alpha)abs$ is increasing if $G(\cdot)$ is stochastically increasing.

To show that the move-along policy is not optimal for a given $\lambda$, $m$, and $\alpha$, it suffices to find a particular $i$ and $j$ such that $\bar{T}_{ij} > \bar{T}_{i+j}$. As an example, if $i = j = 1$, then (2.23) becomes

$$\bar{T}_2 - \bar{T}_{1,1} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k} abs\phi'(x_m)abs \right) [1 - \lambda abs\phi'(x_{k+1})abs](y_k - y_{k+1}) \ . \tag{2.29}$$

If $\lambda > \dfrac{1}{abs\phi'(\alpha)abs}$, then the first term in the sum in (2.29) will be negative. However, it is *not* true that all of the remaining terms become negative for large enough $\lambda$. In particular,

$$abs\phi'(x_2)abs = \int_{0^-}^{\infty} te^{-[\lambda(1-y_1)+\alpha]t} dF(t) < \frac{1}{[\lambda(1-y_1)+\alpha]^2} \ , \tag{2.30}$$

so that $\lambda abs\phi'(x_2)abs < 1$ for large enough $\lambda$, and $\lambda abs\phi'(x_2)abs > \lambda abs\phi'(x_3)abs > \cdots > \lambda abs\phi'(x_\infty)abs > 0$ (see Lemma 2 in Section 3). It therefore is conceivable that the sum (2.29) is positive for *all* $\lambda$. Nevertheless, the following theorem states that in fact for any $\alpha$, the move-along policy is not optimal if $\lambda$ is large enough.

**Theorem 7:** Given any $\alpha$ and $j$ there exist two numbers $i_0$ and $\lambda_0$ such that $\bar{T}_{i+j} < \bar{T}_{ij}$ for $\lambda > \lambda_0$ and $i > i_0$.

The previous results suggest that for any of the situations considered, there exists a threshhold,

$\lambda_0$, such that the move-along policy is optimal *if and only if* $\lambda \le \lambda_0$. To prove this result one must show that if for some $\lambda = \lambda'$, $\overline{T}_{t_f,t_b} \le \overline{T}_{t_f+t_b,0}$ for all positive $t_f$ and $t_b$, then it must also be true for all $\lambda < \lambda'$. (Alternatively, one could show that if for specific $t_f$ and $t_b$, $\overline{T}_{t_f,t_b} \ge \overline{T}_{t_f+t_b,0}$ for $\lambda = \lambda'$, then $\overline{T}_{t_f,t_b} \ge \overline{T}_{t_f+t_b,0}$ for any $\lambda > \lambda'$.) This appears to be difficult, and it is as yet undetermined whether or not this is true.

*3. PROOFS*

The sequences $x_k(s)$, $xTilde_k(s)$, $y_k(s)$, $yTilde_k(s)$ are based on the map $f_s$ defined by

$$f_s(z) = s + \alpha + \lambda[1 - \phi(z)]. \tag{3.1}$$

In particular,

$$x_k(s) = f_s^{(k)}(0), \qquad xTilde_k(s) = f_s^{(k)}(s), \tag{3.2}$$

where $f_s^{(k)}$ is the $k$ times iterated map.

If $\mathrm{Re}(s) \ge 0$, then $f_s$ maps the half plane $\mathrm{Re}(z) \ge 0$ into the half plane $\mathrm{Re}(z) \ge \alpha + \mathrm{Re}(s)$. If $\rho = \lambda m < 1$ then, for $\mathrm{Re}(s) \ge 0$, $f_s$ is a contraction map on $\mathrm{Re}(z) \ge 0$:

$$abs f_s(z_1) - f_s(z_2) abs = abs \lambda[\phi(z_2) - \phi(z_1)] abs \le \lambda m abs z_2 - z_1 abs, \tag{3.3}$$

where the fact that $abs\phi'(z)abs \le abs\phi'[\mathrm{Re}(z)]abs$, which is decreasing in $\mathrm{Re}(z)$, has been used.

Suppose now that $\rho = \lambda m \ge 1$. Since for real $s \ge 0$, $\phi(s)$ is decreasing in $s$, $x_k(s)$ and $xTilde_k(s)$ in (2.8) are increasing in $s$, and $y_k(s)$, $yTilde_k(s)$ are decreasing in $s$. Hence, for $\mathrm{Re}(s) \ge 0$, $f_s$ maps the half plane $\mathrm{Re}(z) \ge x_k(0)$ into the half plane $\mathrm{Re}(z) \ge x_{k+1}(0)$. Since (see Figure 1) $\lambda abs\phi'[x_\infty(0)]abs < 1$, there exist a $\rho *$, $0 < \rho * < 1$, and a $k_0$, with the property that if $k \ge k_0$ then

$$abs f_s^{(k+1)}(z_1) - f_s^{(k+1)}(z_2) abs < \rho * \, abs f_s^{(k)}(z_1) - f_s^{(k)}(z_2) abs \tag{3.4}$$

for all $s$, $z_1$, $z_2$ with $\mathrm{Re}(s) \ge 0$, $\mathrm{Re}(z_i) \ge 0$. For $\rho *$ we could choose

$$\rho * = \frac{1}{2}(1 + \lambda abs\phi'[x_\infty(0)]abs). \tag{3.5}$$

and for $k_0$ we choose

$$k_0 = \min\{k \ge 0 : \lambda abs\phi'[x_k(0)]abs < \rho * \}. \tag{3.6}$$

(3.6) uses the fact that for $s \ge 0$, $x_k(s)$ is increasing in both $k$ and $s$ (see Figure 1).

As a straightforward application of (3.2)-(3.4) we can obtain relations such as

$$abs x_{k_0+k}(s) - x_\infty(s) abs = abs f_s^{(k_0+k)}(0) - f_s^{(k_0+k)}[x_\infty(s)] abs \tag{3.7}$$

$$\le (\rho^*)^k abs f_s^{(k_0)}(0) - f_s^{(k_0)}[x_\infty(s)] abs$$

$$\le (\rho^*)^k \rho^{k_0} abs x_\infty(s) abs,$$

where $\rho$, $\rho$ *, $k_0$, do not depend on $s$. The series in Theorems 1, 2, and 3 therefore converge uniformly on compact subsets of $\{s : \mathrm{Re}(s) \ge 0\}$. In the remainder of this section we will mostly disregard convergence issues.

The following lemma is needed to prove Theorems 1 and 2.

**Lemma 1.**

$$\eta_{G,H}(s) = \int_0^\infty e^{-st} (1 - e^{-\alpha t}) \, dG(t) + \sum_{k=0}^\infty \int_0^\infty e^{-st} e^{-(\alpha+\lambda)t} \frac{(\lambda t)^k}{k!} \eta_{HstarF^{stark}}(s) \, dG(t) \, . \tag{3.8}$$

**Proof:**. Suppose that initially the amount of work in front of the test customer is exactly $t$. If by the time this work has been done by the server event $E$ has occurred, then $T = t$. Otherwise, the test customer first waits time $t$, and then becomes the last customer in a queue with an amount of work in front of him equal to $Y$, the initial amount of work behind him, plus the work required by customers who arrived in the meantime. (3.8) is a formal statement of this observation, and allows the amount of work in front of the test customer to be random, as long as it is independent of the amount of work behind the test customer.□

**Remark.** By defining

$$p_k^{(G)}(s) = \int_0^\infty e^{-(\lambda+\alpha+s)t} \frac{(\lambda t)^k}{k!} \, dG(t)$$

$$= \frac{(-\lambda)^k}{k!} \left( \frac{d}{ds} \right)^k \psi_G(\lambda + \alpha + s) \, , \tag{3.9}$$

we can rewrite (3.8) as

$$\eta_{G,H}(s) = \psi_G(s) - \psi_G(s + \alpha) + \sum_{k=0}^\infty p_k^{(G)}(s) \, \eta_{HstarF^{stark}}(s) \, . \tag{3.10}$$

This result makes it easy to prove Theorems 1 and 2. Theorem 1 is obtained by choosing $Y = 0$ ($H(x) = 1$ for $x \ge 0$) and $G = F^{stari}$.

**Proof of Theorem 1:** Assume that at time zero there is no work behind the test customer, and there are $i$ customers in front of him, none of whom have received any service yet, so that $Y = 0$ and $G = F^{stari}$. From

(3.10) we have for $i \geq 1$:

$$\eta_i(s) = \phi^i(s) - \phi^i(s+\alpha) + \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left( \left( \frac{d}{ds} \right)^k \phi^i(\lambda + \alpha + s) \right) \eta_k(s) \ . \tag{3.11}$$

In addition to (3.11) we also have the boundary condition

$$\eta_0\ (s) = \frac{\alpha}{\lambda + \alpha + s} + \frac{\lambda}{\lambda + \alpha + s}\ \eta_1\ (s)\ . \tag{3.12}$$

Namely, if the test customer is alone in the system, then the first event to occur is either $E$, or the arrival of an ordinary customer.

To prove Theorem 1 we must show that (2.11), (2.12) are the unique solution to (3.11), (3.12). First we substitute (3.12) into (3.11), and obtain for $i \geq 1$,

$$\eta_i(s) = \phi^i(s) - \phi^i(\alpha + s) + \frac{\alpha + \lambda \eta_1(s)}{\lambda + \alpha + s}\ \phi^i(\lambda + \alpha + s) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \left( \left( \frac{d}{ds} \right)^k \phi^i(\lambda + \alpha + s) \right) \eta_k(s) \ . \tag{3.13}$$

This can be rewritten as

$$\eta_i(s) = b_i(s) + \sum_{k=1}^{\infty} P_{i,k}(s)\ \eta_k(s)\ , \tag{3.14}$$

where

$$P_{i,k}(s) = \begin{cases} \dfrac{\lambda \phi^i(\lambda + \alpha + s)}{\lambda + \alpha + s} - \lambda\ \dfrac{d}{ds}\ \phi^i(\lambda + \alpha + s)\ , & \text{if } k = 1 \\[2ex] \dfrac{(-\lambda)^k}{k!} \left( \dfrac{d}{ds} \right)^k \phi^i(\lambda + \alpha + s) & \text{if } k \geq 2\ . \end{cases} \tag{3.15}$$

From (3.9) and (3.15) we see that

$$\begin{aligned} \sum_{k=1}^{\infty} P_{i,k}(s) &= \sum_{k=0}^{\infty} \int_0^{\infty} e^{-(\lambda+\alpha+s)t}\ \frac{(\lambda t)^k}{k!}\ dF^{stari}(s) - \frac{(\alpha + s)\ \phi^i(\lambda + \alpha + s)}{\lambda + \alpha + s} \\ &= \phi^i(\alpha + s) - \frac{(\alpha + s)\ \phi^i(\lambda + \alpha + s)}{\lambda + \alpha + s}\ , \end{aligned} \tag{3.16}$$

and for $i \geq 1$, $\mathrm{Re}(s) \geq 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \left| P_{i,k}\ (s) \right| &\leq \phi^i[\alpha + \mathrm{Re}(s)] - \frac{[\alpha + \mathrm{Re}(s)]\ \phi^i[\alpha + \lambda + \mathrm{Re}(s)]}{\lambda + \alpha + \mathrm{Re}(s)} \\ &< \phi^i(\alpha) \leq \phi\ (\alpha) < 1\ . \end{aligned} \tag{3.17}$$

Hence, the solution to (3.14) is unique. Moreover, the solution to (3.14) can be obtained by choosing $\eta_i^{(0)}(s)$ arbitrarily, and then iterating the contraction map (in $||\cdot||_\infty$)

$$\eta_i^{(n)}(s) = b_i(s) + \sum_{k=1}^{\infty} P_{i,k}(s)\, \eta_k^{(n-1)}(s) \,. \tag{3.18}$$

Clearly, $\lim_{n\to\infty} \eta_i^{(n)}(s) = \eta_i(s)$. This is one of the ways (2.11), (2.12) can be derived. Our original derivation was a form of "clever trying". Since the solution is available, it suffices to verify that (2.11), (2.12) indeed form a solution to (3.11), (3.12). It is easily seen that for any distribution $G$ on $[0,\infty)$, and with $p_k^{(G)}(s)$ defined as in (3.9), and $x_m(s)$ and $y_m(s)$ as in (2.8):

$$\sum_{k=0}^{\infty} p_k^{(G)}(s) y_m^k(s) = \psi_G[x_{m+1}(s)] \,. \tag{3.19}$$

It now is easy to verify that (2.11), (2.12) indeed is a solution to (3.11) - (3.13). This completes the proof of Theorem 1.□

**Proof of Theorem 2**. The proof of Theorem 2 consists of the following steps: (1) Use Theorem 1 and (3.9), (3.10) to obtain an expression for $\eta_G(s)$. (2) Use this expression for $\eta_G(s)$ and (3.9), (3.10) to get an expression for $\eta_{G,H}(s)$ for general $G$, $H$. This gives $\eta_{t_f,t_b}(s)$ as a special case, which immediately gives $\eta_P(s)$.

From (3.9), (3.10), and Theorem 1,

$$\begin{aligned}
\eta_G(s) &= \psi_G(s) - \psi_G(s+\alpha) + \sum_{k=0}^{\infty} p_k^{(G)}(s)\eta_k(s) \\
&= \psi_G(s) - \psi_G(s+\alpha) + \sum_{k=0}^{\infty} p_k^{(G)}(s)\left[\sum_{m=0}^{\infty}[y_m^k(s) - yTilde_{m+1}^k(s)] + \eta_0(s)y_\infty^k(s)\right] \\
&= \psi_G(s) - \psi_G(s+\alpha) + \sum_{m=0}^{\infty} (\psi_G[x_{m+1}(s)] - \psi_G[xTilde_{m+2}(s)]) + \eta_0(s)\psi_G[x_\infty(s)] \\
&= \eta_0(s)\psi_G[x_\infty(s)] + \sum_{m=0}^{\infty} (\psi_G[x_m(s)] - \psi_G[xTilde_{m+1}(s)]) \\
&= \psi_G(s) - [1-\eta_0(s)]\psi_G[x_\infty(s)] - \sum_{m=1}^{\infty} (\psi_G[xTilde_m(s)] - \psi_G[x_m(s)]) \,,
\end{aligned} \tag{3.20}$$

which is the promised expression for $\eta_G(s)$. Substituting for $\eta_{H*F^{*k}}(s)$ in (3.10), and noting that $\psi_{H*F^{*k}}(s) = \psi_H(s)\phi^k(s)$, gives

$$\eta_{G,H}(s) = \psi_G(s) - \psi_G(s+\alpha) \tag{3.21}$$

$$+ \sum_{k=0}^{\infty} p_k^{(G)}(s) \Bigg\{ \eta_0(s)\psi_H[x_\infty(s)]\phi^k[x_\infty(s)] + \sum_{m=0}^{\infty} \Bigg( \psi_H[x_m(s)]\phi^k[x_m(s)] - \psi_H[xTilde_{m+1}(s$$

$$= \psi_G(s) - \psi_G(s+\alpha) + \eta_0(s)\psi_G[x_\infty(s)]\psi_H[x_\infty(s)]$$

$$+ \sum_{k=0}^{\infty} (\psi_G[x_{k+1}(s)]\psi_H[x_k(s)] - \psi_G[xTilde_{k+2}(s)]\psi_H[xTilde_{k+1}(s)])$$

$$= \eta_0(s)\psi_G[x_\infty(s)]\psi_H[x_\infty(s)] + \psi_G(s)$$

$$+ \lim_{n\to\infty} \Bigg[ -\psi_G[xTilde_{n+1}(s)]\psi_H[xTilde_n(s)] + \sum_{k=0}^{n-1} (\psi_G[x_{k+1}(s)]\psi_H[x_k(s)] - \psi_G[xTi$$

$$= \psi_G(s) - [1 - \eta_0(s)]\psi_G[x_\infty(s)]\psi_H[x_\infty(s)] - \sum_{k=0}^{\infty} (\psi_G[xTilde_{k+1}(s)]\psi_H[xTilde_k(s)] - \psi_G$$

For the case $Pr\{X = t_f\} = 1$ and $Pr\{Y = t_b\} = 1$, $\psi_G(s) = e^{-st_f}$, and $\psi_H(s) = e^{-st_b}$, and (3.21) specializes to

$$\eta_{t_f,t_b}(s) = \eta_0(s)e^{-x_\infty(s)(t_f+t_b)} + \sum_{k=0}^{\infty} \left( e^{-x_k(s)t_f - x_{k-1}(s)t_b} - e^{-xTilde_{k+1}(s)t_f - xTilde_k(s)t_b} \right) \tag{3.22}$$

where $x_{-1}(s) \equiv 0$. Averaging over $t_f$ and $t_b$ gives (2.13).□

**Proof of Theorem 3:** To compute the derivative of the expression (2.13), it is necessary to compute

$$\begin{aligned} \frac{d}{ds}[x_k(s) - xTilde_k(s)]vbar_{s=0} &= \frac{d}{ds}(\lambda[yTilde_{k-1}(s) - y_{k-1}(s)])vbar_{s=0} \\ &= -\lambda\phi'(x_{k-1})\frac{d}{ds}[x_{k-1}(s) - xTilde_{k-1}(s)]vbar_{s=0} \\ &= \lambda^k \prod_{m=0}^{k-1} abs\phi'(x_m)abs \frac{d}{ds}[x_0(s) - xTilde_0(s)]vbar_{s=0} \\ &= \lambda^k \prod_{m=0}^{k-1} abs\phi'(x_m)abs \end{aligned} \tag{3.23}$$

where $x_k \equiv x_k(0)$. From (2.13),

$$\frac{d}{ds}\eta_P(s)vbar_{s=0} = \psi_G{}'(0) + [\eta_0(0) - 1]\frac{d}{ds}\psi_P[x_\infty(s), x_\infty(s)]vbar_{s=0} - \overline{T}_0\psi_P(x_\infty, x_\infty)$$

$$+ \sum_{k=0}^{\infty} \frac{d}{ds}(\psi_P[x_{k+1}(s), x_k(s)] - \psi_P[xTilde_{k+1}(s), xTilde_k(s)])vbar_{s=0} \tag{3.24}$$

$$= \psi_G{}'(0) - \overline{T}_0\psi_P[x_\infty(s), x_\infty(s)] + \sum_{k=0}^{\infty} \frac{d}{ds}(\psi_P[x_{k+1}(s), x_k(s)] - \psi_P[xTilde_{k+1}(s), xTilde_k(s)])vbar_{s=0} \; .$$

Using the fact that $x_k(0) = xTilde_k(0) = x_k$,

$$\sum_{k=0}^{\infty} \frac{d}{ds} (\psi_P[x_{k+1}(s), x_k(s)] - \psi_P[xTilde_{k+1}(s), xTilde_k(s)]) vbar_{s=0} \tag{3.25}$$

$$= \sum_{k=0}^{\infty} \left( \frac{\partial \psi_P(s_1, s_2)}{\partial s_2} vbar_{s_1=x_{k+1}, s_2=x_k} \frac{d}{ds} [x_k(s) - xTilde_k(s)] vbar_{s=0} + \frac{\partial \psi_P(s_1, s_2)}{\partial s_1} vbar_{s_1=x_{k+1}, s_2=x_k} \frac{d}{ds} [x_{k+1}(s)$$

$$= \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m) abs \right) \left( \frac{\partial \psi_P(s_1, s_2)}{\partial s_2} vbar_{s_1=x_{k+1}, s_2=x_k} + \lambda abs\phi'(x_k) abs \frac{\partial \psi_P(s_1, s_2)}{\partial s_1} vbar_{s_1=x_{k+1}, s_2=x_k} \right).$$

Combining (2.14), (3.24), and (3.25), and noting that $\psi_G{}'(0) = -\bar{X}$, gives (2.15).

From (2.12),

$$\frac{d}{ds} \eta_0(s) vbar_{s=0} = -\frac{1}{x_\infty} \frac{d}{ds} \left( \sum_{k=0}^{\infty} [x_k(s) - xTilde_k(s)] \right) vbar_{s=0} - \left( \sum_{k=0}^{\infty} [x_k(s) - xTilde_k(s)] \right) vbar_{s=0} \frac{d}{ds} \left( \frac{1}{x_\infty(s)} \right) vbar_{s=0} \tag{3.26}$$

$$= -\frac{1}{x_\infty} \sum_{k=0}^{\infty} \lambda^k \prod_{m=0}^{k-1} abs\phi'(x_m) abs = -\frac{1}{x_\infty} \left[ 1 + \sum_{k=0}^{\infty} \lambda^{k+1} \prod_{m=0}^{k} abs\phi'(x_m) abs \right].$$

□

As a side remark, we outline an alternative derivation of (2.22). In analogy with the recurrence relation (2.14), the following recurrence relation can be derived for the mean delay $\bar{T}_{G,H}$,

$$\bar{T}_{G,H} = \bar{X} + \sum_{k=0}^{\infty} p_k^{(G)} \bar{T}_{H*F^{*k}} \tag{3.27}$$

where $\bar{T}_{H*F^{*k}}$ is the mean delay until service given that the amount of work in front of the test customer is the random variable $Y$ plus $k$ service times and $p_k^{(G)} \equiv p_k^{(G)}(0)$. If $Y = 0$, then

$$\bar{T}_G = \bar{X} + \sum_{k=0}^{\infty} p_k^{(i)} \bar{T}_k \tag{3.28}$$

where $\bar{T}_k = \bar{T}_{F^{*k}}$ and $p_k^{(i)} \equiv p_k^{(F^{stari})}$. The boundary condition, in analogy with (3.12), is

$$\bar{T}_0 = \frac{1}{\lambda + \alpha} + \frac{\lambda}{\lambda + \alpha} \bar{T}_1 \; . \tag{3.29}$$

The contraction mapping technique, which can be used to obtain the solution to (3.9), (3.10), can also be applied to (3.28), (3.29), thereby giving explicit expressions for $\bar{T}_{ij}$ and $\bar{T}_{G,H}$.

Before proving Theorems 4-7, some basic properties of the sequences $x_k$ and $y_k$, which follow directly from the discussion at the beginning of this section, are stated. The sequences $x_k$ and $y_k$ are illustrated graphically in Figure 1.

**Lemma 2.** For real $s \geq 0$, the sequence $x_k(s)$, defined by (2.8), increases monotonically and converges to

$x_\infty(s)$, and the sequences $y_k(s)$ and $abs\phi'[x_k(s)]abs$ each decrease monotonically and converge to $y_\infty(s)$ and $abs\phi'[x_\infty(s)]abs$, respectively. Also, $x_\infty(s) < s + \lambda + \alpha$.

**Proof of Corollary 2:** From (2.15), for the case $\psi_P(s_1, s_2) = e^{-(s_1 t_f + s_2 t_b)}$,

$$\overline{T}_{t_f, t_b} = \overline{T}_0 e^{-s(t_f + t_b)} + t_f + \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) e^{-x_{k+1} t_f - x_k t_b} [t_b - \lambda\phi'(x_k) t_f] \,, \tag{3.30}$$

so that

$$\begin{aligned}\overline{T}_{t_f + t_b, 0} - \overline{T}_{t_f, t_b} &= t_b + \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \left( -\lambda\phi'(x_k)(t_f + t_b) e^{-x_{k+1}(t_f + t_b)} - t_b e^{-x_{k+1} t_f - x_k t_b} + \lambda\phi'(x_k) t_f e^{-x_{k+1} t_f - x_k t_b} \right) \\ &= t_b + \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \left( -t_b e^{-x_{k+1} t_f} [e^{-x_k t_b} + \lambda\phi'(x_k) e^{-x_{k+1} t_b}] + \lambda\phi'(x_k) t_f e^{-x_{k+1} t_f} [e^{-x_k t_b} - e^{-x_{k+1} t_b}] \right) \\ &= \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k-1} abs\phi'(x_m)abs \right) \left( t_b e^{-x_k t_b} (e^{-x_k t_f} - e^{-x_{k+1} t_f}) + \lambda\phi'(x_k) t_f e^{-x_{k+1} t_f} (e^{-x_k t_b} - e^{-x_{k+1} t_b}) \right) .\end{aligned} \tag{3.31}$$

□

The next lemma proves Theorem 4 by showing that if $\lambda m \le 1$, then all terms in the series (2.21) are nonnegative.

**Lemma 3:** The function

$$f(x_k, x_{k+1}) = t_b e^{-x_k t_b} (e^{-x_k t_f} - e^{-x_{k+1} t_f}) + \lambda\phi'(x_k) t_f e^{-x_{k+1} t_f} (e^{-x_k t_b} - e^{-x_{k+1} t_b}) \tag{3.32}$$

is greater than or equal to zero for $t_f \ge 0$, $t_b \ge 0$, and $\lambda abs\phi'(x_k)abs \le 1$.

**Proof:** Observe that

$$f(x_k, x_k) = 0 \,. \tag{3.33}$$

Also,

$$\begin{aligned}\frac{\partial f}{\partial x_{k+1}} &= t_b t_f e^{-x_k t_b - x_{k+1} t_f} + \lambda\phi'(x_k) t_f \left( -t_f e^{-x_{k+1} t_f} (e^{-x_k t_b} - e^{-x_{k+1} t_b}) + t_b e^{-x_{k+1} t_f - x_{k+1} t_b} \right) \\ &= t_b t_f e^{-x_k t_b - x_{k+1} t_f} + \lambda\phi'(x_k) t_f e^{-x_{k+1} t_f} [(t_f + t_b) e^{-x_{k+1} t_b} - t_f e^{-x_k t_b}] \\ &= t_b t_f e^{-x_{k+1} t_f} (e^{-x_k t_b} - \lambda abs\phi'(x_k)abs e^{-x_{k+1} t_b}) + \lambda abs\phi'(x_k)abs t_f^2 e^{-x_{k+1} t_f} (e^{-x_k t_b} - e^{-x_{k+1} t_b}) \,.\end{aligned} \tag{3.34}$$

Lemma 2 states that $x_{k+1} > x_k$, therefore the derivative (3.34) is positive, which implies that $f(x_k, x_{k+1})$ is positive, if $\lambda abs\phi'(x_k)abs \le 1$. Every term in the sum (3.31) is guaranteed to be positive if $\lambda abs\phi'(x_0)abs = \lambda m \le 1$.□

Notice that if $\lambda m = 1$, then the sum (3.31) will be strictly positive, since $abs\phi'(x_k)abs$ is strictly decreasing with $k$. It therefore seems likely that there exists a threshhold $\lambda_0 > 1/m$, such that if $\lambda \le \lambda_0$ the sum (3.31) is positive for all positive $t_f$ and $t_b$.

**Proof of Theorem 5:** For the case $\psi_G(s) = e^{-st_f}$ and $\psi_H(s) = \phi^j(s)$, (2.22) becomes

$$\bar{T}_{G*H} - \bar{T}_{G,H} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k} abs\phi'(x_m)abs \right) f(x_k, x_{k+1}) \tag{3.35a}$$

where

$$f(x_k, x_{k+1}) = \left( jy_k^{j-1}(e^{-x_k t_f} - e^{-x_{k+1} t_f}) - \lambda t_f e^{-x_{k+1} t_f}(y_k^j - y_{k+1}^j) \right). \tag{3.35b}$$

Now,

$$f(0,0) = 0 \tag{3.36}$$

and

$$\frac{\partial f}{\partial x_{k+1}} = \lambda t_f^2 e^{-x_{k+1} t_f}(y_k^j - y_{k+1}^j) + (1 - \lambda abs\phi'(x_{k+1})abs) jt_f e^{-y_{k+1} t_f} y_k^{j-1} , \tag{3.37}$$

which is positive for $k \ge 1$ if $1 - \lambda abs\phi'(\alpha)abs \ge 0$. For $k = 0$,

$$f(x_k, x_{k+1}) = f(0, \alpha) = j(1 - e^{-\alpha t}) - \lambda t e^{-\alpha t}[1 - \phi^j(\alpha)] , \tag{3.38}$$

which is greater than or equal to zero if

$$\lambda \le \frac{e^{\alpha t} - 1}{t} \cdot \frac{j}{1 - \phi^j(\alpha)} . \tag{3.39}$$

It is easily shown that $\dfrac{e^{\alpha t} - 1}{t} \ge \alpha$, with equality as $t$ approaches zero. Furthermore, it can be shown that the function $\dfrac{j}{1 - a^j}$, where $0 \le a < 1$ and $j \ge 1$ is an integer, increases with $j$, so that if

$$\lambda \le \min\left( \frac{1}{abs\phi'(\alpha)abs} , \frac{\alpha}{1 - \phi(\alpha)} \right), \tag{3.40}$$

then $f(x_k, x_{k+1}) \ge 0$ for all $k \ge 0$. Referring to Figure 2, it is clear that

$$\alpha abs\phi'(\alpha)abs < 1 - \phi(\alpha) < \alpha abs\phi'(0)abs = \alpha m , \tag{3.41}$$

which proves the result.□

**Proof of Theorem 6:** Substituting $\psi_H(s) = \phi^j(s)$ into (2.22) gives

$$\overline{T}_{G,H} - \overline{T}_{G*H} = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{m=0}^{k} abs\phi'(x_m)abs \right) f(x_k, x_{k+1}) \tag{3.42}$$

where

$$f(x_k, x_{k+1}) = jy_k^{j-1}[\psi_G(x_k) - \psi_G(x_{k+1})] - \lambda abs\psi_G{}'(x_{k+1})abs(y_k^j - y_{k+1}^j) \ . \tag{3.43}$$

As in the preceding proofs, it is easily shown that for $k \geq 1$, $\partial f / \partial x_k \leq 0$, and hence $f(x_k, x_{k+1}) \geq 0$, if $\lambda abs\phi'(\alpha)abs \leq 1$. For $k = 0$,

$$f(0, \alpha) = j[1 - \psi_G(\alpha)] - \lambda abs\psi_G{}'(\alpha)abs[1 - \phi^j(\alpha)] \ , \tag{3.44}$$

which is greater than or equal to zero if

$$\lambda \leq \frac{j}{1 - \phi^j(\alpha)} \cdot \frac{1 - \psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} \tag{3.45}$$

for all possible $\psi_G(s)$ and $j \geq 1$. Substituting $\psi_G(s) = \phi^{i-1}(s)\phi_\tau(s)$, and noting that the right side of (3.45) is minimized by setting $j = 1$, and is equal to $1/abs\phi'(\alpha)abs$ for $i = 1$ and $\tau = 0$, gives Theorem 6.□

**Proof of Corollary 2:** For this case $\psi_G(s) = \phi^i(s)$, and we show that

$$\frac{1 - \psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} = \frac{1 - \phi^i(\alpha)}{i\phi^{i-1}(\alpha)abs\phi'(\alpha)abs}$$

increases with $i$. Assume that this is false. Then

$$\frac{1 - \phi^i(\alpha)}{i\phi^{i-1}(\alpha)} > \frac{1 - \phi^{i+1}(\alpha)}{(i+1)\phi^i(\alpha)} \tag{3.46}$$

for some $i$. This implies that

$$(i+1)\phi(\alpha) - \phi^{i+1}(\alpha) > i \ . \tag{3.47}$$

The left side assumes its maximum value, however, when $\phi(\alpha) = 1$, therefore this cannot be true. Consequently,

$$\min_{i \geq 1} \frac{1 - \psi_G(\alpha)}{abs\psi_G{}'(\alpha)abs} = \frac{1 - \phi(\alpha)}{abs\phi'(\alpha)abs} \tag{3.48}$$

and Corollary 2 follows from Theorem 6.□

We remark that Corollary 2 can also be proved directly from (2.23). In particular, it is easily shown that

$$f(y_k, y_{k+1}) = jy_k^{j-1}(y_k^i - y_{k+1}^i) - \lambda abs\phi'(\alpha)absiy_{k+1}^{i-1}(y_k^j - y_{k+1}^j) \geq 0 \tag{3.49}$$

for $i \ge 1,\ j \ge 1$, and $\lambda abs \phi'(\alpha) abs \le 1$.

**Proof of Theorem 7:** From (2.23) and Lemma 2

$$\begin{aligned}
\overline{T}_{i+j} - \overline{T}_{ij} &< jm(1-y_1^i) - \lambda imabs\phi'(x_1)abs y_1^{i-1}(1-y_1^j) \\
&\quad + \lambda mabs\phi'(x_1)abs[jy_1^{j-1}(y_1^i - y_2^i) - \lambda abs\phi'(x_2)absiy_2^{i-1}(y_1^j - y_2^j)] \\
&\quad + \lambda^2 mabs\phi'(x_1)absabs\phi'(x_2)abs \sum_{k=2}^{\infty}(\lambda abs\phi'(x_2)abs)^{k-2} jy_2^{j-1}(y_2^i - y_\infty^i) \\
&< jm(1-y_1^i) - \lambda mabs\phi'(x_1)absiy_1^{i-1}(1-y_1^j) + \lambda mabs\phi'(x_1)absjy_1^{i+j-1} \\
&\quad + \lambda^2 abs\phi'(x_1)absabs\phi'(x_2)abs \frac{jmy_2^{j-1}(y_2^i - y_\infty^i)}{1-\lambda abs\phi'(x_2)abs} ,
\end{aligned} \tag{3.50}$$

assuming $\lambda$ is large enough so that $\lambda abs\phi'(x_2)abs < 1$. For fixed $\alpha$ it can be shown that the last term goes to zero faster than $O(1/\lambda)$. Therefore, for large $\lambda$,

$$\overline{T}_{i+j} - \overline{T}_{ij} < jm(1-y_1^i) + \lambda mabs\phi'(x_1)abs y_1^{i-1}[jy_1^j - i(1-y_1^j)] + O(1/\lambda) \tag{3.51}$$

which can be negative only if

$$jy_1^j < i(1-y_1^j) ,$$

or

$$y_1^j < \frac{i}{i+j} . \tag{3.52}$$

Since $y_1 < 1$, $i$ can be selected large enough so that (3.51) is true for any $j$, and if $\lambda$ is greater than some threshold $\lambda_0$, then from (3.51), $\overline{T}_{i+j} - \overline{T}_{ij} < 0$.□

*4. A MODIFIED PROBLEM*

So far we have assumed that the test customer can always give up his place in the queue, and move to the back of the queue. It has been shown that if $\lambda$ is large enough, using this option will decrease the test customer's expected delay until service. Suppose, however, that the test customer *cannot* move to the back of the queue once he is in the queue, but upon reaching the server before $E$ has occured, he can choose to wait outside the queue any amount of time before rejoining the back of the queue. The amount of time the test customer waits is determined according to some policy, i.e., it may be determined by observing the length of the queue. Initially, then, the test customer may wait before joining the queue, but once in the queue he must stay in the queue until he reaches the server. This version of the problem was in fact the original version [Gopinath (1984)], and will be referred to as Problem 2 (P2). The problem considered so far will be referred to as Problem 1 (P1).

**Lemma 4.** Given any $\lambda$, $\alpha$, and $m$, if the move-along policy is optimal for P1, then it is also optimal for P2.

**Proof:** Any allowable policy for P2 can be effectively duplicated by a policy for P1 (but not vice versa). Therefore the optimal policy for P1 must perform at least as well as the optimal policy for P2.□

Theorems 4-6 and Corollary 2 therefore also apply to P2. Because any policy for P1 cannot in general be duplicated by a policy for P2, the converse to Lemma 4 may not be true. That is, if the move-along policy is *not* optimal for P1, it is unknown whether or not this implies that the move-along policy is not optimal for P2. The following Theorem states the analogous result for P2 as was stated in Theorem 6.

**Theorem 8:** For P2, given any $\alpha$ and $m$, there exists a $\lambda_0$ such that if $\lambda \geq \lambda_0$, the move-along policy is not optimal.

**Proof:** Assume that initially there are $i$ customers in the queue, and that the test customer must decide to either join the queue immediately, or wait until either there are $i' > i$ customers in the queue, or until $E$ occurs, whichever occurs first. If the test customer chooses to wait, the mean delay until service is

$$\overline{T}_i{}' = p_{i,i'} D + (1 - p_{i,i'})[\overline{T}_{i'} + \tau_{i,i'}(\lambda)] \tag{4.1}$$

where $p_{i,i'}$ is the probability that $E$ occurs before the queue length becomes $i'$, $D$ is the mean delay until service given that $E$ occurs before the queue length becomes $i'$, and $\tau_{i,i'}(\lambda)$ is the mean time it takes to go from a queue length of $i$ to $i'$ (not including the test customer). Since $D$ must be less than the mean delay given that $E$ occurs after the queue length becomes $i'$, waiting outside the queue reduces the mean delay if

$$\overline{T}_i - \overline{T}_{i'} > \tau_{i,i'}(\lambda) \tag{4.2}$$

for some $i'$. Clearly,

$$\lim_{\lambda \to \infty} \tau_{i,i'}(\lambda) = 0. \tag{4.3}$$

Using (2.17) and Lemma 2 gives the lower and upper bounds

$$\bar{T}_i > \bar{T}_i = im + y_\infty^i \bar{T}_0 + \lambda m i y_1^{i-1} abs \phi'(x_1) abs \qquad (4.4)$$

and

$$\bar{T}_i < \bar{T}_i = im + y_\infty^i \bar{T}_0 + \lambda m i y_1^{i-1} abs \phi'(x_1) abs + \lambda^2 i m y_2^{i-1} abs \phi'(x_1) abs abs \phi'(x_2) abs \sum_{k=2}^{\infty} (\lambda abs \phi'(x_2) abs)^{k-2} \ . \qquad (4.5)$$

The infinite series can be summed provided that $\lambda abs \phi'(x_2) abs < 1$. From (2.30) this is always true if $\lambda$ is large enough. Consequently, (4.5) becomes

$$\bar{T}_i = im + y_\infty^i \bar{T}_0 + \lambda m i y_1^{i-1} abs \phi'(x_1) abs + \frac{\lambda^2 m abs \phi'(x_1) abs abs \phi'(x_2) abs i y_2^{i-1}}{1 - \lambda abs \phi'(x_2) abs} \ . \qquad (4.6)$$

The condition (4.2) is satisfied if $\bar{T}_i - \bar{T}_{i'} > \tau_{i,i'}(\lambda)$, or

$$(i' - i)m + \bar{T}_0 (y_\infty^{i'} - y_\infty^i) + \lambda m abs \phi'(x_1) abs (i' y_1^{i'-1} - i y_1^{i-1}) + \frac{m \lambda^2 abs \phi'(x_1) abs abs \phi'(x_2) abs i' y_2^{i'-1}}{1 - \lambda abs \phi'(x_2) abs} + \tau_{i,i'}(\lambda) < 0 \qquad (4.7)$$

For fixed $\alpha$, $m$, $i$, and $i' > 1$,

$$\lim_{\lambda \to \infty} [\lambda^2 abs \phi'(x_2) abs y_2^{i'-1}] = 0 \ . \qquad (4.8)$$

Consequently, for large $\lambda$ the left hand side of (4.7) becomes

$$(i' - i)m + \bar{T}_0 (y_\infty^{i'} - y_\infty^i) + \lambda m abs \phi'(x_1) abs (i' y_1^{i'-1} - i y_1^{i-1}) \ ,$$

which is negative for large enough $\lambda$ if

$$i' y_1^{i'-1} < i y_1^{i-1} \ . \qquad (4.9)$$

Since the function $f(u) = i u^{i-1}$ decreases with $i$ when $u < 1$ and $i$ is large enough, (4.9) is true for large enough $i'$. Consequently, for any $\alpha$, $m$, and initial number of customers $i$, if $\lambda$ is large enough, the observer can reduce his delay until service by waiting for the number of customers in the queue to increase to $i'$. Therefore if $\lambda$ is greater than some threshold value $\lambda_0$, then the condition (4.2) is true for some $i$ and $i'$, and the move-along policy is not optimal.□

## *5. EXAMPLES*

We conclude with two specific examples, namely the $M/D/1$ and $M/M/1$ queues.

### *Deterministic Service Time (M/D/1)*

For this case

$$F(t) = \begin{cases} 0 & \text{if } t \le m \\ 1 & \text{if } t > m \end{cases}, \tag{5.1}$$

and

$$\phi(s) = e^{-sm}$$

$$\phi'(s) = -\, m e^{-sm} \tag{5.2c}$$

$$\phi_\tau(s) = e^{-s(m-\tau)}, \qquad \phi_\tau{}'(s) = -\,(m-\tau)e^{-s(m-\tau)} \;.$$

To compute $\overline{T}_{G,H}$, it is necessary to compute the sequence

$$x_{k+1} = \lambda(1 - e^{-x_k}) + \alpha \;, \qquad x_0 = 0 \;. \tag{5.3}$$

For the case $\psi_G(s) = \phi^{i-1}(s)\phi_\tau(s) = e^{-s(im-\tau)}$, and $\psi_H(s) = e^{-sjm}$, (2.22) becomes

$$\overline{T}_{G,H} = im - \tau + e^{-x_\infty[(j+i)m-\tau]}\overline{T}_0 + \sum_{k=0}^{\infty} \lambda^k [\, jm + \lambda(im-\tau)e^{-x_k m}] \exp\left(-jmx_k - (im-\tau)x_{k+1} - m\sum_{l=1}^{k-1} x_l\right) \tag{5.4a}$$

where

$$\overline{T}_0 = \frac{1}{x_\infty}\left[1 + \sum_{k=0}^{\infty} (\lambda m)^{k+1} \exp\left(-m\sum_{l=0}^{k} x_l\right)\right]. \tag{5.4b}$$

From Theorem 5, assuming that the test customer can move to the back of the queue at any time, the move-along policy is optimal if

$$\lambda \le \frac{\alpha}{1 - e^{-\alpha m}} \;, \tag{5.5}$$

and from Corollary 2, the move-along policy is optimal for the discrete-epoch problem if

$$\lambda \le \frac{1}{m}\, e^{m\alpha} \;. \tag{5.6}$$

*Exponential Service Time (M/M/1)*

For this case

$$F(t) = F_\tau(t) = 1 - e^{-t/m}, \tag{5.7}$$

and

$$\phi(s) = \phi_\tau(s) = \frac{1}{1+sm} \,,$$

$$\phi'(s) = \phi_\tau{}'(s) = -\,\frac{m}{(1+sm)^2} \,, \tag{5.8b}$$

and

$$x_{k+1} = \frac{\lambda m x_k}{m x_k + 1} + \alpha \ . \tag{5.9}$$

For the case $\psi_G(s) = \phi^{i-1}(s)\phi_\tau(s) = \phi^i(s)$, and $\psi_H(s) = \phi^j(s)$, the expression for $\overline{T}_{G,H}$ becomes

$$\overline{T}_{ij} = im + \left(\frac{1}{1 + mx_\infty}\right)^{j+i} \overline{T}_0 + \sum_{k=0}^{\infty} \lambda^k \left(\prod_{l=0}^{k} \frac{m}{(1 + mx_l)^2}\right) \frac{1}{(1 + mx_k)^{j-1}(1 + mx_{k+1})^i} \left\{ j + \frac{\lambda mi}{(1 + mx_{k+1})(1 + mx_k)} \right\} \tag{5.10a}$$

where

$$\overline{T}_0 = \frac{1}{x_\infty} \left[ 1 + \sum_{k=0}^{\infty} (\lambda m)^{k+1} \left( \prod_{l=0}^{k} \frac{1}{(1 + mx_l)^2} \right) \right]. \tag{5.10b}$$

In the case of exponentially distributed service times the situations described in subsections (1.3) (minimal information and complete freedom) and (1.4) (minimal information and limited freedom) become identical. From Theorem 6 or Corollary 2 the move-along policy is guaranteed to be optimal if

$$\lambda \leq \frac{(1 + m\alpha)^2}{m} \ . \tag{5.11}$$

From Theorem 5, if the test customer knows the service times of the customers ahead of him, then the move-along policy is optimal if

$$\lambda \leq \frac{1 + m\alpha}{m} \ . \tag{5.12}$$

## *6. UNANSWERED QUESTIONS*

Assuming the conjecture stated in Section 1 is true, then Theorems 4 through 6 give lower bounds on the critical levels $\lambda_k^*$. Further improvements to these bounds have not yet been obtained.

The following Theorem applies to the case where the queueing system has a *finite capacity* $C < \infty$, where incoming customers are blocked and disappear if there are already $C$ customers in the system (not including the test customer).

**Theorem 9:** For an $M/M/1$ queue with $C \leq 4$, the condition $\overline{T}_{ij} < \overline{T}_{i+j}$ is *always* true, independent of $\lambda$, $m$, and $\alpha$.

**Proof:** For finite $C$, a finite set of linear difference equations for the mean delay $\overline{T}_{ij}$ can be written down by inspection from which the theorem is easily verified. Details are ommitted.□

It is not known if Theorem 9 is true for any $C > 4$. For a very large capacity queueing system, Theorem 7 must apply. Consequently, there must exist a threshhold $\underline{\lambda}$ (not necessarily finite), which is a function of $C$, such that the move-along policy is optimal if $\lambda \le \underline{\lambda}$. How does $\underline{\lambda}$ behave as $C$ goes to infinity?

Perhaps the most interesting generalization of the problem studied here is the case where other customers in the queue are also waiting for events to occur before they can be served. For example, all customers may be waiting for independent events, each of which occurs after an exponentially distributed amount of time, and each may decide to follow the move-along policy. Is the move-along policy optimal for a particular test customer?

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