Power and Signature Optimization for Forward-Link CDMA with Multiple Antennas

Hao Bi
Motorola Inc.
Libertyville, Illinois 60048
Email: hao.bi@motorola.com

Michael L. Honig
Department of ECE
Northwestern University
Evanston, Illinois 60208
Email: mh@ece.nwu.edu

***Abstract*— Signature sequences and associated powers are optimized jointly with linear receivers for a multi-user forward-link CDMA system with multiple transmit and receive antennas. The performance criterion is sum capacity over all users. For the model considered, the optimal signatures are sinusoids so that multi-carrier signaling, in which each carrier is assigned to a single user, maximizes the achievable rate. An optimal assignment of carriers to users appears to be difficult to determine in general, but can be efficiently approximated numerically. The asymptotic sum capacity for large number of users and antennas is characterized at high SNRs. The accuracy of these results for moderate SNRs is illustrated through comparisons with numerically optimized power allocations.**

I. Introduction

Adding antennas to transmitters and receivers in a cellular system can improve performance and increase capacity [1]–[3]. Recently, there has been substantial effort devoted to determining the forward-link sum capacity with multiple users and multiple antennas [4]–[7]. It has been shown that the sum capacity can be achieved with dirty paper coding [8] and superposition coding at the transmitter, and successive decoding at the receiver.

In this paper, we examine the sum capacity of the forward-link with multiple antennas at the transmitter and receiver, and *linear* receivers. This constraint is motivated by practical considerations, and simplifies the optimal coding scheme. The channels are assumed to be frequency-selective and are known at both the transmitter and receiver. We start with a Code-Division Multiplexing (CDM) scheme in which each user is assigned a set of signature sequences for each transmit antenna. Each mobile has a *linear* receiver, which detects the data symbols across the assigned signature sequences. We jointly optimize the set of signature sequences, associated powers, and receiver filters to maximize the sum mutual information subject to a total power constraint. This is an extension of our prior work in [9], [10], which applies to a single receive antenna.

Assuming circulant channel matrices, the optimal received signature sequences after passing through the channel are orthogonal, which leads to a multi-carrier signaling scheme. Here we assume that each signature, which corresponds to a particular frequency, is assigned to a single user. Because the signatures are orthogonal, there is no need for superposition coding. Each signature is associated with a matrix of antenna gains, corresponding to the Multi-Input/Multi-Output (MIMO) spatial channel. The optimal matrix transmit and receive filters for each signature (i.e., discrete frequency) create multiple (uncoupled) spatial sub-channels. The optimization then reduces to allocating powers across users, signatures, and spatial sub-channels to maximize the sum mutual information subject to a total power constraint.[1] This problem is difficult to solve in general, but reduces to a convex optimization problem if users can time-share signatures.

The forward-link sum capacity per signature is then analyzed for a large number of signatures, or discrete frequencies K, at high SNRs, assuming channel gains are *i.i.d.* across frequencies, users, and antennas. Specifically, we characterize the sum mutual information per signature as a function of the number of users U, the number of transmit antennas M_t, and the number of receive antenna M_r. Numerical examples are presented, and a comparison with numerical optimizations for finite-size systems show that the asymptotic analysis is accurate for a moderate number of users and antennas.

II. Forward-Link Model

We consider the Code-Division Multiple Access (CDMA) forward-link with short signatures having duration equal to the symbol duration. Each user can be assigned multiple signatures which carry independent symbol streams. With appropriate cyclic extensions on signatures and zero-padding on receiver filters, the $K \times 1$ received vector of chip matched-filter outputs on antenna i for a particular user u is given by

$$\mathbf{r}_{u,i} = \sum_{j=1}^{M_t} \mathbf{H}^u_{i,j}\mathbf{S}_j\mathbf{A}_j\mathbf{b}_j + \mathbf{w}^u_i \tag{1}$$

where M_t is the number of transmit antennas, $\mathbf{S}_j = \left(\mathbf{s}^j_1\ \mathbf{s}^j_2\ \cdots\ \mathbf{s}^j_K\right)$ is the matrix of signatures from transmit antenna j, $\mathbf{A}_j = diag(a_{j,1}, \cdots, a_{j,K})$ is the corresponding

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[1]Since the sum rate with single-user decoding and linear receivers equals the corresponding sum mutual information with multi-user cooperative decoding, this achieves the forward-link sum capacity [11].

$K \times K$ diagonal matrix of amplitude gains across signatures, and $\mathbf{b}_j = (b_{j,1}, \cdots, b_{j,K})^{\mathrm{T}}$ is the corresponding vector of symbols, where $|b_{j,k}|^2 = 1$. The $K \times K$ circulant channel matrix $\mathbf{H}_{i,j}^u$ between transmit antenna j and receive antenna i has as its first row $[h_{i,j}^u(L) \cdots h_{i,j}^u(0) \cdots 0]$, which is the (time-reversed) channel impulse response between transmit antenna j and receive antennas i for user u. Finally, $\mathbf{w}_i^u$ is the $K \times 1$ Gaussian noise vector on receive antenna i for user u, which has covariance matrix $\sigma_i^2 \mathbf{I}_{K\times K}$.

Letting $\underline{\mathbf{r}}_u = \left(\mathbf{r}_1^{\mathrm{T}} \cdots \mathbf{r}_{M_r}^{\mathrm{T}}\right)^{\mathrm{T}}$ denote the stacked vector of signal vectors across M_r receiver antennas at mobile station u, we have

$$\begin{aligned} \underline{\mathbf{r}}_u &= \mathcal{H}_u \mathbb{S} \mathbf{A} \mathbf{b} + \mathbf{w}_u && (2) \\ &= \mathcal{H}_u \mathcal{S} \mathbf{T} \mathbf{A} \mathbf{b} + \mathbf{w}_u && (3) \end{aligned}$$

where $\mathbb{S} = diag\left(\mathbf{S}_1 \cdots \mathbf{S}_{M_t}\right)$ ($KM_r \times KM_r$ block-diagonal matrix), $\mathbf{A} = diag\left(\mathbf{A}_1 \cdots \mathbf{A}_{M_t}\right)$, $\underline{\mathbf{b}} = \left(\mathbf{b}_1^{\mathrm{T}} \cdots \mathbf{b}_{M_t}^{\mathrm{T}}\right)^{\mathrm{T}}$ is the vector containing the $M_t K$ symbols transmitted over the M_t transmit antennas, and $\underline{\mathbf{w}}_u = \left(\mathbf{w}_1^{\mathrm{T}} \cdots \mathbf{w}_{M_r}^{\mathrm{T}}\right)^{\mathrm{T}}$ is the noise vector across all receive antennas.

The composite $KM_r \times KM_t$ channel matrix is

$$\mathcal{H}_u = \begin{pmatrix} \mathbf{H}_{1,1}^u & \cdots & \mathbf{H}_{1,M_t}^u \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{M_r,1}^u & \cdots & \mathbf{H}_{M_r,M_t}^u \end{pmatrix}, \qquad (4)$$

where $\mathbf{H}_{i,j}^u$ denotes the channel matrix from transmit antenna j to the ith receive antenna of user u. We have written $\mathbb{S} = \mathcal{S}\mathbf{T}$ where $\mathcal{S} = diag\left(\mathbf{S} \cdots \mathbf{S}\right)$, $\mathbf{S}$ denotes a non-singular $K \times K$ matrix, and $\mathbf{T}$ is an $M_t K \times M_t K$ matrix. This form will be convenient in what follows.

With multiple transmit and receive antennas it is possible to transmit reliably different symbols from different transmit antennas on the same signature. Let $\bar{\mathbf{b}}_k = (b_{1,k} \cdots b_{M_t,k})^{\mathrm{T}}$ and $\bar{\mathbf{A}}_k = diag(a_{1,k} \cdots a_{M_t,k})$ represent, respectively, the vector of symbols transmitted on signature k across the M_t transmit antennas, and the gain matrix for these symbols. We define the stacked vector of symbols across all signatures as $\bar{\mathbf{b}} = \left(\bar{\mathbf{b}}_1^{\mathrm{T}} \cdots \bar{\mathbf{b}}_K^{\mathrm{T}}\right)^{\mathrm{T}}$ and the corresponding amplitude matrix $\bar{\mathbf{A}} = diag\left(\bar{\mathbf{A}}_1 \cdots \bar{\mathbf{A}}_K\right)$. Let $\mathbf{E}_K = \left(\mathbf{e}_1 \mathbf{e}_{1+K} \cdots \mathbf{e}_{1+(M_t-1)K} \cdots \mathbf{e}_K \mathbf{e}_{2K} \cdots \mathbf{e}_{M_t K}\right)$ and $\mathbf{E}_{M_t} = \left(\mathbf{e}_1 \mathbf{e}_{1+M_t} \cdots \mathbf{e}_{1+(K-1)M_t} \cdots \mathbf{e}_{M_t} \mathbf{e}_{2M_t} \cdots \mathbf{e}_{M_t K}\right)$ denote two $M_t K \times M_t K$ permutation matrices, where $\mathbf{e}_j$ is a unit vector of length $M_t K$ with the jth element equal to one and others equal to zero. Note that $\mathbf{E}_K^{\mathrm{T}} = \mathbf{E}_{M_t}$. Furthermore, $\mathbf{E}_K$ and $\mathbf{E}_{M_t}$ are unitary matrices, and shuffle either the rows (when postmultiplied) or columns (when premultiplied) of a matrix. Hence,

$$\underline{\mathbf{A}\mathbf{b}} = \mathbf{E}_{M_t}^{\mathrm{T}} \bar{\mathbf{A}} \bar{\mathbf{b}} \qquad (5)$$

and (3) can be rewritten as

$$\mathbf{r}_u = \mathcal{H}_u \mathcal{S} \mathbf{T} \mathbf{E}_{M_t}^{\mathrm{T}} \bar{\mathbf{A}} \bar{\mathbf{b}} + \mathbf{w}_u \qquad (6)$$

In what follows, we will denote $M_{in} = \min(M_r, M_t)$ and $M_{ax} = \max(M_r, M_t)$.

III. Joint Power and Signature Optimization

Let $\widetilde{\mathbf{b}} = \left(\widetilde{\mathbf{b}}_{u_1,1}^{\mathrm{T}} \cdots \widetilde{\mathbf{b}}_{u_K,K}^{\mathrm{T}}\right)^{\mathrm{T}}$ denote the vector of soft receiver outputs over all users, where u_k denotes the user to whom signature k is assigned. The soft receiver output corresponding to signature k is

$$\begin{aligned} \widetilde{\mathbf{b}}_{u_k,k} &= \mathbf{G}_{u_k,k}^{\dagger} \mathbf{r}_{u_k} \\ &= \mathbf{G}_{u_k,k}^{\dagger} \mathcal{H}_{u_k} \mathcal{S} \mathbf{T} \mathbf{E}_{M_t}^{\mathrm{T}} \bar{\mathbf{A}} \bar{\mathbf{b}} + \mathbf{G}_{u_k,k}^{\dagger} \mathbf{w}_{u_k} && (7) \end{aligned}$$

where $\mathbf{G}_{u_k,k}$ denotes the receiver filter at mobile station u_k for symbols transmitted on signature k from all M_t transmitter antennas. We therefore have

$$\begin{aligned} \widetilde{\mathbf{b}} &= \begin{pmatrix} \mathbf{G}_{u_1,1}^{\dagger} \mathcal{H}_{u_1} \\ \vdots \\ \mathbf{G}_{u_K,K}^{\dagger} \mathcal{H}_{u_K} \end{pmatrix} \mathcal{S} \mathbf{T} \mathbf{E}_{M_t}^{\mathrm{T}} \bar{\mathbf{A}} \bar{\mathbf{b}} + \begin{pmatrix} \mathbf{G}_{u_1,1}^{\dagger} \mathbf{w}_{u_1} \\ \vdots \\ \mathbf{G}_{u_K,K}^{\dagger} \mathbf{w}_{u_K} \end{pmatrix} \\ &= \mathbf{F}\mathbf{x} + \mathbf{v} && (8) \end{aligned}$$

where

$$\mathbf{F} = \begin{pmatrix} \mathbf{G}_{u_1,1}^{\dagger} \mathcal{H}_{u_1} \mathcal{S} \mathbf{T} \mathbf{E}_{M_t}^{\mathrm{T}} \\ \vdots \\ \mathbf{G}_{u_K,K}^{\dagger} \mathcal{H}_{u_K} \mathcal{S} \mathbf{T} \mathbf{E}_{M_t}^{\mathrm{T}} \end{pmatrix} \qquad (9)$$

$$\mathbf{x} = \bar{\mathbf{A}} \bar{\mathbf{b}} \qquad (10)$$

$$\mathbf{v} = \begin{pmatrix} \mathbf{G}_{u_1,1}^{\dagger} \mathbf{w}_{u_1} \\ \vdots \\ \mathbf{G}_{u_K,K}^{\dagger} \mathbf{w}_{u_K} \end{pmatrix} \qquad (11)$$

The sum mutual information over all users is

$$I(\mathbf{x}; \widetilde{\mathbf{b}}) = \log_2 \left| \left(\mathbf{R}_{\mathbf{xx}}^{-1} + \mathbf{F}^{\dagger} \mathbf{R}_{\mathbf{vv}}^{-1} \mathbf{F} \right) \mathbf{R}_{\mathbf{xx}} \right| \qquad (12)$$

where $\mathbf{R}_{\mathbf{xx}}$ and $\mathbf{R}_{\mathbf{vv}}$ are the covariance matrices for $\mathbf{x}$ and $\mathbf{v}$, respectively, i.e.,

$$\mathbf{R}_{\mathbf{xx}} = \bar{\mathbf{A}}^2 \qquad (13)$$

$$\mathbf{R}_{\mathbf{vv}} = diag\left(\sigma_{u_1}^2 \mathbf{G}_{u_1,1}^{\dagger} \mathbf{G}_{u_1,1} \cdots \sigma_{u_K}^2 \mathbf{G}_{u_K,K}^{\dagger} \mathbf{G}_{u_K,K} \right) \qquad (14)$$

We wish to maximize the mutual information $I(\mathbf{x}; \widetilde{\mathbf{b}})$, given by (12), over $\mathbf{S}$, $\mathbf{T}$, $\mathbf{G}_{u_k,k}$, $k = 1, \cdots, K$, and $\bar{\mathbf{A}}$. This can be achieved through the following procedure [13]–[15]:

1) Choose the signature matrix $\mathbf{S}$, receiver filters $\mathbf{G}_{u_k,k}$, $k = 1, \cdots, K$, and pre-coding filter $\mathbf{T}$ to diagonalize $\mathbf{F}^{\dagger} \mathbf{R}_{\mathbf{vv}}^{-1} \mathbf{F}$; and
2) Optimize the power allocation over signatures and transmit antennas.

Since $\mathbf{H}_{i,j}^u$, is circulant, it can be diagonalized by the DFT matrix $\mathbf{\Phi}$ as

$$\mathbf{H}_{i,j}^u = \mathbf{\Phi} \mathbf{\Lambda}_{i,j}^u \mathbf{\Phi}^{\dagger} \qquad (15)$$

where $\mathbf{\Lambda}_{i,j}^u = \mathrm{diag}(\lambda_{i,j,1}^u \cdots \lambda_{i,j,K}^u)$ and $\lambda_{i,j,k}^u = 1/\sqrt{K} \sum_{l=0}^{L} h_{i,j}^u (L-l) e^{-j2\pi \frac{lk}{K}}$. Let $\Gamma_{u,k}$ be the $M_r \times M_t$ matrix consisting of $\lambda_{i,j,k}^u, 1 \le i \le M_r, 1 \le j \le M_t$, i.e.,

$$\mathbf{\Gamma}_{u,k} = \begin{pmatrix} \lambda_{1,1,k}^u & \cdots & \lambda_{1,M_t,k}^u \\ \vdots & \ddots & \vdots \\ \lambda_{M_r,1,k}^u & \cdots & \lambda_{M_r,M_t,k}^u \end{pmatrix} \qquad (16)$$

which has singular value decomposition

$$\mathbf{\Gamma}_{u,k} = \mathbf{\Theta}_{u,k}\mathbf{\Delta}_{u,k}\mathbf{\Psi}^{\dagger}_{u,k} \tag{17}$$

where $\mathbf{\Theta}_{u,k}$ and $\mathbf{\Psi}_{u,k}$ are $M_r \times M_r$ and $M_t \times M_t$ unitary matrices, respectively; and $\mathbf{\Delta}_{u,k}$ is an $M_r \times M_t$ diagonal matrix with diagonal elements $\delta_{u,k,i} \geq 0$, where $1 \leq i \leq M_{in}$.

Therefore, $\left(\mathbf{R}_{\mathbf{xx}}^{-1} + \mathbf{F}^{\dagger}\mathbf{R}_{\mathbf{vv}}^{-1}\mathbf{F}\right)\mathbf{R}_{\mathbf{xx}}$ can be diagonalized by choosing the signatures, pre-coding filter, and receiver filter as

$$\mathbf{S} = \mathbf{\Phi} \tag{18}$$

$$\mathbf{T} = \mathbf{E}_K \begin{pmatrix} \mathbf{\Psi}_{u_1,1} & & \\ & \ddots & \\ & & \mathbf{\Psi}_{u_K,K} \end{pmatrix} \mathbf{E}_K^{\mathrm{T}} \tag{19}$$

and

$$\mathbf{G}_{u,k} = \begin{pmatrix} \phi_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \phi_k & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \phi_k \end{pmatrix} \mathbf{\Theta}_{u,k} \begin{pmatrix} \mathbf{I}_{M_{in}\times M_{in}} \\ \mathbf{0}_{(M_r - M_{in})\times M_{in}} \end{pmatrix} \tag{20}$$

where $\mathbf{0}$ is $K \times 1$ zero vector.

The problem of signature and power allocation can be formulated as

$$\max_{u_k, P_k} I(\mathbf{x}; \widetilde{\mathbf{b}}) = \sum_{k=1}^{K} I(u_k, P_k) \tag{21}$$

$$\text{subject to} \quad \sum_{k=1}^{K} P_k \leq P \tag{22}$$

$$P_k \geq 0, \; 1 \leq k \leq K \tag{23}$$

where $I(u_k, P_k)$ denotes the mutual information associated with signature, or discrete frequency k, assigned to user u_k with power P_k. With multiple transmit and receive antennas, for each signature k assigned to user u_k there is a set of M_{in} parallel spatial sub-channels with channel gains $\delta_{u_k,i}, 1 \leq i \leq M_{in}$.[2] Hence,

$$I(u_k, P_k) = \sum_{i=1}^{M_{in}} \log_2\left(1 + \frac{\delta^2_{u_k,i}P_{u_k,i}}{\sigma_u^2}\right) \tag{24}$$

where

$$\sum_{i=1}^{M_{in}} P_{u_k,i} \leq P_k. \tag{25}$$

Given P_k, $I(u_k, P_k)$ is maximized by water-pouring over the spatial sub-channels, i.e.,

$$P_{u_k,i} = \left(z - \frac{\sigma_u^2}{\delta^2_{u_k,i}}\right)^+, \quad 1 \leq i \leq M_{in} \tag{26}$$

where $(x)^+$ denotes the maximum of x and 0, and z is the water level, which satisfies (25). To maximize the mutual information $I(\mathbf{x}; \widetilde{\mathbf{b}})$, the signature k should be assigned to the "best" user, i.e., the user with the largest rate $I(u_k, P_k)$. If there is only one (spatial) channel gain associated with signature k, then the best user is simply the one with the largest channel gain [9], [10]. With multiple spatial sub-channels per signature the assignment is complicated by the fact that the user with the best rate (determined by water pouring over each user's set of spatial channels) can change, depending on the amount of power allocated to signature k. This appears to require an exhaustive search over all U^K user assignments.

In order to avoid this exhaustive search, a convex relaxation technique can be applied by assuming that each signature can be time shared among multiple users [16]. Let $w_{u,k}$ denote the fraction of time user u transmits on the kth signature, and let $P_{u,k}$ denote the power assigned to user u on signature k. Then (21) can be recast as

$$\max_{w_{u,k}, P_{u,k}} \quad \sum_{k=1}^{K}\sum_{u=1}^{U} w_{u,k} I(u, \frac{P_{u,k}}{w_{u,k}}) \tag{27}$$

$$\text{s.t.} \quad \sum_{k=1}^{K}\sum_{u=1}^{U} P_{u,k} \leq P \tag{28}$$

$$P_{u,k} \geq 0, \; 1 \leq u \leq U, 1 \leq k \leq K \tag{29}$$

$$\sum_{u=1}^{U} w_{u,k} \leq 1, 1 \leq k \leq K \tag{30}$$

It is easy to verify that the objective in (27) is convex. Hence a local maximum is also the global maximum.

For a given set of $w_{u,k}$'s, the optimal powers $P_{u,k,i}$ can be obtained by a variant of water-pouring, i.e.,

$$P_{u,k,i} = w_{u,k}\left(z - \frac{\sigma_u^2}{\delta^2_{u,k,i}}\right)^+ \tag{31}$$

where z is the water level that satisfies $\sum_{u=1}^{U}\sum_{k=1}^{K}\sum_{i=1}^{M_{in}} P_{u,k,i} \leq P$. Similarly, for a given set of powers $P_{u,k,i}$, the optimal $w_{u,k}$'s satisfy

$$\mu - \sum_{i=1}^{M_{in}} \left[\ln\left(1 + \frac{P_{u,k,i}\delta^2_{u,k,i}}{w_{u,k}\sigma_u^2}\right) - \frac{1}{\frac{w_{u,k}\sigma_u^2}{P_{u,k,i}\delta^2_{u,k,i}} + 1}\right] = 0 \tag{32}$$

where μ is chosen to satisfy $\sum_{u=1}^{U} w_{u,k} \leq 1$. Hence, (27-30) can be solved by iterating (31) and (32) until the sum mutual information converges.

The weight $w_{u,k}$ can be also interpreted as the fraction of the bandwidth associated with signature k, which is allocated to user u. Given a fixed total bandwidth, as K increases, the bandwidth associated with each signature decreases, so that the increase in aggregate rate due to signature-sharing among multiple users also decreases. In the limit as $K \to \infty$, no signature sharing occurs, and (21) and (27) become equivalent. Therefore, for sufficiently large K, assigning signature k to the user with the largest $w_{u,k}$ gives negligible performance loss.

[2] In what follows, the subscript (u_k, i) is used if signature k is assumed to be assigned to a particular user u_k. Otherwise, if the assignment of signature k has not been determined, or if the assignment is not unique, then the subscript (u, k, i) is used.

IV. SUM CAPACITY AT HIGH SNRS

Although it appears to be difficult to calculate the optimal power and signature allocations and the sum capacity in general, it is possible to give analytical characterizations at high SNRs. In what follows, we analyze the achievable sum mutual information for a large number of users and antennas. As for the single receive antenna forward-link models in [9], [10], the sum mutual information $I(\mathbf{x};\tilde{\mathbf{b}})$ maximized over signatures and receivers is the sum capacity. That is, although $I(\mathbf{x};\tilde{\mathbf{b}})$ corresponds to the situation in which the receivers coordinate and jointly decode all transmitted symbols, the optimized transmitted and received signatures are orthogonal. Hence the output of $\mathbf{G}_{u_k,k}$ contains no information about other users' symbols, and the sum rate with single-user decoding is the same as that achieved with joint decoding.

We assume that $\sigma_u^2 = N_0$, i.e., the noise variance is the same across all mobiles on all signatures and receive antennas, and that the channel gains across signatures (discrete frequencies) and pairs of transmit and receive antennas are independent complex Gaussian random variables $CN(0,1)$. Hence, the components of $\mathbf{\Gamma}_{u,k}$ in (16) are independent $CN(0,1)$ random variables.

Combining (24)-(26) gives

$$I(u_k,P_k) = \sum_{n=1}^{N} \log_2 \delta_{u_k,n}^2 + N\log_2\left(\frac{P_k}{N_0 N} + \frac{\sum_{n=1}^{N}\frac{1}{\delta_{u_k,n}^2}}{N}\right) \quad (33)$$

where N denotes the number of spatial sub-channels in use, i.e., no power is allocated to spatial sub-channel $n > N$, and it is assumed without loss of generality that $\delta_{u_k,n} \geq \delta_{u_k,n+1}$ for $1 \leq n \leq M_{in} - 1$.

At high SNRs we have $\frac{P_k}{N_0} \gg \sum_{n=1}^{N}\frac{1}{\delta_{u_k,n}^2}$, and asymptotically (33) becomes

$$I(u_k,P_k) = M_{in}\log_2\left(1 + \frac{\left(\prod_{i=1}^{M_{in}}\delta_{u_k,i}^2\right)^{\frac{1}{M_{in}}}}{M_{in}}\frac{P_k}{N_0}\right), \quad (34)$$

which is M_{in} times the mutual information for signature k with transmit $SNR = \frac{P_k}{N_0}$ and effective channel gain $\left(\prod_{i=1}^{M_{in}}\delta_{u,k,i}^2\right)^{\frac{1}{M_{in}}}/M_{in}$. Let $y_k = \max_{1\leq u\leq U}\left(\prod_{i=1}^{M_{in}}\delta_{u,k,i}^2\right)^{\frac{1}{M_{in}}}/M_{in}$. The maximum sum mutual information, i.e., the solution to (21 - 23) at high SNR, is

$$\max I(\mathbf{x};\widetilde{\mathbf{b}}) = M_{in}\sum_{k=1}^{K}\log_2\left(1 + \frac{P_k}{N_0}y_k\right) \quad (35)$$

$$P_k = \left(z - \frac{N_0}{y_k}\right)^+ \quad (36)$$

where z is chosen to satisfy (22). Therefore, the transmit power per signature is

$$P = \frac{1}{K}\sum_{k=1}^{K}(z - \frac{N_0}{y_k})^+ \quad (37)$$

and the sum mutual information per signature per spatial sub-channel is

$$R = \frac{\max I(\mathbf{x};\widetilde{\mathbf{b}})}{KM_{in}} = \frac{1}{K}\sum_{k=1}^{K}\log_2\left[\max\left(\frac{y_k}{N_0}z, 1\right)\right] \quad (38)$$

As the number of signatures $K \to \infty$, the empirical distribution of y_k over signatures approaches its probability density function (pdf) $p_Y(y)$. Hence as $K \to \infty$, P and R converge in probability to

$$P = \int_{\frac{N_0}{z}}^{\infty}\left(z - \frac{N_0}{y}\right)p_Y(y)\,dy \quad (39)$$

and

$$R = \int_{\frac{N_0}{z}}^{\infty}\log_2\left(\frac{y}{N_0}z\right)p_Y(y)\,dy. \quad (40)$$

Let $y'_{u,k} = \left(\prod_{i=1}^{M_{in}}\delta_{u,k,i}^2\right)^{\frac{1}{M_{in}}}/M_{in}$ and $\bar{y}_{u,k} = \ln y'_{u,k}$. As $M_{in} \to \infty$, the distribution of $\bar{y}_{u,k}$ converges to a normal distribution $\mathcal{N}(\mu,\sigma^2)$ [17] with

$$\mu = -\mathcal{C} + \sum_{i=1}^{M_{ax}-M_{in}}\frac{1}{i} + \frac{1}{M_{in}}\sum_{i=1}^{M_{in}-1}\frac{i}{M_{ax}-i} - \ln M_{in} \quad (41)$$

$$\sigma^2 = \frac{1}{M_{in}}\left(\frac{\pi^2}{6} - \sum_{i=1}^{M_{ax}-1}\frac{1}{i^2}\right) + \frac{1}{M_{in}^2}\sum_{i=1}^{M_{in}-1}\frac{i}{(M_{ax}-M_{in}+i)^2} \quad (42)$$

where $\mathcal{C} = 0.5772$ is Euler's constant. Hence, $y'_{u,k}$ has a log-normal distribution.

We now characterize the behavior of the sum mutual information per signature with a large number of users. From extreme value theory [12, Sec. 2.3.3], as the number of users $U \to \infty$, the distribution of $y_k = \max_{1\leq u\leq U} y'_{u,k}$ satisfies

$$\lim_{U\to\infty}\text{Prob}(y < \mu_U + \beta_U x) = e^{-e^{-x}} \quad (43)$$

where μ_U and β_U are determined as

$$\mu_U = e^{\sqrt{2\ln U} - \frac{\ln\ln U + \ln 4\pi}{2\sqrt{2\ln U}}\sigma + \mu} \quad (44)$$

$$\beta_U = \frac{\sigma}{\sqrt{2\ln U}}\mu_U \quad (45)$$

Let ρ_y and σ_y denote the mean and standard deviation of y_k, respectively. Then,

$$\rho_y = \mu_U + \mathcal{C}\beta_U \quad (46)$$

$$\sigma_y = \frac{\beta_U\pi}{\sqrt{6}}. \quad (47)$$

Let

$$\bar{R} = \log_2\left(\frac{P}{N_0}\rho_y\right) \quad (48)$$

Theorem 1:

$$\lim_{U\to\infty}(R - \bar{R}) = 0. \quad (49)$$

That is, the difference between R and $\bar{R}$ can be made arbitrarily small by increasing the number of users. For large U it therefore suffices to study the behavior of $\bar{R}$.

$$\bar{R}_a = \log_2 e \left\{ \ln \frac{P}{N_0} - \ln M_{in} + \sum_{i=1}^{M_{ax}-M_{in}} \frac{1}{i} + \frac{1}{M_{in}} \sum_{i=1}^{M_{in}-1} \frac{i}{M_{ax}-i} + \sqrt{2 \ln U \left[\frac{1}{M_{in}} \left(\frac{\pi^2}{6} - \sum_{i=1}^{M_{ax}-1} \frac{1}{i^2} \right) + \frac{1}{M_{in}^2} \sum_{i=1}^{M_{in}-1} \frac{i}{(M_{ax}-M_{in}+i)^2} \right]} \right\} \quad (50)$$

Substituting (41-42) and (44-46) into (48), it follows that $\lim_{U,M_{ax},M_{in}\to\infty} \bar{R}/\bar{R}_a = 1$, where $\bar{R}_a$ is defined in (50). (This is independent of the way in which each of the variables tends to infinity.) Also, the sum mutual information per signature $R' = M_{in}R$ satsfies $\lim_{U,M_{ax},M_{in}\to\infty} R'/(M_{in}\bar{R}_a) = 1$. Letting $\alpha = M_{ax}/M_{in}$, for large M_{in} (41) and (42) can be rewritten as [17]

$$\mu = 2\ln\left(\sqrt{\alpha}+1\right) + (\alpha+1)\ln\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha}+1}\right) + (\alpha-1)\ln\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha}-1}\right) \quad (51)$$

$$\sigma^2 = \frac{1}{M_{in}^2} \ln\left(\frac{\alpha}{\alpha-1}\right) \quad (52)$$

and (50) can be rewritten as (53) on the next page.

We note that when there are multiple transmit and receive antennas, R' increases as $O(\ln U)$, compared with the $O(\ln \ln U)$ increase when there is only one receive antenna [9], [10]. When $M_t = M_r = M$, (hence $M_{ax} = M_{in} = M$), (53) becomes

$$M_{in}\bar{R}_a = M\left[\ln\frac{P}{N_0} - \ln M\right] + \sum_{i=1}^{M-1}\frac{i}{M-i} + \sqrt{2\ln U\left[M\left(\frac{\pi^2}{6} - \sum_{i=1}^{M-1}\frac{1}{i^2}\right) + \sum_{i=1}^{M-1}\frac{1}{i}\right]} \quad (54)$$

Furthermore, with $\alpha = 1$ we have $\mu = 0$,

$$\sigma^2 = \frac{\ln M + \mathcal{C} + 1}{M^2}, \quad (55)$$

and (53) becomes

$$M\bar{R}_a = M \ln\frac{P}{N_0} + \sqrt{2\ln U \ln M} \quad (56)$$

Figure 1 shows the sum mutual information per signature R' versus SNR with $U = 30$ users. Different curves are shown for different numbers of transmit and receive antennas. The asymptotic curves are computed from (50), and the numerical results are obtained by solving the convex optimization problem (27-30), which is used to determine the assignment of signatures to users, and using (33) to compute the rate for each signature. The asymptotic and numerical results converge as the SNR increases. The figure shows that for $M = 4$ and $M = 8$ the asymptotic results accurately approximate the numerical results for $SNR \geq 10$ dB.

Figure 2 shows the sum mutual information per signature R' versus the number of transmit antennas M_t with $U = 10$ and $M_r = 4$. Different curves are shown for different SNRs. The asymptotic results accurately predict the numerical results.

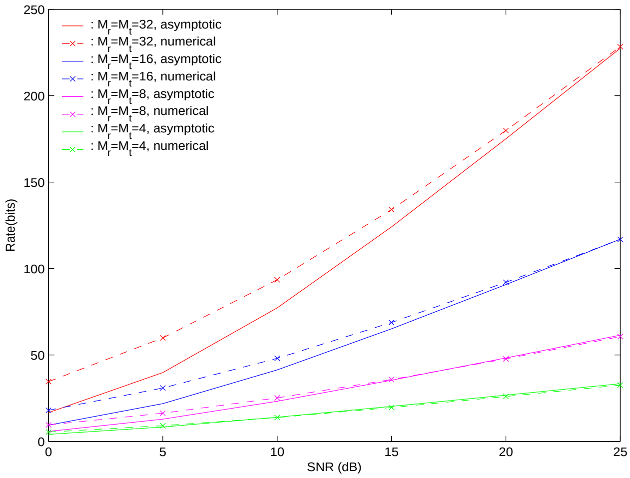

Fig. 1. Sum mutual information per signature versus SNR with 30 users.

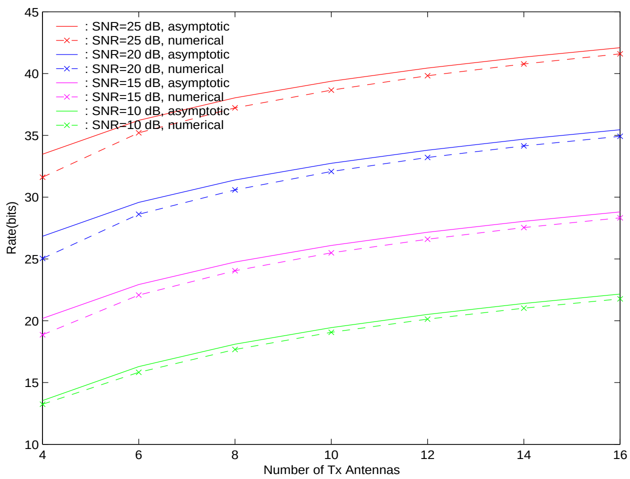

Fig. 2. Sum mutual information per signature versus the number of transmit antennas with 10 users and $M_r = 4$.

V. Conclusions

Signatures and powers have been jointly optimized for a frequency-selective, forward-link CDMA model with multiple transmit and receive antennas, and linear receivers. The optimal signatures lead to multi-carrier signaling on each transmit antenna. The assignment of carriers to users is complicated

$$
\begin{aligned}
M_{in}\bar{R}_a &= \log_2 e \left\{ M_{in} \ln \frac{P}{N_0} + \sqrt{2 \ln U \ln M_{in}} + \right. \\
&\left. M_{in} \left[2 \ln\left(\sqrt{\alpha}+1\right) + (\alpha+1)\ln\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha}+1}\right) + (\alpha-1)\ln\left(\frac{\sqrt{\alpha}}{\sqrt{\alpha}-1}\right) \right] \right\} \qquad (53)
\end{aligned}
$$

by the property that the distribution of powers over the spatial modes, and the corresponding rate, depend on the power assigned to each signature, which in turn depends on the carrier assignment. The optimal solution can be efficiently approximated by allowing time-sharing of signatures.

To gain further insight into the dependence of the sum capacity on system parameters, we have analyzed the sum capacity at high SNRs, and for a large number of users and antennas. The assumption that the channel gains are *i.i.d.* across signatures (discrete frequencies) and pairs of antennas enables a characterization of the asymptotic growth rate of the sum capacity with users and antennas. Namely, the sum capacity grows as $M_{in} \log\ (P/N_0) + \sqrt{2 \log\ U \log\ M_{in}}$. Comparisons of the asymptotic results for high SNRs with numerical optimization of signatures and powers show that the asymptotic results are accurate at moderate SNRs with a moderate number of antennas.

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