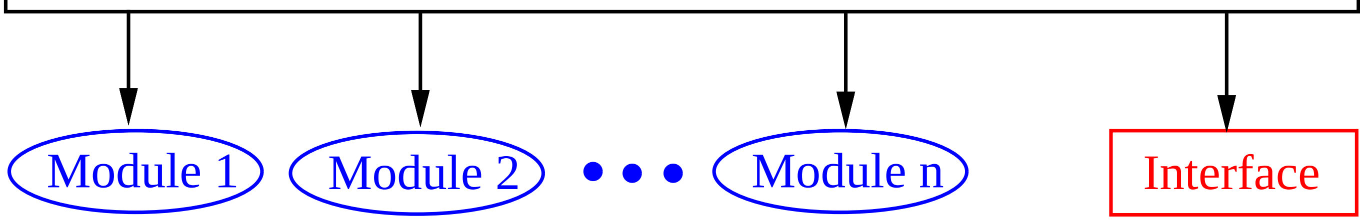
Partitioning

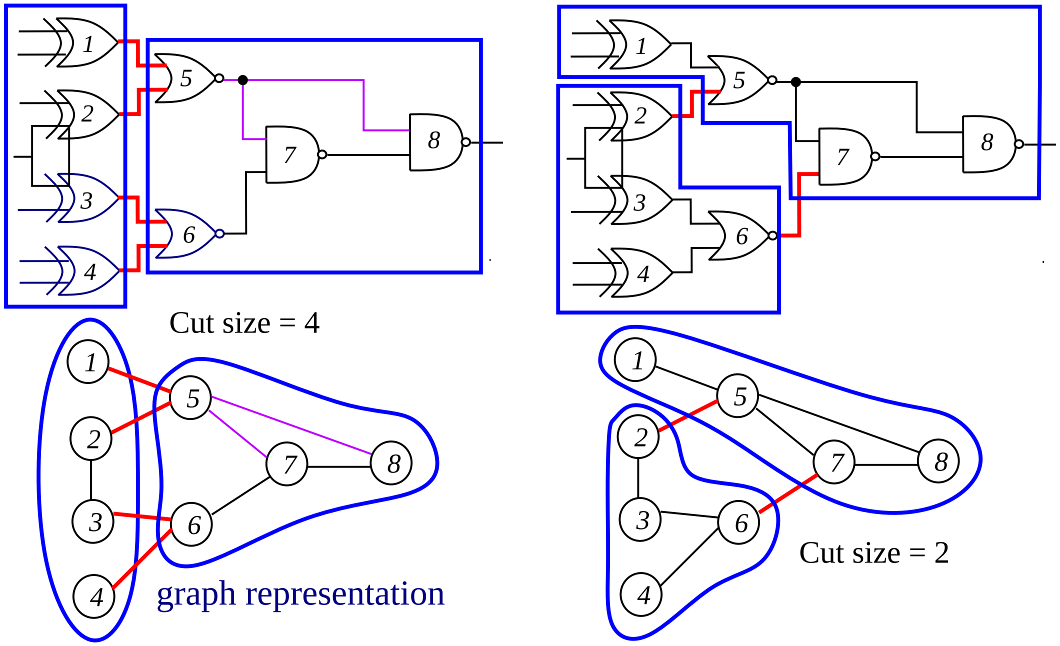
system design

- Decomposition of a complex system into smaller subsystems.
- Each subsystem can be designed independently speeding up the design process.
- Decomposition scheme has to minimize the interconnections among the subsystems.
- Decomposition is carried out hierarchically until each subsystem is of managable size.



Circuit Partitioning

- **Objective:** Partition a circuit into parts such that every component is within a prescribed range and the # of connections among the components is minimized.
  - More constraints are possible for some applications.
- Cutset? Cut size? Size of a component?



Problem Definition: Partitioning

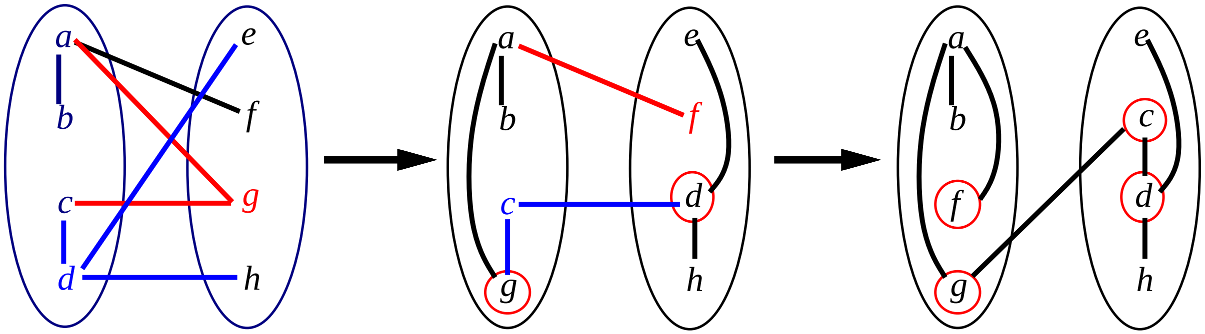
- k-**way partitioning:** Given a graph $G(V, E)$, where each vertex $v \in V$ has a **size** $s(v)$ and each edge $e \in E$ has a **weight** $w(e)$, the problem is to divide the set V into k disjoint subsets V_1, V_2, ..., V_k, such that an objective function is optimized, subject to certain constraints.
- **Bounded size constraint:** The size of the i-th subset is bounded by B_i ($\sum_{v \in V_i} s(v) \leq B_i$).
  - Is the partition balanced?
- **Min-cut cost between two subsets:** Minimize $\sum_{\forall e=(u,v) \wedge p(u) \neq p(v)} w(e)$, where $p(u)$ is the partition # of node u.
- The 2-way, balanced partitioning problem is NP-complete, even in its simple form with identical vertex sizes and unit edge weights.

Kernighan-Lin Algorithm

- Kernighan and Lin, “An efficient heuristic procedure for partitioning graphs,” The Bell System Technical Journal, vol. 49, no. 2, Feb. 1970.
- An **iterative**, **2-way**, **balanced** partitioning (bi-sectioning) heuristic.
- Till the cut size keeps decreasing
  - Vertex pairs which give the largest decrease or the smallest increase in cut size are exchanged.
  - These vertices are then **locked** (and thus are prohibited from participating in any further exchanges).
  - This process continues until all the vertices are locked.

Kernighan-Lin Algorithm: A Simple Example

- Each edge has a unit weight.



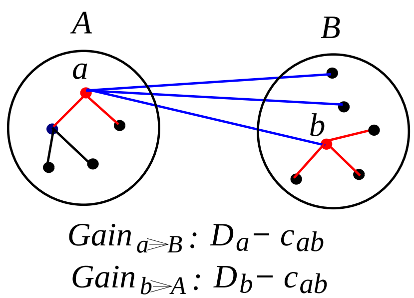
Step #	Vertex pair	Cost reduction	Cut cost
0	-	0	5
1	{d, g}	3	2
2	{c, f}	1	1
3	{b, h}	-2	3
4	{a, e}	-2	5

- Questions: How to compute cost reduction? What pairs to be swapped?
  - Consider the change of internal & external connections.

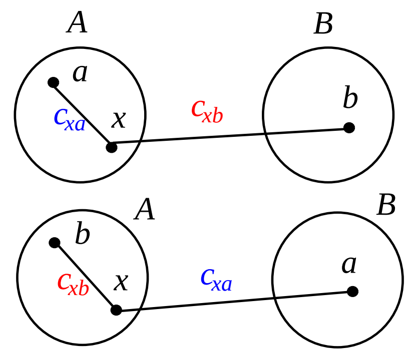
Properties

- Two sets A and B such that $|A| = n = |B|$ and $A \cap B = \emptyset$.
- **External cost** of $a \in A$: $E_a = \sum_{v \in B} c_{av}$.
- **Internal cost** of $a \in A$: $I_a = \sum_{v \in A} c_{av}$.
- D-value of a vertex a: $D_a = E_a - I_a$ (cost reduction for moving a).
- Cost reduction (gain) for swapping a and b: $g_{ab} = D_a + D_b - 2c_{ab}$.
- If $a \in A$ and $b \in B$ are interchanged, then the new D-values, D', are given by

$$
\begin{aligned}
D'_x &= D_x + 2c_{xa} - 2c_{xb}, \forall x \in A - \{a\} \\
D'_y &= D_y + 2c_{yb} - 2c_{ya}, \forall y \in B - \{b\}.
\end{aligned}
$$



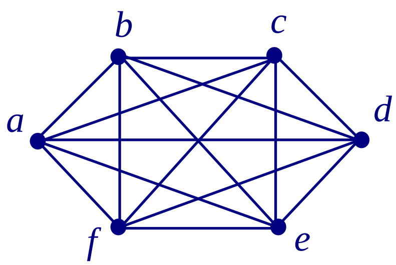
Internal cost vs. External cost



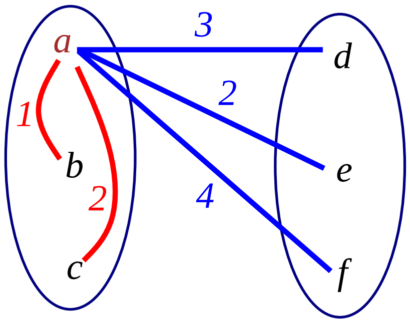
before swap	after swap	$\triangle C$
$-c_{xa}$	$+c_{xa}$	$+2c_{xa}$
$+c_{xb}$	$-c_{xb}$	$-2c_{xb}$

updating D−values

Kernighan-Lin Algorithm: A Weighted Example



	a	b	c	d	e	f
a	0	1	2	3	2	4
b	1	0	1	4	2	1
c	2	1	0	3	2	1
d	3	4	3	0	4	3
e	2	2	2	4	0	2
f	4	1	1	3	2	0



costs associated with a

Initial cut cost = (3+2+4)+(4+2+1)+(3+2+1) = 22

- Iteration 1:

$I_a = 1 + 2 = 3;$ $E_a = 3 + 2 + 4 = 9;$ $D_a = E_a - I_a = 9 - 3 = 6$

$I_b = 1 + 1 = 2;$ $E_b = 4 + 2 + 1 = 7;$ $D_b = E_b - I_b = 7 - 2 = 5$

$I_c = 2 + 1 = 3;$ $E_c = 3 + 2 + 1 = 6;$ $D_c = E_c - I_c = 6 - 3 = 3$

$I_d = 4 + 3 = 7;$ $E_d = 3 + 4 + 3 = 10;$ $D_d = E_d - I_d = 10 - 7 = 3$

$I_e = 4 + 2 = 6;$ $E_e = 2 + 2 + 2 = 6;$ $D_e = E_e - I_e = 6 - 6 = 0$

$I_f = 3 + 2 = 5;$ $E_f = 4 + 1 + 1 = 6;$ $D_f = E_f - I_f = 6 - 5 = 1$

Weighted Example (cont'd)

- Iteration 1:

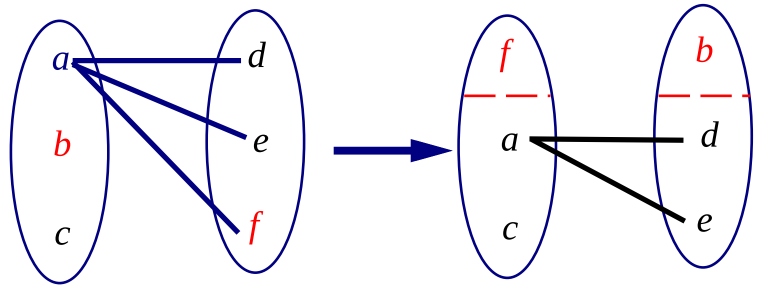
$$\begin{array}{lll} I_a = 1+2 = 3; & E_a = 3+2+4 = 9; & D_a = E_a - I_a = 9-3 = 6 \\ I_b = 1+1 = 2; & E_b = 4+2+1 = 7; & D_b = E_b - I_b = 7-2 = 5 \\ I_c = 2+1 = 3; & E_c = 3+2+1 = 6; & D_c = E_c - I_c = 6-3 = 3 \\ I_d = 4+3 = 7; & E_d = 3+4+3 = 10; & D_d = E_d - I_d = 10-7 = 3 \\ I_e = 4+2 = 6; & E_e = 2+2+2 = 6; & D_e = E_e - I_e = 6-6 = 0 \\ I_f = 3+2 = 5; & E_f = 4+1+1 = 6; & D_f = E_f - I_f = 6-5 = 1 \end{array}$$

- $g_{xy} = D_x + D_y - 2c_{xy}$.

$$\begin{aligned} g_{ad} &= D_a + D_d - 2c_{ad} = 6+3-2\times 3 = 3 \\ g_{ae} &= 6+0-2\times 2 = 2 \\ g_{af} &= 6+1-2\times 4 = -1 \\ g_{bd} &= 5+3-2\times 4 = 0 \\ g_{be} &= 5+0-2\times 2 = 1 \\ g_{bf} &= 5+1-2\times 1 = 4 \ (maximum) \\ g_{cd} &= 3+3-2\times 3 = 0 \\ g_{ce} &= 3+0-2\times 2 = -1 \\ g_{cf} &= 3+1-2\times 1 = 2 \end{aligned}$$

- Swap b and f! ($\hat{g}_1 = 4$)

Weighted Example (cont'd)



- $D'_x = D_x + 2c_{xp} - 2c_{xq}, \forall x \in A - \{p\}$ (swap p and q, $p \in A$, $q \in B$)

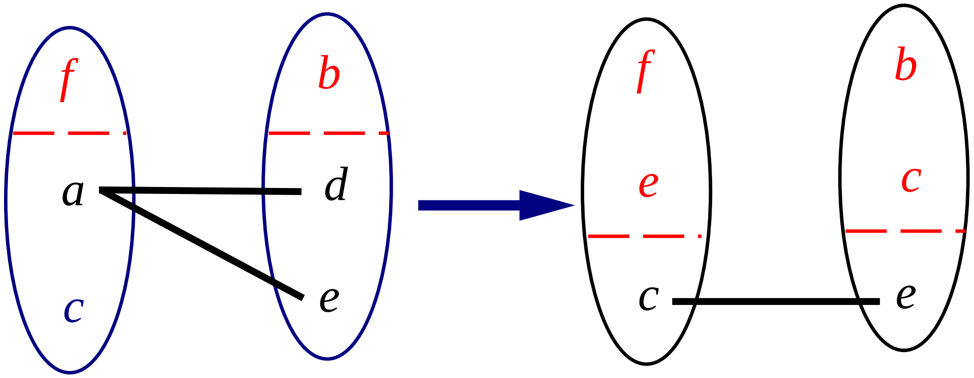
$$\begin{aligned} D'_a &= D_a + 2c_{ab} - 2c_{af} = 6 + 2 \times 1 - 2 \times 4 = 0 \\ D'_c &= D_c + 2c_{cb} - 2c_{cf} = 3 + 2 \times 1 - 2 \times 1 = 3 \\ D'_d &= D_d + 2c_{df} - 2c_{db} = 3 + 2 \times 3 - 2 \times 4 = 1 \\ D'_e &= D_e + 2c_{ef} - 2c_{eb} = 0 + 2 \times 2 - 2 \times 2 = 0 \end{aligned}$$

- $g_{xy} = D'_x + D'_y - 2c_{xy}$.

$$\begin{aligned} g_{ad} &= D'_a + D'_d - 2c_{ad} = 0 + 1 - 2 \times 3 = -5 \\ g_{ae} &= D'_a + D'_e - 2c_{ae} = 0 + 0 - 2 \times 2 = -4 \\ g_{cd} &= D'_c + D'_d - 2c_{cd} = 3 + 1 - 2 \times 3 = -2 \\ g_{ce} &= D'_c + D'_e - 2c_{ce} = 3 + 0 - 2 \times 2 = -1 \ (maximum) \end{aligned}$$

- Swap c and e! ($\hat{g}_2 = -1$)

Weighted Example (cont'd)



- $D''_x = D'_x + 2c_{xp} - 2c_{xq}, \forall x \in A - \{p\}$

$$\begin{aligned} D''_a &= D'_a + 2c_{ac} - 2c_{ae} = 0 + 2 \times 2 - 2 \times 2 = 0 \\ D''_d &= D'_d + 2c_{de} - 2c_{dc} = 1 + 2 \times 4 - 2 \times 3 = 3 \end{aligned}$$

- $g_{xy} = D''_x + D''_y - 2c_{xy}$.

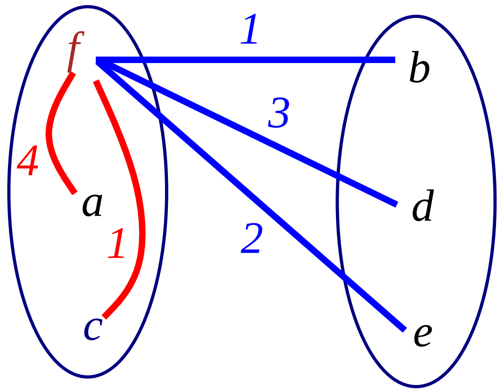
$$g_{ad} = D''_a + D''_d - 2c_{ad} = 0 + 3 - 2 \times 3 = -3 (\hat{g}_3 = -3)$$

- Note that this step is redundant ($\sum_{i=1}^{n} \hat{g}_i = 0$).

- Summary: $\widehat{g}_1 = g_{bf} = 4$, $\widehat{g}_2 = g_{ce} = -1$, $\widehat{g}_3 = g_{ad} = -3$.
- Largest partial sum $\max \sum_{i=1}^{k} \widehat{g}_i = 4$ $(k = 1) \Rightarrow$ Swap b and f.

Weighted Example (cont'd)

	a	b	c	d	e	f
a	0	1	2	3	2	4
b	1	0	1	4	2	1
c	2	1	0	3	2	1
d	3	4	3	0	4	3
e	2	2	2	4	0	2
f	4	1	1	3	2	0



Initial cut cost = (1+3+2)+(1+3+2)+(1+3+2) = 18 (22−4)

- Iteration 2: Repeat what we did at Iteration 1 (Initial cost$= 22 - 4 = 18$).
- Summary: $\widehat{g_1} = g_{ce} = -1$, $\widehat{g_2} = g_{ab} = -3$, $\widehat{g_3} = g_{fd} = 4$.
- Largest partial sum $=$ max $\sum_{i=1}^{k} \widehat{g}_i = 0$ $(k = 3) \Rightarrow$ Stop!

Algorithm: Kernighan-Lin(G)
Input: $G = (V, E), |V| = 2n$.
Output: Balanced bi-partition A and B with ''small'' cut cost.

begin
Bipartition G into A and B such that $|V_A| = |V_B|$, $V_A \cap V_B = \emptyset$, and $V_A \cup V_B = V$.
repeat
 Compute D_v, $\forall v \in V$.
 for $i = 1$ **to** n **do**
  Find a pair of unlocked vertices $v_{ai} \in V_A$ and $v_{bi} \in V_B$ whose exchange makes the largest decrease or smallest increase in cut cost;
  Mark v_{ai} and v_{bi} as locked, store the gain $\widehat{g}_i$, and compute the new D_v, for all unlocked $v \in V$;
 Find k, such that $G_k = \sum_{i=1}^{k} \widehat{g}_i$ is maximized;
 if $G_k > 0$ **then**
  Move $v_{a1}, \ldots, v_{ak}$ from V_A to V_B and $v_{b1}, \ldots, v_{bk}$ from V_B to V_A;
 Unlock v, $\forall v \in V$.
until $G_k \leq 0$;
end

Time Complexity

- Line 4: Initial computation of D: $O(n^2)$
- Line 5: The **for**-loop: $O(n)$
- The body of the loop: $O(n^2)$.
  - Lines 6–7: Step i takes $(n - i + 1)^2$ time.
- Lines 4–11: Each pass of the repeat loop: $O(n^3)$.
- Suppose the repeat loop terminates after r passes.
- The total running time: $O(rn^3)$.

Extensions of K-L Algorithm

- **Unequal sized subsets** (assume $n_1 < n_2$)
  1. Partition: $|A| = n_1$ and $|B| = n_2$.
  2. Add $n_2 - n_1$ dummy vertices to set A. Dummy vertices have no connections to the original graph.
  3. Apply the Kernighan-Lin algorithm.
  4. Remove all dummy vertices.

- **Unequal sized "vertices"**
  1. Assume that the smallest "vertex" has unit size.
  2. Replace each vertex of size s with s vertices which are fully connected with edges of infinite weight.
  3. Apply the Kernighan-Lin algorithm.

- **k-way partition**
  1. Partition the graph into k equal-sized sets.
  2. Apply the Kernighan-Lin algorithm for each pair of subsets.
  3. Time complexity? Can be reduced by recursive bi-partition.

A "Better" Implementation of K-L Algorithm

- Sort the D-values in a non-increasing order:
  $D_{a_1} \geq D_{a_2} \geq \ldots \geq D_{a_n}$
  $D_{b_1} \geq D_{b_2} \geq \ldots \geq D_{b_n}$

- Start with a_1, compute $g_{a_1,b_i}, \forall b_i$
  Start with a_2, compute $g_{a_2,b_i}, \forall b_i$
  $\vdots$
  whenever $D_{a_i} + D_{b_j} \leq$ Maximum gain found so far (Quit!).

- Partition $A = \{a, b, c\}$: $D_a = 6$; $D_b = 5$; $D_c = 3$;
  Partition $B = \{d, e, f\}$: $D_d = 3$; $D_f = 1$; $D_e = 0$;
  Compute g's
  $g_{ad} = 3 \rightarrow g_{af} = -1 \rightarrow g_{ae} = 2$
  $g_{bd} = 0 \rightarrow g_{bf} = 4 \rightarrow g_{be} = 1$
  $g_{cd} = 0 \rightarrow$ No need to compute g_{cf} (Quit!)
  since $D_c + D_f \leq g_{bf} = 4$.

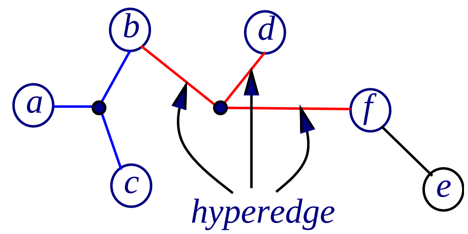
- Note that the overall time complexity remains $O(rn^3)$.

Drawbacks of the Kernighan-Lin Heuristic

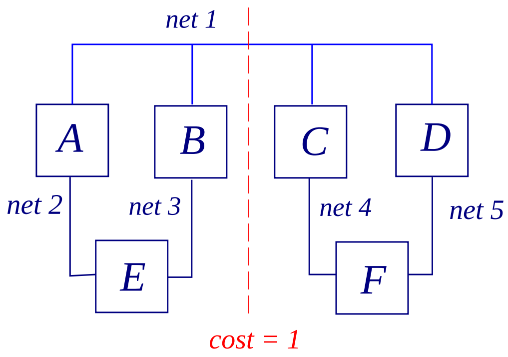
- The K-L heuristic **handles only unit vertex weights**.
  - Vertex weights might represent block sizes, different from blocks to blocks.
  - Reducing a vertex with weight $w(v)$ into a clique with $w(v)$ vertices and edges with a high cost increases the size of the graph substantially.
- The K-L heuristic **handles only exact bisections**.
  - Need dummy vertices to handle the unbalanced problem.
- The K-L heuristic **cannot handle hypergraphs**.
  - Need to handle multi-terminal nets directly.
- The **time complexity of a pass is high**, $O(n^3)$.

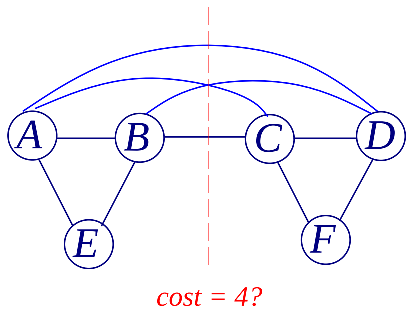
Coping with Hypergraph

- A hypergraph $H = (N, L)$ consists of a set N of vertices and a set L of hyperedges, where each hyperedge corresponds to a **subset** N_i of distinct vertices with $|N_i| \geq 2$.



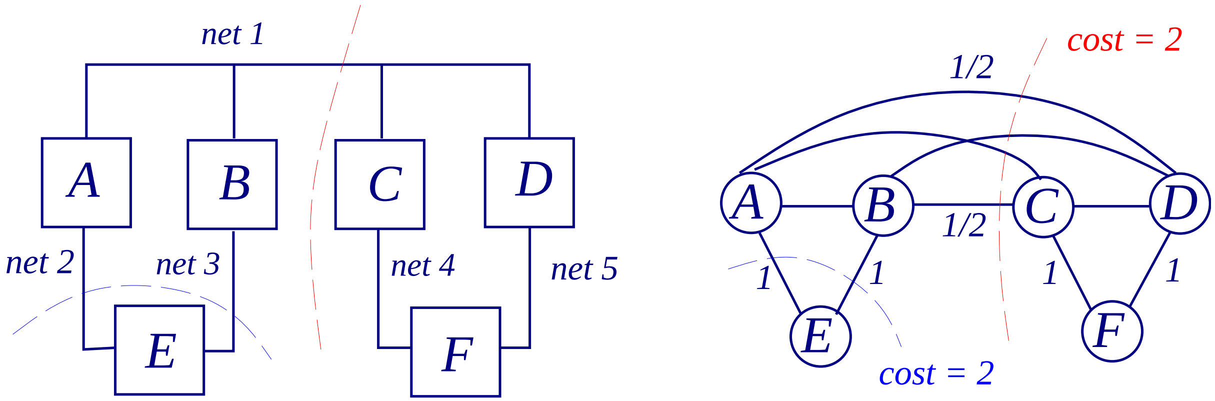
- Schweikert and Kernighan, "A proper model for the partitioning of electrical circuits," 9th Design Automation Workshop, 1972.
- For multi-terminal nets, **net cut** is a more accurate measurement for cut cost (i.e., deal with hyperedges).
  - $\{A, B, E\}, \{C, D, F\}$ is a good partition.
  - Should not assign the same weight for all edges.


net 1
A
B
C
D
net 2
net 3
net 4
net 5
E
F
cost = 1

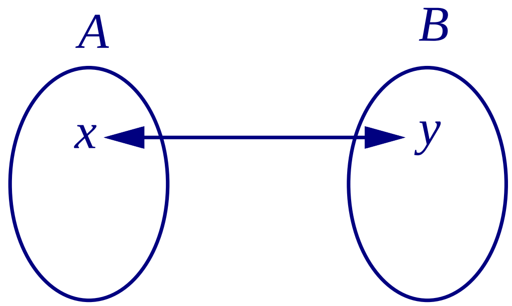

A
B
C
D
E
F
cost = 4?

Net-Cut Model

- Let $n(i)$ = # of cells associated with Net i.
- Edge weight $w_{xy} = \frac{2}{n(i)}$ for an edge connecting cells x and y.



- Easy modification of the K-L heuristic.



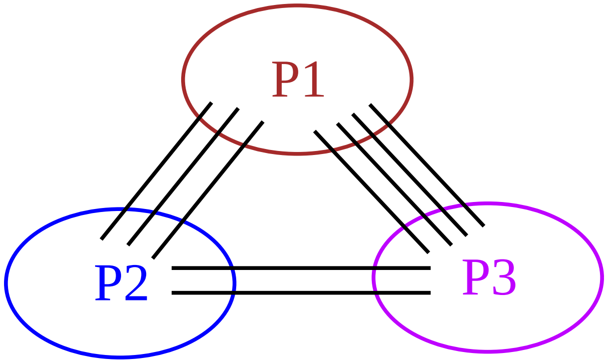
D_x: *gain in moving x to B*

D_y: *gain in moving y to A*

$$g_{xy} = D_x + D_y - \mathit{Correction}(x, y)$$

Network Flow Based Partitioning

- Min-cut balanced partitioning: Yang and Wong, ICCAD-94.
  - Based on max-flow min-cut theorem.

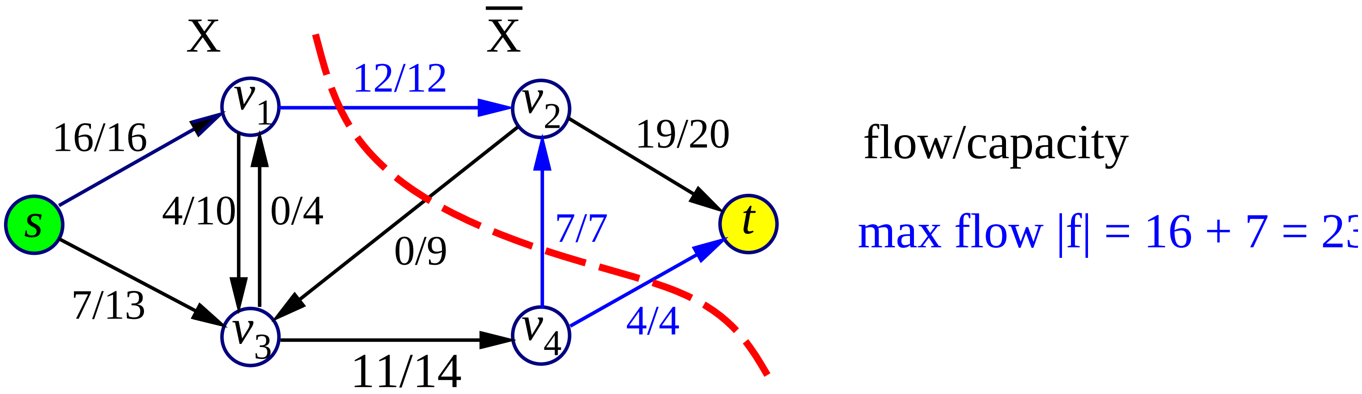


- Gate replication for partitioning: Yang and Wong, ICCAD-95.
- Performance-driven multiple-chip partitioning: Yang and Wong, FPGA'94, ED&TC-95.
- Multi-way partitioning with area and pin constraints: Liu and Wong, ISPD-97.
- Multi-resource partitioning: Liu, Zhu, and Wong, FPGA-98.
- Partitioning for time-multiplexed FPGAs: Liu and Wong, ICCAD-98.

Flow Networks

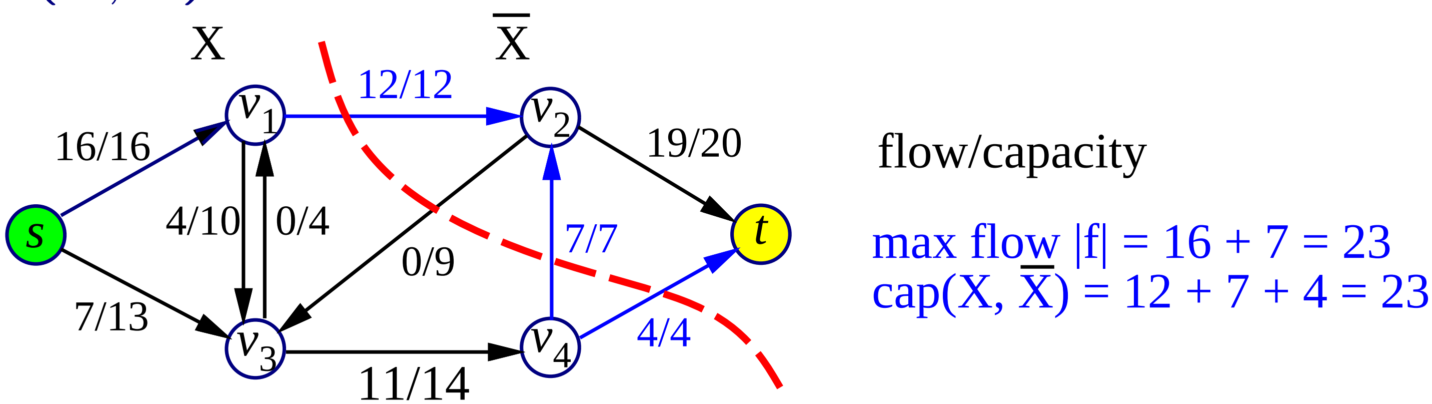
- A **flow network** $G = (V, E)$ is a **directed** graph in which each edge $(u, v) \in E$ has a **capacity** $c(u, v) > 0$.
- There is exactly one node with no incoming (outgoing) edges, called the **source** s (**sink** t).
- A **flow** $f : V \times V \to R$ satisfies
  - **Capacity constraint:** $f(u, v) \leq c(u, v), \forall u, v \in V$.
  - **Skew symmetry:** $f(u, v) = -f(v, u), \forall u, v \in V$.
  - **Flow conservation:** $\sum_{v \in V} f(u, v) = 0, \forall u \in V - \{s, t\}$.
- The **value** of a flow f: $|f| = \sum_{v \in V} f(s, v) = \sum_{v \in V} f(v, t)$

- **Maximum-flow problem:** Given a flow network G with source s and sink t, find a flow of maximum value from s to t.



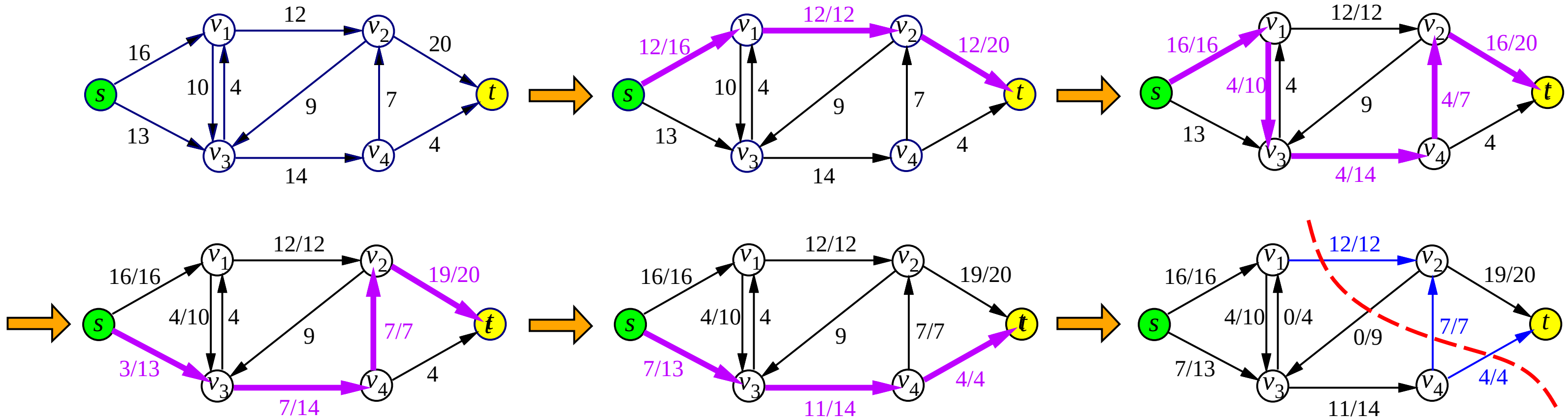
Max-Flow Min-Cut

- A **cut** $(X, \bar{X})$ of flow network $G = (V, E)$ is a partition of V into X and $\bar{X} = V - X$ such that $s \in X$ and $t \in \bar{X}$.
  - **Capacity of a cut:** $cap(X, \bar{X}) = \sum_{u \in X, v \in \bar{X}} c(u, v)$. (Count only **forward** edges!)
- **Max-flow min-cut theorem** Ford & Fulkerson, 1956.
  - f is a max-flow $\iff |f| = cap(X, \bar{X})$ for some min-cut $(X, \bar{X})$.



Network Flow Algorithms

- An **augmenting path** p is a simple path from s to t with the following properties:
  - For every edge $(u, v) \in E$ on p in the **forward** direction (a **forward edge**), we have $f(u, v) < c(u, v)$.
  - For every edge $(u, v) \in E$ on p in the **reverse** direction (a **backward edge**), we have $f(u, v) > 0$.
- f is a max-flow $\iff$ no more augmenting path.



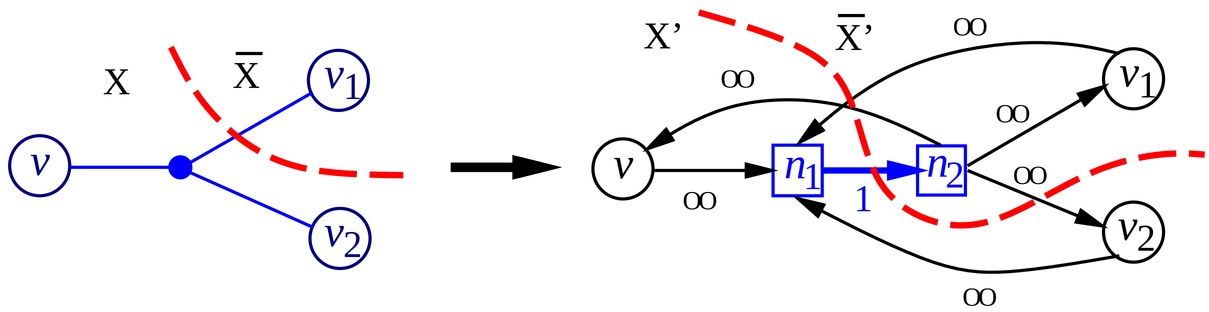
- First algorithm by Ford & Fulkerson in 1959: $O(|E||f|)$; First **polynomial-time** algorithm by Edmonds & Karp in 1969: $O(|E|^2|V|)$; Goldberg & Tarjan in 1985: $O(|E||V| \lg(|V|^2/|E|))$, etc.

Network Flow Based Partitioning

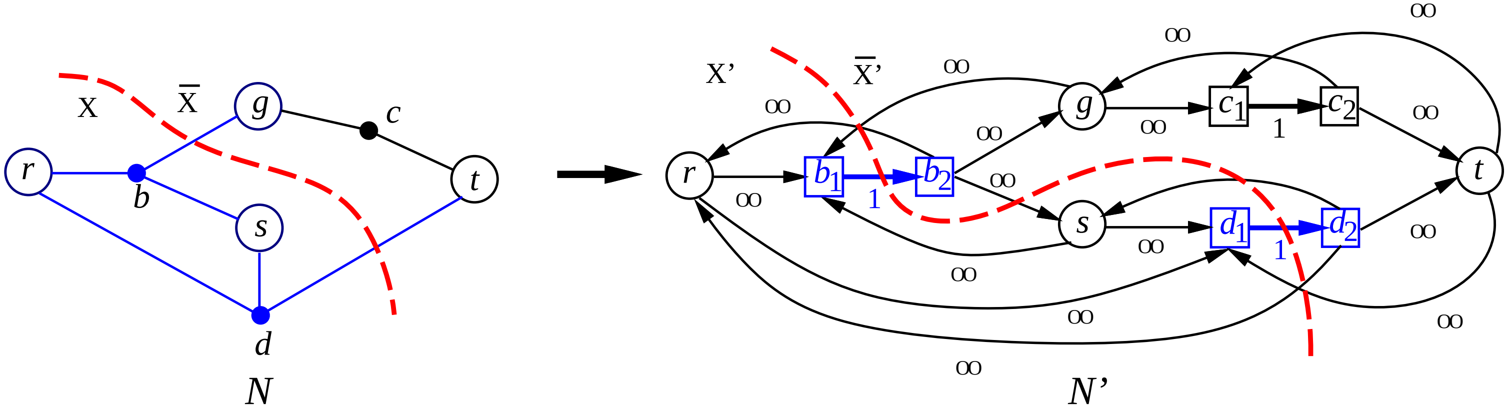
- Why was the technique not wisely used in partitioning?
  - Works on graphs, not hypergraphs.
  - Results in unbalanced partitions; repeated min-cut for balance: $|V|$ max-flows, time-consuming!
- Yang & Wong, ICCAD-94.
  - Exact **net** modeling by flow network.
  - Optimal algorithm for min-net-cut bipartition (unbalanced).
  - Efficient implementation for repeated min-net-cut: same asymptotic time as **one** max-flow computation.

Min-Net-Cut Bipartition

- Net modeling by flow network:



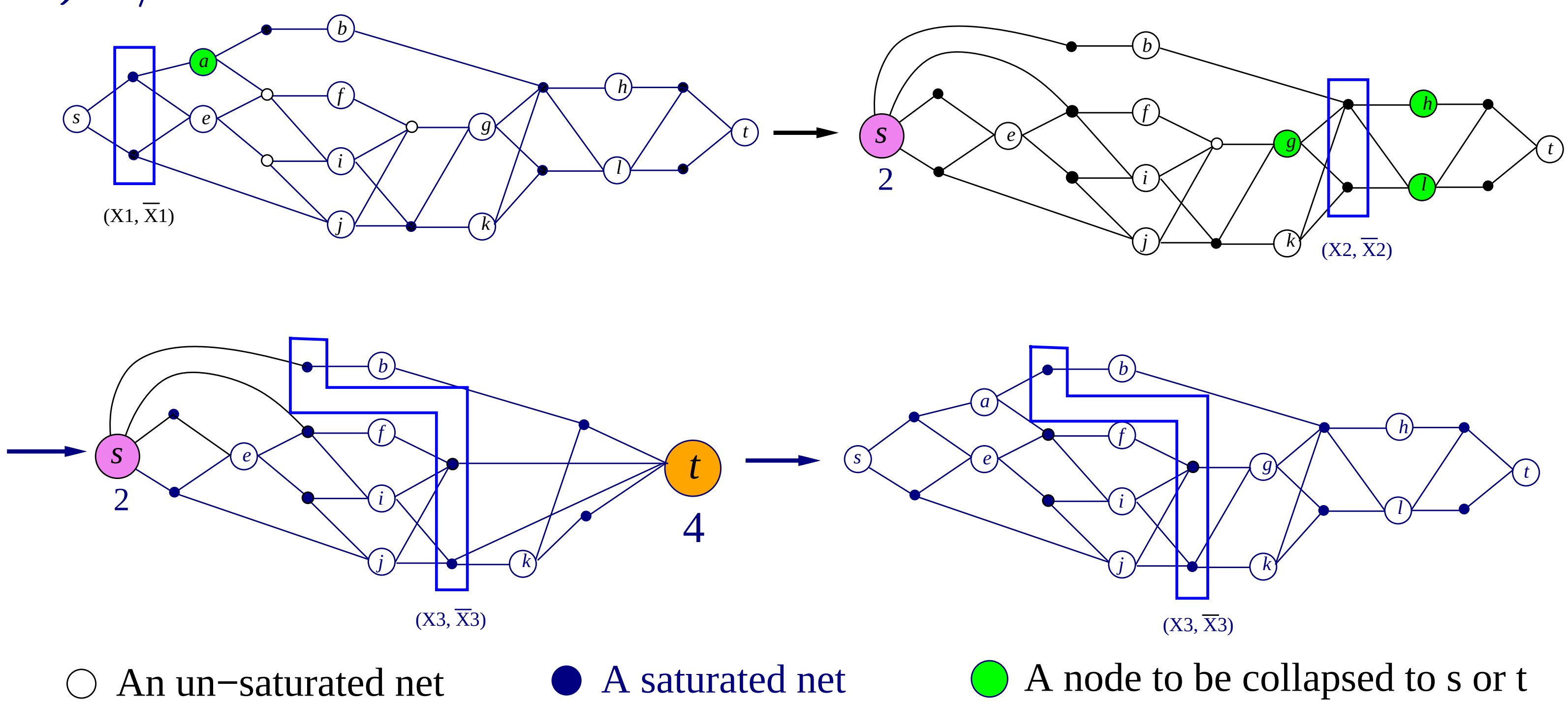
- A min-net-cut $(X, \bar{X})$ in $N \iff$ A min-capacity-cut $(X', \bar{X}')$ in N'.
- Size of flow network: $|V'| \leq 3|V|$, $|E'| \leq 2|E| + 3|V|$.
- Time complexity: O(min-net-cut-size) $\times |E|$ = O($|V||E|$).



X
X̄
g
c
r
b
t
s
d
N
X'
X̄'
∞
g
c1
c2
1
r
b1
b2
t
s
d1
d2
N'

Repeated Min-Cut for Balanced Bipartition (FBB)

- Allow component weights to deviate from $(1-\epsilon)W/2$ to $(1+\epsilon)W/2$.



Incremental Flow

- Repeatedly compute max-flow: very time-consuming.
- No need to compute max-flow from scratch in each iteration.
- Retain the flow function computed in the previous iteration.
- Find additional flow in each iteration. Still correct.
- FBB time complexity: $O(|V||E|)$, same as **one** max-flow.
  - At most $2|V|$ augmenting path computations.
    - At each augmenting path computation, either an augmenting path is found, or a new cut is found, and at least 1 node is collapsed to s or t.
    - At most $|f| \leq |V|$ augmenting paths found, since bridging edges have unit capacity.

– An augmenting path computation: $O(|E|)$ time.